SOLUTIONS TO TEAM PROBLEMS 01/98

ANSWERS:

T1:	$5 \pm \sqrt{15}$ and $5 - \sqrt{57}$	T5:	$(1/\sqrt{3}, 2/\sqrt{3})$
T2:	16	T6:	500
T3:	$a = 1000001$ and $b = 2$	T7:	39
T4:	457/1998	T8:	299997/8

T1. Suppose first that P is on segment $\overline{CD}$. The condition $\angle APC + \angle BPD = 90°$ implies that $\angle APB = 90°$. Let $Q = (5,5)$ and note that Q is the midpoint of segment $\overline{AB}$. Since $\angle APB = 90°$, we deduce that P is on the circle centered at Q of radius QB. Hence, $QP = QB$ and we get

$$(x_0 - 5)^2 + 5^2 = 6^2 + 2^2 \quad \Longrightarrow \quad x_0 = 5 \pm \sqrt{15}.$$

Note that the points $(5 \pm \sqrt{15}, 0)$ are on $\overline{CD}$.

Now, suppose P is on the x-axis but not on $\overline{CD}$. Let $B' = (11, -3)$. The condition $\angle APC + \angle BPD = 90°$ implies that $\angle APB' = 90°$. Let $Q' = (5,2)$, the midpoint of $\overline{AB'}$. Then P is on the circle centered at Q' of radius $Q'B'$. Thus,

$$(x_0 - 5)^2 + 2^2 = 6^2 + 5^2 \quad \Longrightarrow \quad x_0 = 5 \pm \sqrt{57}.$$

The point $(5 + \sqrt{57}, 0)$ is on $\overline{CD}$ and, hence, is not an answer. The point $(5 - \sqrt{57}, 0)$ is not on $\overline{CD}$ and, hence, is an answer. Thus, the possible values for x_0 are $5 \pm \sqrt{15}$ and $5 - \sqrt{57}$.

T2. We describe a path by listing the points it traverses in order. A path (as described) traversing 16 edges is $FABCDEFGJFIJDHJBG$. It remains to show that more than 16 of the 18 edges cannot be traversed. Each of the 6 points B, D, F, G, H, and I has an odd number of edges emanating from it. Suppose we are given a point in a path and the given point is neither the first nor the last point in the path. The number of edges in the path containing the given point as an endpoint must be even since for every edge traversed going to the point there must be another edge traversed going from the point. Thus, in every path, at least 4 of the 6 points B, D, F, G, H, and I must have the property that there is an edge not in the path that has the point as an endpoint. Since an edge has two endpoints, it follows that there must be at least two edges not traversed in every path. Therefore, each path as described in the problem can traverse at most 16 edges.

T3. We use that $(\sqrt{k+1} + \sqrt{k})(\sqrt{k+1} - \sqrt{k}) = 1$ to rewrite the sum as a "telescoping" sum. We obtain

$$\sum_{k=2}^{1000000} \frac{1}{\sqrt{k} + \sqrt{k+1}} = \sum_{k=2}^{1000000} \left(\sqrt{k+1} - \sqrt{k}\right) = \sqrt{1000001} - \sqrt{2}.$$

Thus, apparently $a = 1000001$ and $b = 2$. It can easily be shown that this choice for a and b is unique, but this is not required by the problem.

T4. The primes dividing 10! are 2, 3, 5, and 7. For d a positive integer, define $A(d)$ to be the number of integers in $S = \{1, 2, 3, \ldots, 1998\}$ which are divisible by d. Then $A(d)$ is the number of positive integral multiples of d which are ≤ 1998. It follows that $A(d) = [1998/d]$ where $[x]$ denotes the greatest integer $\leq x$. By the principle of inclusion-exclusion, the number of integers not divisible by any of 2, 3, 5, and 7 is

$$\begin{aligned} &A(1) - A(2) - A(3) - A(5) - A(7) + A(6) + A(10) + A(14) \\ &\qquad + A(15) + A(21) + A(35) - A(30) - A(42) - A(70) - A(105) + A(210). \end{aligned}$$

Given $A(d) = [1998/d]$, the above can be computed quickly with a calculator. It's value is 457. Therefore, the probability that an element of S chosen at random is not divisible by any of 2, 3, 5, and 7 is $457/1998$.

