SOLUTIONS TO TEAM PROBLEMS 02/97

T1. The number of times a prime p divides $n!$ is given by

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots,$$

where $[x]$ denotes the greatest integer $\le x$ (note the sum above is finite since $[n/p^k]$ is 0 if k is large). We deduce that if 2^r is the highest power of 2 dividing 30!, then $r=[30/2]+[30/4]+\cdots=15+7+3+1=26$. Similarly, if 3^s is the highest power of 3 dividing 30!, then $s=[30/3]+[30/9]+\cdots=10+3+1=14$. Continuing, we obtain

$$30! = 2^{26}\times 3^{14}\times 5^7\times 7^4\times 11^2\times 13^2\times 17\times 19\times 23\times 29.$$

The number of divisors is obtained by multiplying one more than each of the exponents together, so the number of divisors is $27\times 15\times 8\times 5\times 3\times 3\times 2\times 2\times 2\times 2=\boxed{2332800}$.

T2. Draw line segments joining the $(0,0)$ to each of $(2\sqrt{3},2)$ and $(2\sqrt{3},-2)$ and joining $(16,0)$ to each of $(11,5\sqrt{3})$ and $(11,-5\sqrt{3})$. The answer is easily obtained by summing the appropriate areas of triangles and trapezoids and subtracting the sum of the areas of two sectors. The answer is $\boxed{-8+80\sqrt{3}-36\pi}$.

T3. The thisforlackofabettername points are seen to be those of the form (x,y) where $x\ge |y|$ and $x+y$ is even. The condition $x\ge|y|$ comes from noting that $(0,0)$ is the initial point of a path and other points are obtained by adding at least as much to the x-coordinate as the absolute value of what is added to the y-coordinate. The condition $x+y$ is even is seen, for example, by induction. That each of these points can be obtained is also easy to see (to get such an (x,y), simply consider the path from $(0,0)$ to $((x-y)/2,(y-x)/2)$ along a straight line and then from $((x-y)/2,(y-x)/2)$ to (x,y) along a straight line). The number of such (x,y) satisfying $0\le x\le 10$ and $0\le y\le 10$ is $\boxed{36}$.

T4. Since $PA-PC=(PA-PB)+(PB-PC)$, it suffices to consider the case that only $PA-PB$ and $PB-PC$ are integers. The sum of the lengths of two sides of a triangle is at least as big as the length of the third side so that $PA+AB\ge PB$ (by considering ΔABP) and $PB+BC\ge PC$ (by considering ΔBCP). Note that equality could occur if A, B, and P are collinear of if B, C, and P are collinear, respectively. Similarly, we get $PB+AB\ge PA$ and $PC+BC\ge PB$. The conditions in the problem imply $AB=1$ and $BC=2$. It follows that $-1\le PA-PB\le 1$ and $-2\le PB-PC\le 2$. Since $PA-PB$ and $PB-PC$ are integers, we have $PA-PB\in\{-1,0,1\}$ and $PB-PC\in\{-2,-1,0,1,2\}$. The condition $PA-PB=\pm1$ occurs precisely when P is on the line passing through A and B. The condtion $PA-PB=0$ occurs precisely when P is on the perpendicular bisector of line segment $\overline{AB}$. The condition $PB-PC=\pm2$ occurs precisely when P is on the line passing through B and C. The condition $PB-PC=\pm1$ occurs precisely when

P is on a specific hyperbola with foci at B and C (that the absolute value of the difference in the distances PB and PC is constant defines geometrically a hyperbola). The condtion $PB - PC = 0$ occurs precisely when P is on the perpendicular bisector of line segment $\overline{BC}$. We need both $PA - PB \in \{-1, 0, 1\}$ and $PB - PC \in \{-2, -1, 0, 1, 2\}$ to occur so that in the end we are interested in the number of intersection points between one of the two lines determined by $PA - PB \in \{-1, 0, 1\}$ and one of the two lines and one hyperbola determined by $PB - PC \in \{-2, -1, 0, 1, 2\}$. One counts the number of intersection points directly. For example, the line passing through A and B and the hyperbola defined above intersect in two points. It is of relevance when counting to note that the point $Q = (0, 1/2)$ is not on the hyperbola (this can be seen by computing $QB - QC$ and verifying that it is not ± 1). We deduce that there are $\boxed{6}$ points P as in the problem.

