SOLUTIONS TO TEAM PROBLEMS
FEBRUARY, 2001

Answers:	1. $\sqrt{13}$	4. $\sqrt{3}/2$	7. 6
	2. 0.56395 (or 56.395%)	5. 240131	8. 224
	3. 2098960	6. -432	

1. Since $(3/\sqrt{13})^2 + (2/\sqrt{13})^2 = 1$, there is an angle ϕ such that $\cos\phi = 3/\sqrt{13}$ and $\sin\phi = 2/\sqrt{13}$. Hence,

$$\begin{aligned} 3\cos\theta - 2\sin\theta &= \sqrt{13}\left(\frac{3}{\sqrt{13}}\cos\theta - \frac{2}{\sqrt{13}}\sin\theta\right) \\ &= \sqrt{13}\big(\cos\phi\cos\theta - \sin\phi\sin\theta\big) = \sqrt{13}\cos(\phi+\theta). \end{aligned}$$

The value of $3\cos\theta - 2\sin\theta$ is apparently maximized then when $\theta = -\phi$. This maximum value is $\sqrt{13}$. (An alternative approach would be to use Calculus.)

2. The probability that the 6 people choose different numbers is

$$\frac{19}{20} \times \frac{18}{20} \times \frac{17}{20} \times \frac{16}{20} \times \frac{15}{20} = 0.43605$$

as the first person can choose any number, then the second person would have 19 numbers to choose, the third 18 numbers to choose, etc. Hence, the answer is $1 - 0.43605 = 0.56395$.

3. Let $N = 2^{6972593}$. Note that N has a non-zero unit's digit (since 5 does not divide N). Hence, $N - 1$ and N have the same number of digits. The number of digits of N is the smallest integer larger than

$$\log_{10} N = 6972593 \times \log_{10} 2 = 2098959.64\ldots.$$

The answer is 2098960.

4. Let A, B, and C be the vertices of the equilateral triangle. Note that $AB = AC = BC = 1$. It follows that the areas of ΔAPB, ΔAPC, and ΔBPC are $h_1/2$, $h_2/2$, and $h_3/2$ (in some order). Hence, $h_1 + h_2 + h_3$ must be twice the area of ΔABC, which is $\sqrt{3}/2$.

5. Let $g(x) = f(x) - 2x + 1$. The given information about the values of $f(x)$ implies $g(1) = g(2) = g(3) = g(4) = g(5) = 0$. Also, $g(x)$ is a fifth degree polynomial with leading coefficient 2001. It follows that

$$g(x) = 2001(x-1)(x-2)(x-3)(x-4)(x-5).$$

Hence, $g(6) = 2001 \times 5! = 240120$. Since $g(6) = f(6) - 11$, we deduce that $f(6) = 240131$.

6. Observe that

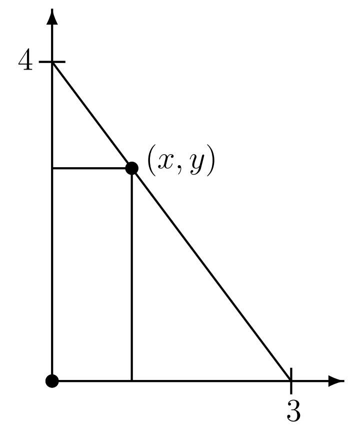
$$2x^2 + y^2 - 2xy + 12y + 72 = (x + 6)^2 + (x - y - 6)^2.$$

It follows that $2x^2 + y^2 - 2xy + 12y + 72 \leq 0$ if and only if $x + 6 = 0$ and $x - y - 6 = 0$ (strict inequality never holds). Solving, we deduce $x = -6$ and $y = -12$. Hence, $x^2 y = -432$.

7. The picture to the right represents one-half of the triangle and one-half of R placed in a xy-coordinate system. The point (x, y) is on the line $y = (-4x + 12)/3$. The area A of one-half of R is therefore $x(-4x + 12)/3$. As x varies from 0 to 3, the equation

$$A = \frac{x(-4x + 12)}{3} = \frac{-4x^2 + 12x}{3}$$

represents part of a parabola (in the variables x and A). Its maximum value is at its vertex $(3/2, 3)$. The answer is therefore $2 \times 3 = 6$.



8. The given information implies

$$\frac{10^r - 1}{N} = \frac{10^r}{N} - \frac{1}{N} = d_1 d_2 \ldots d_r$$

is an integer. Hence, N divides $10^r - 1$. The converse is also true; in other words, if N divides $10^r - 1$, then $1/N$ can be written as $0.\overline{d_1 d_2 \ldots d_r}$ (this follows by noting that the digits after the decimal of $10^r/N$ and $1/N$ must cancel in order for their difference to be an integer). In particular, we deduce that k is the minimal positive integer such that N divides $10^k - 1$ and (from the information given in the problem) if N divides $10^r - 1$, then k divides r. Since

$$\left(10^{16} + 1\right) N = \left(10^{16} + 1\right)\left(10^{96} - 10^{80} + 10^{64} - 10^{48} + 10^{32} - 10^{16} + 1\right) = 10^{112} + 1,$$

we obtain that N divides $10^{112} + 1$. Also, we deduce that N divides

$$10^{224} - 1 = \left(10^{112} + 1\right)\left(10^{112} - 1\right).$$

Therefore, k must divide 224. Note that $N > 10^{95}$. Hence, it is not possible for N to divide $10^r - 1$ for any positive integer $r \leq 95$. This implies that $k = 112$ or $k = 224$. Since N divides $10^{112} + 1$, we obtain that N cannot divide $10^{112} - 1$ (otherwise, it would divide 2 the difference between $10^{112} + 1$ and $10^{112} - 1$). Thus, $k = 224$.