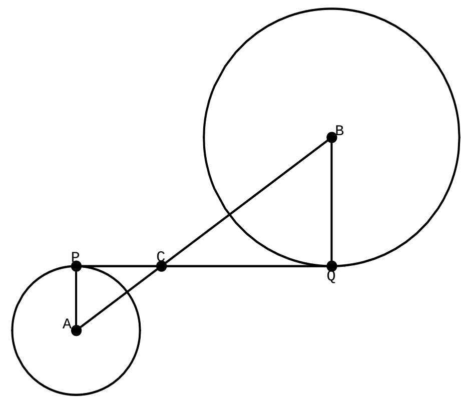
SOLUTIONS TO TEAM PROBLEMS 02/95

T1. Let $\mathcal{C}_1$ be the circle of radius 3 and $\mathcal{C}_2$ the circle of radius 4. Let A be the center of $\mathcal{C}_1$, and let B be the center of $\mathcal{C}_2$. Let C be the intersection of ℓ and $\overleftrightarrow{AB}$. Set $x = AC$ and note $BC = 25 - x$. Then ΔAPC is similar to ΔBQC, and we get

$$\frac{x}{3} = \frac{25-x}{4}.$$



Hence, $AC = x = (3/7)25 = 75/7$ and $BC = 25 - x = (4/7)25 = 100/7$. Therefore, the distance from P to Q is

$$\begin{aligned}&\sqrt{(3 \times 25/7)^2 - 9} + \sqrt{(4 \times 25/7)^2 - 16} \\ &\quad = \frac{3}{7}\sqrt{25^2 - 7^2} + \frac{4}{7}\sqrt{25^2 - 7^2} = \sqrt{25^2 - 7^2} = \sqrt{576} = 24.\end{aligned}$$

T2. $(1+2+4+8)(1+3+17+19)(1+7+31+61) = 15 \times 40 \times 100 = 60000$

T3. (Use the Euclidean Algorithm.) Let $d = \gcd(106577, 1071089)$. Then d divides $1071089 - (10 \times 106577) = 5319$ and, hence, $106577 - (20 \times 5319) = 197$. One checks that 197 divides 5319 and therefore both 106577 and 1071089. The answer is 197.

T4. Completing the square twice, $f(x,y) = (x-3y)^2 + (2y-1)^2 - 3$. Thus, $f(x,y) \geq -3$ for all pairs (x,y) and achieves this value when $y = 1/2$ and $x = 3/2$. The answer is $(x,y) = (1/2, 3/2)$.

T5. The sum can be evaluated via

$$\begin{aligned}\sum_{n=1}^{98} \frac{1}{n(n+1)(n+2)(n+3)} &= \sum_{n=1}^{98} \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right) \\ &= \frac{1}{3}\left(\frac{1}{6} - \frac{1}{99 \times 100 \times 101}\right).\end{aligned}$$

Since 6 divides $99 \times 100 \times 101$, the least common denominator of the difference above is $99 \times 100 \times 101 = 999900$. It is easy to see that the resulting numerator will not be divisible by any of the prime divisors of $99 \times 100 \times 101$ including 3. The answer is $3 \times 999900 = 2999700$.

T6. The answer is 6. After Dave and Michael choose 6, the value of S is 12. No matter what number k Dave chooses after this point, Michael can choose $7 - k$. Every turn by both Dave and Michael will increase the sum by 7. After Michael has four more turns, $S = 12 + (4 \times 7) = 40$, so Michael wins with this strategy. No other choice for Michael's first turn will do (if Michael picks $\ell \in \{1, 2, 3, 4, 5\}$, Dave can choose $6 - \ell$ for his second move and continue by choosing $7 - k$ for each integer k Michael chooses; Dave will be able to win).