T5. The equation of the ellipse must be of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some numbers a, b, c, d, e, and f. Since the ellipse is symmetric about the origin and passes through the points $(1, 0)$, $(0, 1)$, and $(1, 1)$, it must also pass through the points $(-1, 0)$, $(0, -1)$, and $(-1, -1)$. Plugging these 6 points into the equation of the ellipse above, we obtain the equations:

(i) $a + d + f = 0$	(iv) $a - d + f = 0$
(ii) $c + e + f = 0$	(v) $c - e + f = 0$
(iii) $a + b + c + d + e + f = 0$	(vi) $a + b + c - d - e + f = 0$

The equations (i) and (iv) imply $d = 0$. The equations (ii) and (v) imply $e = 0$. Now, (i) and (ii) imply $a = -f$ and $c = -f$. Plugging in these values for d, e, a, and c into (iii) gives $b = f$. Thus, the equation of the ellipse is $-fx^2 + fxy - fy^2 + f = 0$. This equation would not describe an ellipse unless $f \neq 0$. Thus, we can divide by $-f$ to deduce that the ellipse can be described by the equation $x^2 - xy + y^2 - 1 = 0$. Hence, if (x, y) is a point on the ellipse, then

$$(*) \qquad \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2 = x^2 - xy + y^2 = 1.$$

The two terms on the left-hand side are clearly non-negative. Since their sum is 1, each must be ≤ 1. Thus, the largest possible value for y is $2/\sqrt{3}$ which occurs when the second term is 1. When $x = y/2 = 1/\sqrt{3}$, $(*)$ clearly holds so that the desired point on the ellipse is $(1/\sqrt{3}, 2/\sqrt{3})$.

T6. The number k which is the coefficient of x^{125} in $f(x)$ can be determined by taking minus the sum of all products of 875 roots of $f(x)$. The roots of $f(x)$ are the integers from 1 to 1000. The product of 875 roots of $f(x)$ divisible by the least power of 2 is the one obtained by using the 500 odd positive integers ≤ 1000, the 250 positive integers ≤ 1000 that are divisible by 2 and not 4, and the 125 positive integers ≤ 1000 that are divisible by 4 and not 8. The product of these 875 roots of $f(x)$ is divisible by 2^{500} and not by 2^{501}.

Every other product of 875 roots of $f(x)$ must be divisible by a larger power of 2 (think about it if this is not clear) and, hence, by 2^{501}. We deduce that there are integers s and t with s odd such that

$$k = 2^{500}s + 2^{501}t = 2^{500}(s + 2t).$$

Since s is odd, we deduce $s + 2t$ is odd and the answer is $n = 500$.

T7. Let $N = 15^{\left(15^{15}\right)} + 15$. Clearly, N is divisible by both 3 and 5, so these must be two of the four least primes dividing N. Also, N is even, so N is divisible by 2. It remains to determine the next least prime dividing N. For each prime $p > 5$, we compute the smallest positive integer, say $e(p)$, such that $15^{e(p)} \equiv 1 \pmod{p}$. We then compute the value of 15^{15} modulo $e(p)$, call it $r(p)$. Then there is an integer $q(p)$ such that $15^{15} = q(p)e(p) + r(p)$. (One can make use of a calculator for these computations.) It follows that

$$N = 15^{\left(15^{15}\right)} + 15 = 15^{q(p)e(p)+r(p)} + 15 = \left(15^{e(p)}\right)^{q(p)} 15^{r(p)} + 15 \equiv 15^{r(p)} + 15 \pmod{p}.$$