T5. Using the identities

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \text{ and } \cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

with $A = n\theta$ and $B = \theta$, one obtains

$$(*) \qquad \cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(\theta)\cos(n\theta).$$

We take $g_1(x) = x$ and $g_2(x) = 2x^2 - 1$ so that $\cos(\theta) = g_1(\cos(\theta))$ and $\cos(2\theta) = g_2(\cos(\theta))$. By setting $g_{n+1}(x) = 2xg_n(x) - g_{n-1}(x)$ for $n \geq 2$, we get from $(*)$ that $\cos(m\theta) = g_m(\cos(\theta))$ for all positive integers m. The formula $g_{n+1}(x) = 2xg_n(x) - g_{n-1}(x)$ for $n \geq 2$ (together with the values given for $g_1(x)$ and $g_2(x)$) allows one to compute $g_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$. Letting $u = \cos(\theta)$ with $\theta = 10°$, we deduce that

$$\frac{1}{2} = \cos(60°) = \cos(6\theta) = g_6(\cos(\theta)) = 32u^6 - 48u^4 + 18u^2 - 1.$$

Thus, for $u = \cos(\theta)$ with $\theta = 10°$, we have

$$0 = 64u^6 - 96u^4 + 36u^2 - 3 = (2u)^6 - 6(2u)^4 + 9(2u)^2 - 3.$$

It follows that $2\cos(10°) = 2u$ is a root of $\boxed{x^6 - 6x^4 + 9x^2 - 3}$. (The answer is unique.)

T6. Let A denote the die with one garnet face, and let R denote the garnet face. Let B denote the die with two garnet faces; denote them by S_1 and S_2. Let C denote the die with three garnet faces; denote them by T_1, T_2, and T_3. The possibilities for the roll of the three dice are:

	Two Faces Known	Third Face	# of Cases
1.	R and an S_i	T_1, T_2, or T_3	6
2.	R and an S_i	one of 3 blue faces of die C	6
3.	R and a T_j	S_1 or S_2	6
4.	R and a T_j	one of 4 blue faces of die B	12
5.	an S_i and a T_j	R	6
6.	an S_i and a T_j	one of 5 blue faces of die A	30

There are 66 cases above exactly 18 of which occur with the third face turning up garnet. The probability is 18/66 or (simplified) $\boxed{3/11}$.

T7. If $g(x)$ is a reciprocal polynomial of degree r (and non-zero constant term), then $x^r g(1/x) = g(x)$. Suppose $g(x)$ is a reciprocal polynomial of degree r dividing $u(x) = x^{1234} - x^3 - x + 1$ so that $u(x) = g(x)h(x)$ for some polynomial $h(x)$ of degree $1234 - r$. Then

$$\begin{aligned} x^{1234} - x^{1233} - x^{1231} + 1 = x^{1234}u(1/x) &= x^{1234}g(1/x)h(1/x) \\ &= x^r g(1/x)x^{1234-r}h(1/x) = g(x)\big(x^{1234-r}h(1/x)\big). \end{aligned}$$

Thus, $g(x)$ is a factor of $w(x) = x^{1234} - x^{1233} - x^{1231} + 1$. Since $g(x)$ divides each of $u(x)$ and $w(x)$, we deduce that it divides

$$(*) \qquad \begin{aligned} x^3 w(x) - (x^3 - x^2 - 1)u(x) &= x^3 + (x^3 - x^2 - 1)(x^3 + x - 1) \\ &= x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1 = (x-1)^2(x^2+1)(x^2+x+1). \end{aligned}$$

It follows that $g(x)$ must divide this last expression. In particular, the only possible factors of $g(x)$ are $x-1$, x^2+1, and x^2+x+1. Each of these is a factor of $u(x) = x^{1234} - x^3 - x + 1$ which can be verified as follows. First, $u(1) = 0$ and $u(i) = 0$ imply that $x-1$ and x^2+1 are factors. Since x^2+x+1 is a factor of x^3-1 and x^3-1 is a factor of

$$x\left(x^{3\times 411} - 1\right) - \left(x^3 - 1\right) = x^{1234} - x^3 - x + 1,$$

we get x^2+x+1 is a factor of $u(x)$.

Next, we note that $(x-1)^2$ does not divide $u(x)$. Two ways to see this are:

(i) Use that if $(x-1)^2$ is a factor of $u(x)$, then $u'(1) = 0$ but $u'(1) = 1230 \neq 0$. (This is a Calculus approach; a non-Calculus approach follows.)

(ii) Use that if $(x-1)^2$ is a factor of $u(x)$, then $u(x)^2 = (x-1)^2 v(x)$ for some polynomial $v(x)$ so that $u(x+1) = x^2 v(x+1)$. This implies that the coefficient of x in $u(x+1)$ is 0. The Binomial Theorem implies, however, that the coefficient of x in $u(x+1) = (x+1)^{1234} - (x+1)^3 - (x+1) + 1$ is 1230.