T7. It suffices to count the number of real roots of

$$\begin{aligned} f(x) &= \left(x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1\right) - \left(x^5 + x^3 + x\right) \\ &= x^{12} + x^{10} + x^8 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1. \end{aligned}$$

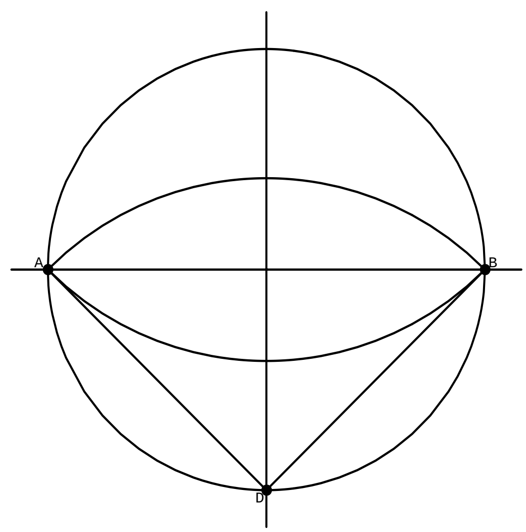
Clearly, $f(0) \neq 0$. Since

$$f(-x) = x^{12} + x^{10} + x^8 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,$$

Descartes Rule of Signs gives that $f(x)$ has no negative roots. Observe that

$$\begin{aligned} (x+1)f(x) &= (x+1)\left(x^{12} + x^{10} + x^8 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right) \\ &= x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + 1. \end{aligned}$$

By Descartes Rule of Signs, $(x+1)f(x)$ and, hence, $f(x)$ has no positive real roots. The answer is 0.



T8. Consider the circle $\mathcal{C}'$ centered at $D = (0, -3)$ and passing through A and B. The measure of angle $\angle ADB$ is $\pi/2$, from which it follows that for any point P on the smallest arc (along $\mathcal{C}'$) from A to B, the measure of angle $\angle APB$ is $3\pi/4$. Similarly, any point P on the smallest arc from A to B along a circle centered at $(0, 3)$ is such that the measure of angle $\angle APB$ is $3\pi/4$. Let R be the region strictly between these two arcs. Then P is a point in R if and only if the measure of $\angle APB$ is $> 3\pi/4$. The answer is therefore the ratio of the area of $\mathcal{C}$ minus the area of R to the area of $\mathcal{C}$. The area of $\mathcal{C}$ is 9π. Since $DB = 3\sqrt{2}$ and the measure $\angle ADB = \pi/2$, the area of R is

$$2 \times \left(\frac{1}{2}\left(\frac{\pi}{2}\right)(3\sqrt{2})^2 - \frac{1}{2}(3\sqrt{2})^2\right) = 18\left(\frac{\pi}{2} - 1\right) = 9\pi - 18.$$

After a little simplification, we deduce that the probability is $2/\pi$.

T9. Observe that $\binom{m}{k} = \binom{m}{m-k}$. Thus,

$$(x+1)^{50}(x+1)^{50} = \left(\sum_{k=0}^{50}\binom{50}{k}x^k\right)\left(\sum_{k=0}^{50}\binom{50}{k}x^k\right) = \left(\sum_{k=0}^{50}\binom{50}{k}x^k\right)\left(\sum_{k=0}^{50}\binom{50}{k}x^{50-k}\right).$$

The coefficient of x^{50} in this last product (when expanded) is

$$\binom{50}{0}^2 + \binom{50}{1}^2 + \binom{50}{2}^2 + \cdots + \binom{50}{50}^2.$$

On the other hand, $(x+1)^{50}(x+1)^{50} = (x+1)^{100}$, so the coefficient of x^{50} must be $\binom{100}{50}$. An answer is $a = 100$ and $b = 50$.

T10. Let t be the positive integer for which 2^t divides 1995! and 2^{t+1} does not divide 1995!. So we may write $1995! = 2^t m$ where m is an odd integer. Let

$$S = \sum_{j=1}^{1995} \frac{1995!}{j}.$$

Here, S is a sum of integers. Observe that the term $1995!/1024 = 2^{t-10}m$ is divisible by 2^{t-10} and not by 2^{t-9}. Also, since no other denominator in the sum S above is divisible by 1024, every other term is divisible by 2^{t-9}. Write

$$S - \frac{1995!}{1024} = 2^{t-9}k,$$

where k is an integer. Then

$$\frac{a}{b} = \frac{1}{1995!}S = \frac{1}{2^t m}\left(2^{t-9}k + 2^{t-10}m\right) = \frac{1}{2^t m}2^{t-10}\left(2k+m\right) = \frac{2k+m}{2^{10}m}.$$

Hence,

$$2^{10}ma = b(2k+m).$$

Since m is odd, so is $2k+m$, and we deduce that b is even. Only one of a or b is even, so a is odd. The above equation implies therefore that $r = 10$.