SOLUTIONS TO TEAM PROBLEMS 02/96

T1. Since $x^4+4=(x^2-2x+2)(x^2+2x+2)$, we obtain

$$625^2+4=25^4+4=\left(25^2-2\times 25+2\right)\left(25^2+2\times 25+2\right)=577\times 677.$$

The primes must be 577 and 677.

T2. When P is located at $A=(x,1/2)$, suppose the center of the circle is located at $B=(u,1)$. Then u will equal the distance the center of the circle has traveled since P was at the origin which will also be equal to the arclength from $(x,1/2)$ to $C=(u,0)$ (the point where the circle touches the x-axis). It follows that $\angle ABC=u$ (in radians). Hence,

$$\sin u=\frac{u-x}{AB}=u-x \quad \text{and} \quad \cos u=\frac{1/2}{AB}=\frac{1}{2}.$$

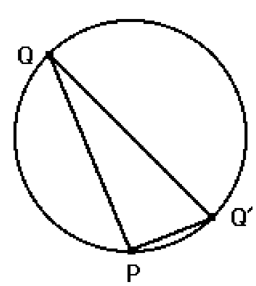
Hence, $u=\pi/3$ and $x=u-\sin u=(\pi/3)-\sqrt{3}/2$. (Note that $\cos u=1/2$ for other values of u but each of these will result in a value of $x>3$.)

T3. Use the "elementary symmetric functions." The sum of the roots of $a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_0$, where $a_n\neq 0$, is $\sigma_1=-a_{n-1}/a_n$. The sum of the roots taken two at a time is $\sigma_2=a_{n-2}/a_n$. Also, note that the sum of the roots squared is $\sigma_1^2-2\sigma_2$. In the present problem, we deduce

$$\alpha_1+\alpha_2+\alpha_3=0 \quad \text{and} \quad \alpha_1^2+\alpha_2^2+\alpha_3^2=(0)^2-2(-1)=2.$$

Since each α_j satisfies $\alpha_j^3-\alpha_j+1=0$ (so that $\alpha_j^3=\alpha_j-1$), we deduce $\alpha_j^4=\alpha_j^2-\alpha_j$ for each j. Therefore,

$$\alpha_1^4+\alpha_2^4+\alpha_3^4=(\alpha_1^2+\alpha_2^2+\alpha_3^2)-(\alpha_1+\alpha_2+\alpha_3)=2-0=2.$$



T4. Note that the problem did not ask for the average of the values of the square of the distance from P to Q. This would be a considerably easier question with unfortunately the same answer (the average of the distances being $4/\pi$ is given to clarify this distinction). To answer the question, imagine choosing first the point P and then the point Q. The distance from P to Q is determined then simply by the placement of Q relative to P. In other words, we can view P as being a fixed point on the circle. For each point Q on the circle, consider the point Q' such that $\overline{QQ'}$ is a diameter of the given circle.

Then $\angle QPQ' = \pi/2$ and $Q'Q = 2$ so that $QP^2 + Q'P^2 = Q'Q^2 = 4$. Thus, the average of QP^2 and $Q'P^2$ is 2. As P varies, we can deduce that the average of the distances squared is 2.

T5. Suppose $A = (x, y)$. The midpoint of $\overline{AB}$ being $(1, 2)$ implies that $B = (2-x, 4-y)$. Now the midpoint of $\overline{BC}$ being $(3, 0)$ implies that $C = (4+x, -4+y)$. Continuing in this manner, we deduce that $D = (12-x, 6-y)$, $E = (-2+x, 2+y)$, and $A = (8-x, 6-y)$. Comparing our two expressions for A, we deduce $x = 4$ and $y = 3$. Hence, $A = (4, 3)$.

T6. Call a polynomial $f(x)$ *palindromic* if it satisfies $x^{\deg f} f(1/x) = f(x)$ (the coefficients read the same in either direction). Observe that if $f(x)$ and $g(x)$ are both palindromic then so is their product (just use the definition). It follows that the polynomial $F(x) = (x^2 - x + 1)^3 (x^3 + 2x^2 + 2x + 1)^5$ is palindromic. Since $a_0 = 1$, we deduce

$$a_1 + a_2 + \cdots + a_{10} = \frac{1}{2}(a_0 + a_1 + \cdots + a_{21} - 2a_0) = \frac{1}{2}(F(1) - 2) = \frac{1}{2}(6^5 - 2) = 3887.$$

T7. The given information and the fact that $a_4 = 19$ imply that the first three primes to consider are 13, 17, and 23. One checks directly that $2^k + 3$ modulo 13 is 0 when $k = 10$ (i.e., $2^4 \equiv 3 \pmod{13} \implies 2^8 \equiv 9 \pmod{13} \implies 2^{10} + 3 \equiv 39 \equiv 0 \pmod{13}$). On the other hand, a direct computation gives that $2^k + 3$ modulo 17 is never 0 and $2^k + 3$ modulo 23 is never 0. The primes are 17 and 23.

T8. Bert's offer is better. A corresponding statement would be, "You will not give me ten dollars and you will not give me a billion dollars."

T9. There are 8^2 one-by-one squares, 7^2 two-by-two squares, 6^2 three-by-three squares, and so on. The answer is

$$1^2 + 2^2 + \cdots + 8^2 = 204.$$

T10. A rectangle will be determined by the endpoints of one of its diagonals. There are 9^2 possible endpoints. Choose one of these at random. The opposite endpoint of a diagonal cannot lie directly to the North, South, East, or West of the chosen endpoint (where we picture here the chessboard being oriented in an obvious manner). Thus, there are 8^2 possibilities for the second endpoint of a diagonal. This gives us $9^2 \times 8^2$ ways of choosing the endpoints. Each rectangle will be counted four times in the process, as the first endpoint chosen could correspond to any one of the four corners. This gives a total of $9^2 \times 8^2/4 = 1296$ rectangles.

By the way, there is a connection here with the answer to **T9**. Note that

$$1^3 + 2^3 + \cdots + 8^3 = (1 + 2 + \cdots + 8)^2 = 1296.$$