TEAM PROBLEMS 02/95

T1. The distance between the centers of two circles in a plane is 25. The radius of one circle is 3 and the radius of the other circle is 4. A line ℓ is tangent to one circle at point P and tangent to the other at point Q. There is a point on the line segment joining the two centers of the circles that is also on ℓ. What is the distance between P and Q? Simplify your answer.

T2. What is the sum of all numbers of the form $a \times b \times c$ where a is from the set $\{1, 2, 4, 8\}$, b is from the set $\{1, 3, 17, 19\}$, and c is from the set $\{1, 7, 31, 61\}$?

T3. What is the greatest common divisor of the numbers 106577 and 1071089?

T4. For any two real numbers x and y, define

$$f(x, y) = x^2 + 13y^2 - 6xy - 4y - 2.$$

For what value of the pair (x, y) is $f(x, y)$ as small as possible?

T5. If the value of

$$\sum_{n=1}^{98} \frac{1}{n(n+1)(n+2)(n+3)}$$

is written as a reduced fraction, then what is the value of its denominator?

T6. Dave and Michael decide to play the following game. They begin with $S = 0$. They take turns picking a number from the set $\{1, 2, 3, 4, 5, 6\}$. On each turn any of the 6 numbers can be chosen. The number is added to S and S is then replaced by the sum. The first person who chooses a number which when added to S totals 40 is the winner. Dave begins by picking the number 6. What number is now the best choice for Michael to pick?

T7. In how many points do the graphs of

$$y = x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1 \quad \text{and} \quad y = x^5 + x^3 + x$$

intersect?

T8. Let $A = (-3, 0)$ and $B = (3, 0)$. Let $\mathcal{C}$ denote the circle $x^2 + y^2 = 9$ (so $\overline{AB}$ is a diameter of $\mathcal{C}$). What is the probability that a random point P inside $\mathcal{C}$ is such that $\angle APB \leq 3\pi/4$. (Here, $\angle APB$ represents an angle with radian measure in $[0, \pi]$.)

T9. For m a positive integer and $k \in \{0, 1, 2, \ldots, m\}$, define $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$. It is known that the coefficient of x^k in $(x+1)^m$ is $\binom{m}{k}$. Using this fact, find positive integers a and b each less than 1000 satisfying

$$\binom{50}{0}^2 + \binom{50}{1}^2 + \binom{50}{2}^2 + \cdots + \binom{50}{50}^2 = \binom{a}{b}.$$

T10. Let a and b be integers for which

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1994} + \frac{1}{1995} = \frac{a}{b}.$$

Suppose that exactly one of a and b is even. Determine which of a and b is even and determine the largest integer r such that 2^r divides ab.