MATH 174, ANSWERS TO PRACTICE PROBLEMS FOR *After* TEST 2

1. (a) $u_2 = 4$, $u_3 = 24$, and $u_4 = 112$
   (b) $-2^n + \dfrac{4^n}{2}$ (or $2^{2n-1} - 2^n$)

2. The answer is (c) since $(x-6)(x-2) = x^2 - 8x + 12$.

3. The answer is $u_{n+1} = u_n + 2u_{n-1}$ since $(x-2)(x+1) = x^2 - x - 2$.

4. Take a number $M'' = MM'$, and take a number x_0'' greater than both x_0 and x_0'. If $x \geq x_0''$, then $x \geq x_0$ and $x \geq x_0'$ so that both $|f(x)| \leq M|g(x)|$ and $|g(x)| \leq M'|h(x)|$ hold. We deduce that

   $$|f(x)| \leq M|g(x)| \leq MM'|h(x)| = M''|h(x)| \qquad \text{for all } x \geq x_0''.$$

   Hence, $f(x) = O(g(x))$.

5. (a) $x = O(x^2)$ since $x \leq x^2$ for every $x \geq 1$
   (b) $x^3 \neq O(x^2)$ since for every $M > 0$ and every $x > M$, we have $x^3 > Mx^2$
   (c) $(x-1)^2 = O(x^2)$ since $(x-1)^2 \leq x^2$ for every $x \geq 1$
   (d) $(x+1)^2 = O(x^2)$ since $(x+1)^2 \leq (2x)^2 = 4x^2$ for every $x \geq 1$
   (e) $(\log_2 x)^{24} = O(x^2)$ since $(\log_2 x)^k = O(x)$ for every k and since $x = O(x^2)$

6. (a) $\displaystyle\sum_{k=1}^{n} k^4 = O(n^5)$ since

   $$\sum_{k=1}^{n} k^4 \leq (\text{the number of terms}) \times (\text{the largest term}) \leq n \times n^4 = n^5 \qquad \text{for all } n \geq 1.$$

   (b) $\displaystyle\sum_{k=1}^{n} k^5 \neq O(n^5)$; to see this consider any M. If $n > 64M$ (so if n is large enough), then there are at least $n/2$ terms in the sum (the largest terms) that are at least as large as $(n/2)^5$. We obtain that for $n > 64M$,

   $$\sum_{k=1}^{n} k^5 \geq (n/2) \times (n/2)^5 = \frac{n}{64} n^5 > Mn^5.$$

   It follows that $\displaystyle\sum_{k=1}^{n} k^5 \neq O(n^5)$.

   (c) $\displaystyle\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = O(n^5)$ since for all $n \geq 3$ we have

   $$\begin{aligned}\sum_{k=1}^{n} k(k+1)(k+2)(k+3) &\leq (\text{the number of terms}) \times (\text{the largest term}) \\ &\leq n \times n(n+1)(n+2)(n+3) \leq n \times n(2n)(2n)(2n) = 8n^5.\end{aligned}$$

   (d) $\displaystyle\sum_{k=1}^{n} 2^k \neq O(n^5)$; the sum is at least as big as its largest term which is 2^n. Since $2^n > n^6$ for if n is large enough, then for every M we have $2^n > n^6 \geq Mn^5$ for n large. It follows that $\displaystyle\sum_{k=1}^{n} 2^k \neq O(n^5)$.

7. The degree of A is 2, the degree of B is 4, and the degree of C is 3.

8. $CBADCEDFGHIBEG$ (there are many correct answers, but the first and final letters must be C and G in some order)

9. No, it contains K_5 as a subgraph and, hence, cannot be planar.