MATH 174, ANSWERS TO PRACTICE PROBLEMS FOR TEST 2

1. $1+3+3^2+3^3+\cdots+3^{99}+3^{100}=\sum_{k=1}^{101} 3^{k-1}$

2. $1-3+3^2-3^3+\cdots-3^{99}+3^{100}=\sum_{k=1}^{101} (-3)^{k-1}$

3. $\frac{1}{2}-\frac{1}{22}=\frac{5}{11}$

4. 2460

5. $\frac{1}{5}\left(2^{12}-1\right)$

6. $(A\cap B)\cup(A\cap C)$

7. $(A\cup B)^c=A^c\cap B^c$
   $(A\cap B)^c=A^c\cup B^c$

8. False

9. False

10. (a) $\{a\}$
    (b) $\{c\}$

11. $\{(1,1),(1,2),(2,1),(2,2)\}$

12. Yes. Every integer n can be written uniquely in the form $4q+r$ where q is an integer and $r\in\{0,1,2,3\}$. The integer n is in B precisely when $r=0$ and is in C precisely when $r=2$. Since $4q+0=2(2q)$, $4q+1=2(2q)+1$, $4q+2=2(2q+1)$, and $4q+3=2(2q+1)+1$, we see that n is odd precisely when r is 1 or 3. In other words, n is in A precisely when $r=1$ or $r=3$. It follows that every integer belongs to exactly one of A, B, or C so that A, B, and C form a partition of $\mathbb{Z}$.

13. $\left\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\right\}$

14. 21

15. $A[56]$

16. 60

17. 36

18. 60

19. -9

20. $1\quad 11\quad 55\quad 165\quad 330\quad 462\quad 462\quad 330\quad 165\quad 55\quad 11\quad 1$

21. $-2^5\binom{8}{5}=-1792$

22. (a) 49
    (b) 36
    (c) 4
    (d) $\sum_{k=1}^{8} k^2=204$

23. Let $P(n)$ be the statement that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n}} \geq \boxed{\sqrt{n}}.$$

We prove that $P(n)$ is true for every positive integer n by using $\boxed{\text{induction}}$. We first show that $\boxed{P(1)}$ is true. Since $\sqrt{1}=1$, we see that $1/\sqrt{1} \geq \sqrt{1}$. Thus, $\boxed{P(1)}$ is in fact true. Next, we suppose that k is an integer ≥ 1 such that $\boxed{P(k)}$ is true. This is called our induction hypothesis. Thus, our induction hypothesis is asserting that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{k-1}}+\frac{1}{\sqrt{k}} \geq \boxed{\sqrt{k}}.$$

From the induction hypothesis, we obtain that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{k-1}}+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \geq \sqrt{k}+\frac{1}{\sqrt{k+1}}.$$

Observe that $\sqrt{k(k+1)} > \sqrt{k^2} = k$ so that

$$\sqrt{k(k+1)}+1 > k+1.$$

Dividing by $\sqrt{k+1}$, we deduce that

$$\boxed{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \boxed{\sqrt{k+1}}.$$

We deduce that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{k-1}}+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \geq \boxed{\sqrt{k+1}}.$$

We have shown that if $P(k)$ is true, then $\boxed{P(k+1)}$ is true. This completes the induction argument. Therefore, $P(n)$ is true for every integer $n \geq 1$. ■