SOLUTIONS TO TEAM PROBLEMS
February, 2000

Answers:			
	1. $\sqrt{26}$	4. $\sqrt{10}/5$	7. $(842401, 233640)$
	2. -2001000	5. $(-1/5, 7/5)$ and $(1, -1)$	8. $(-3+\sqrt{41})/4$
	3. $\sqrt{13}$	6. 2	

1. Squaring the expression in question gives 26. So it must equal $\sqrt{26}$.

2. The polynomial in question is monic and has degree $1+2+3+\cdots+2000 = 2001000$. The coefficient of $x^{2000999}$ is, therefore, minus the sum of the roots counted to their multiplicity. Noting that $n^2 - (n+1)^2 = -n-(n+1)$, we deduce this coefficient is

$$1^2 - 2^2 + \cdots + 1999^2 - 2000^2 = -1 - 2 - \cdots - 1999 - 2000 = -2001000.$$

3. If x is one of the possible values of BC, then by the Law of Cosines we obtain $AC^2 = AB^2 + x^2 - 2xAB\cos 60^\circ$. Simplifying, we deduce $x^2 - 9x + 17 = 0$. The positive difference of these solutions is the value of $\ell - s$. By the quadratic formula, we see that $\ell - s = \sqrt{13}$.

4. Let h denote the height and b the base of the right triangle. Then $h^2 + b^2 = 10^2$ so that $b^2 = 100 - h^2$. We want to know the probability that $(1/2)bh > 15$ (where $0 \le h \le 10$). Since $(1/2)bh > 15$ precisely when

$$\bigl(100 - h^2\bigr)h^2 = b^2h^2 > 900,$$

we deduce that $(1/2)bh > 15$ precisely when h is between the two positive real roots of $x^4 - 100x^2 + 900$. Since $x^4 - 100x^2 + 900 = (x^2-90)(x^2-10)$, the two positive real roots are $\sqrt{10}$ and $\sqrt{90} = 3\sqrt{10}$ (both between 0 and 10). The probability is therefore

$$\frac{1}{10}\Bigl(3\sqrt{10} - \sqrt{10}\Bigr) = \frac{\sqrt{10}}{5}.$$

5. If (x, y) is such a point, then $x^2 + y^2 = 2$. Furthermore, the points $(0, 0)$, (x, y), and $(4, 2)$ form the vertices of a right triangle so that

$$x^2 + y^2 + (x-4)^2 + (y-2)^2 = 4^2 + 2^2.$$

This last equation is equivalent to $x^2 + y^2 - 4x - 2y = 0$. Since $x^2 + y^2 = 2$, we obtain $4x + 2y = 2$ so that $y = 1 - 2x$. Using $x^2 + y^2 = 2$ again, we obtain $x^2 + (1-2x)^2 = 2$ so that $5x^2 - 4x - 1 = 0$. Since $5x^2 - 4x - 1 = (5x+1)(x-1)$ and $y = 1 - 2x$, we deduce that the two points are $(-1/5, 7/5)$ and $(1, -1)$.

Alternatively, one can note that such a point (x, y) must lie on the circle centered at the midpoint of $(0, 0)$ and $(4, 2)$ and passing through $(0, 0)$. It follows that $(x-2)^2+(y-1)^2 = 2^2+1^2$. Hence, $x^2-4x+y^2-2y = 0$. Using $x^2+y^2 = 2$, one obtains $y = 1-2x$ as before (and follows the argument above).

6. Observe that $95 = 3 \times 2^5 - 1$. Therefore, by the binomial theorem,

$$95^{2000} = \left(3 \times 2^5 - 1\right)^{2000} = (-1)^{2000} + 2000(-1)^{1999}(3 \times 2^5) + 2^{10} \times m,$$

where m is some integer (the point here is that 2^{10} is a factor of each of the remaining terms in the binomial expansion of $\left(3\times 2^5-1\right)^{2000}$). Since $(-1)^{2000}+2000(-1)^{1999}(3\times 2^5) = -191999$, we obtain

$$95^{2000} = 1024 \times 188 - 191999 + 2^{10}(m - 188) = 513 + 2^{10}(m - 188).$$

It follows that the 10 right-most bits of the binary expansion of 95^{2000} are the same as that of 513. Since $513 = 2^9 + 1 = (1000000001)_2$, the sum of the 10 right-most bits is 2.

7. Since $649^2 - 13 \times 180^2 = 1$, we see that

$$\left(649 + 180\sqrt{13}\right)\left(649 - 180\sqrt{13}\right) = 1.$$

Squaring both sides, we obtain

$$\left(649 + 180\sqrt{13}\right)^2\left(649 - 180\sqrt{13}\right)^2 = 1.$$

Since $\left(649+180\sqrt{13}\right)^2 = 842401+233640\sqrt{13}$ and $\left(649-180\sqrt{13}\right)^2 = 842401-233640\sqrt{13}$, we obtain

$$842401^2 - 13 \times 233640^2 = \left(842401 + 233640\sqrt{13}\right)\left(842401 - 233640\sqrt{13}\right) = 1.$$

Since each of 842401 and 233640 is $< 10^8$, the answer is $(a, b) = (842401, 233640)$.

8. Observe that if $x \geq 0.9$, then $x^n + 2x^2 + 3x - 4 > 2x^2 + 3x - 4 \geq 2(0.9)^2 + 3(0.9) - 4 > 0$. Thus, any real root of $x^n + 2x^2 + 3x - 4$ must be < 0.9. In particular, $0 < \alpha_n < 0.9$. It follows that α_n^n approaches 0 as n approaches infinity. Since α_n is a root of x^n+2x^2+3x-4, we deduce that $2\alpha_n^2+3\alpha_n-4$ is approaching 0. By considering the graph of $y = 2x^2+3x-4$, one sees that $2\alpha_n^2+3\alpha_n-4$ cannot be near 0 unless α_n is near a root of $2x^2+3x-4$. The roots of $2x^2+3x-4$ are $(-3+\sqrt{41})/4$ and $(-3-\sqrt{41})/4$, and the latter of these is negative (so α_n cannot be near it). It follows that α_n is approaching the root $(-3+\sqrt{41})/4$ as n approaches infinity.