We compute $15^{r(p)} + 15$ modulo p directly to determine if p divides N. For example, if $p = 13$, then we compute $e(13) = 12$ and $r(13) = 3$ (the latter may we require two steps with a calculator, first computing the value of 15^3 modulo 12 and then raising this value to the fifth power modulo 12). Then we compute the value of $15^{r(p)} + 15 = 15^3 + 15$ modulo $p = 13$. For $p = 13$, we obtain the number 10 so $N \equiv 10 \pmod{13}$ which implies N is not divisible by 13. Letting p vary, we derive:

$$\begin{aligned}
e(7) = 1,\ r(7) = 0 \quad &\Longrightarrow \quad N \equiv 15^0 + 15 \equiv 2 \pmod{7},\\
e(11) = 5,\ r(11) = 0 \quad &\Longrightarrow \quad N \equiv 15^0 + 15 \equiv 5 \pmod{11},\\
e(13) = 12,\ r(13) = 3 \quad &\Longrightarrow \quad N \equiv 15^3 + 15 \equiv 10 \pmod{13},\\
e(17) = 8,\ r(17) = 7 \quad &\Longrightarrow \quad N \equiv 15^7 + 15 \equiv 6 \pmod{17},\\
e(19) = 18,\ r(19) = 9 \quad &\Longrightarrow \quad N \equiv 15^9 + 15 \equiv 14 \pmod{19},\\
e(23) = 22,\ r(23) = 1 \quad &\Longrightarrow \quad N \equiv 15^1 + 15 \equiv 7 \pmod{23},\\
e(29) = 28,\ r(29) = 15 \quad &\Longrightarrow \quad N \equiv 15^{15} + 15 \equiv 0 \pmod{29}.
\end{aligned}$$

We deduce that the least prime $p > 5$ dividing N is 29. Thus, the answer is $2+3+5+29 = 39$. There are ways of simplifying the computations, for example by taking advantage of Fermat's Little Theorem which implies that $15^{p-1} \equiv 1 \pmod{p}$ when $p > 5$.

T8. We use that $\cos\theta = \left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right)/2$ (this can be derived from $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$). Thus,

$$(**) \qquad \cos^4\theta = \frac{1}{16}\left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right)^4 = \frac{1}{16}\left(e^{4\mathrm{i}\theta} + 4e^{2\mathrm{i}\theta} + 6 + 4e^{-2\mathrm{i}\theta} + e^{-4\mathrm{i}\theta}\right).$$

Suppose that k is a non-zero integer and n is a positive integer for which n does not divide k. Then

$$\sum_{j=1}^{n} e^{2\pi\mathrm{i}kj/n} = e^{2\pi\mathrm{i}k/n}\,\frac{e^{2\pi\mathrm{i}k} - 1}{e^{2\pi\mathrm{i}k/n} - 1}$$

where the latter is obtained by taking $x = e^{2\pi \mathrm{i}k/n}$ in the polynomial identity

$$x + x^2 + x^3 + \cdots + x^n = \frac{x\big(x^n - 1\big)}{x - 1}.$$

Since k is an integer, $e^{2\pi \mathrm{i}k} - 1 = \cos(2\pi k) + \mathrm{i}\sin(2\pi k) - 1 = 0$. Hence, the numerator above is 0. (Note that the condition that n does not divide k implies the denominator $e^{2\pi \mathrm{i}k/n} - 1 \neq 0$.) Thus,

$$(***) \qquad \sum_{j=1}^{n} e^{2\pi \mathrm{i}kj/n} = 0 \qquad \text{provided } n \text{ does not divide } k.$$

Observe that if $k < n$, then n cannot divide k. We consider $n > 4$. We take $\theta = 2\pi j/n$ and sum from $j = 1$ to n on both sides of $(**)$. The left-hand side becomes $\displaystyle\sum_{j=1}^{n} \cos^4\left(\frac{2\pi j}{n}\right)$. The right-hand side can be expressed as

$$\frac{1}{16}\left(\sum_{j=1}^{n} e^{2\pi \mathrm{i}\times 4j/n} + 4\sum_{j=1}^{n} e^{2\pi \mathrm{i}\times 2j/n} + 6\sum_{j=1}^{n} 1 + 4\sum_{j=1}^{n} e^{2\pi \mathrm{i}\times(-2)j/n} + \sum_{j=1}^{n} e^{2\pi \mathrm{i}\times(-4)j/n}\right).$$

The middle sum has value n and, using $(***)$, we deduce that each of the remaining sums is 0. Hence,

$$\sum_{j=1}^{n} \cos^4\left(\frac{2\pi j}{n}\right) = \frac{1}{16}(6n) = \frac{3n}{8} \qquad \text{for } n > 4.$$

Taking $n = 99999$, we derive the answer to the problem, namely $299997/8$.