Either way, we deduce that $(x-1)^2$ is not a factor of $u(x)$. It follows that $g(x)$ divides $(x-1)(x^2+1)(x^2+x+1)$. Since $g(x)$ is reciprocal, the leading coefficient and constant term must be equal. It follows that $x-1$ is not a factor of $g(x)$. (Note that if $(x-1)^2 = x^2-2x+1$ could be a factor of $g(x)$, the situation would be different.) The reciprocal factor of $u(x)$ of largest degree is now seen to be $(x^2+1)(x^2+x+1) = \boxed{x^4 + x^3 + 2x^2 + x + 1}$.

T8. Observe that $S < \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, so that the decimal representation of S in the problem is justified (i.e., there are no non-zero digits to the left of the decimal). We work with

$$10^{24}S = d_1 d_2 d_3 \ldots d_{24}.d_{25}d_{26}\ldots.$$

For $n \geq 5$, the value of $10^{24}/2^{n!}$ is less than or equal to $10^{24}/2^{5!+(n-5)}$ so that

$$10^{24}\sum_{n=5}^{1997}\frac{1}{2^{n!}} \leq \frac{10^{24}}{2^{5!}}\sum_{n=5}^{1997}\frac{1}{2^{n-5}} < \frac{10^{24}}{2^{120}}\sum_{n=0}^{\infty}\frac{1}{2^{n}} = \frac{10^{24}}{2^{119}} < \frac{10^{24}}{(10^{3/10})^{119}} < 1.$$

On the other hand, for $n \leq 4$, the value of $10^{24}/2^{n!}$ is an integer, namely $2^{24-n!}5^{24}$. Observe that for $n \leq 3$, the number $10^{24}/2^{n!} = 2^{24-n!}5^{24}$ is divisible by $10^{24-3!} = 10^{18}$ so that the right 18 most digits of $10^{24}/2^{n!}$ are all 0. The sum of the three terms $10^{24}/2^{n!}$ for $1 \leq n \leq 3$ in $10^{24}S$ therefore contribute nothing to the first 18 digits to the left of the decimal in $10^{24}S$. It remains to consider $10^{24}/2^{n!}$ with $n = 4$. We have $10^{24}/2^{4!} = 5^{24}$. We compute the right most five digits of 5^{24}. This can be done easily with a calculator. For example, $5^8 = 390625$. We are only interested in the last 5 right most digits, so to calculate the last 5 digits of 5^{12} we can simply compute the value of 90625×625 which ends with 40625. If the calculator can calculate 40625×40625 exactly, then we can get the last 5 digits of 10^{24} by computing the last 5 digits in this product. Alternatively, one can multiply by 625 three times (each time dropping all but the last 5 digits). This gives that the last 5 digits of 5^{24} are 90625. The information above now implies that $d_{20} = 9$, $d_{21} = 0$, $d_{22} = 6$, $d_{23} = 2$, and $d_{24} = 5$. Hence, the sum in the problem is $9 + 0 + 6 + 2 + 5 = \boxed{22}$.

T9. Let $B' = (13, 4)$, and let P denote the intersection of the x-axis with the line passing through A and B'. We claim that with P so chosen, $AP - BP$ gives the desired maximum. To see this observe that if Q is any point on the x-axis, then $AQ - BQ = AQ - B'Q$. In particular, $AP - BP = AB' = \sqrt{12^2 + 5^2} = 13$. On the other hand, if $Q \neq P$, then since the sum of the lengths of two sides of a triangle is greater than the third, we obtain $AB' + B'Q > AQ$ so that $AQ - BQ = AQ - B'Q < AB' = 13$. Hence, the answer is $\boxed{13}$.

T10. Let $\alpha = 62831853071795864769252 8/10^{23}$, and observe that $\alpha \approx 2\pi$. Since $\alpha > 1$, we cannot use the estimate given in the problem for $\cos x$ directly. However, since $\cos\alpha = \cos(2\pi - \alpha)$, we can use the estimate by setting $x = 2\pi - \alpha$. Since $2\pi \times 10^{23} = 628318530717958647692528.6766559\ldots$, we obtain that

$$\cos\alpha = \cos(2\pi - \alpha) = 1 - \frac{1}{2}\left(\frac{0.6766559\ldots}{10^{23}}\right)^2 + E\left(\frac{0.6766559\ldots}{10^{23}}\right).$$

From the information given in the problem, $E(x) < 1/10^{92}$. Since $\frac{1}{2}(0.6766559\ldots)^2 = 0.22893\ldots$, it follows that $\cos\alpha = 1 - u + v$ where $u < 1$, the decimal expansion of u consists of 46 zeroes to the right of the decimal followed by the digits 22893, and the decimal expansion of v consists of at least 92 zeroes to the right of the decimal. Hence, $\cos\alpha = 0.999\ldots9977106\ldots$, where 46 nines occur between the decimal and the first digit seven. The answer is $\boxed{1}$.