SOLUTIONS TO TEAM PROBLEMS
January, 2003

Answers:

1. $A < B < C < D$
2. Alan shakes 2 hands
   Alice shakes 2 hands
   Bernice shakes 4 hands
   Bob shakes 0 hands
   Calvin shakes 3 hands
   Cathy shakes 1 hand
3. 233
4. 6
5. GOOD or GO.OD or good or go.od
6. 1/9
7. 3
8. $\log 2$ or $\log_e 2$ or $\ln 2$

1. Let $f(x) = \log\log x$. The base of the logarithm is not important. We will use the natural logarithm here. Since $f(x)$ is an increasing function, we may consider $f(A)$, $f(B)$, $f(C)$, and $f(D)$ instead of A, B, C, and D. The number A is a tower (say) of twelve $\sqrt{2}$'s. It is increased by replacing the highest exponent $\sqrt{2}$ (at the top) with 2. Since $\sqrt{2}^2 = 2$, we deduce $A \le 2$ so that $f(A) \le \log\log 2 < 0$. Note that

$$f(B) = \log\log B = \log\left(10^{100}\log 10\right) = 100\log 10 + \log\log 10 = 231.09\ldots.$$

Similarly,

$$f(C) = 2^{2^{2^2}}\log 2 + \log\log 2 = 2^{16}\log 2 + \log\log 2 = 45425.727\ldots.$$

Also, $f(D) = \sqrt{3}^t\log\sqrt{3} + \log\log\sqrt{3}$ where t consists of a tower of five $\sqrt{3}$'s. The value of t can be estimated with a calculator by considering

$$\log t = \sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}}}}\log\sqrt{3} = 5.359\ldots.$$

Thus,

$$f(D) > \sqrt{3}^{e^{5.359}}\log\sqrt{3} + \log\log\sqrt{3} > 2\times 10^{50}.$$

Thus, $f(A) < f(B) < f(C) < f(D)$. The answer is $A < B < C < D$ (no wonder the problem said to put the answer in this form).

2. Let $h(x)$ denote the number of hands person x shakes. The most hands that anyone shakes in this problem is 4. Since Alice, Bernice, Bob, Calvin and Cathy shook a different number of hands, they shook 0, 1, 2, 3, and 4 hands in some order. Also, the given information in the problem implies

$$h(\text{Bob}) \le 3, \quad h(\text{Alan}) \ge 1, \quad \text{and} \quad 1 \le h(\text{Calvin}) \le 3.$$

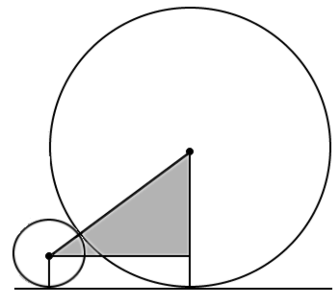
Observe that Alice cannot shake 4 hands since then nobody shakes 0 hands. Also, if Cathy shakes 4 hands, then nobody shakes 0 hands. It follows that Bernice shakes 4 hands. From this, we deduce that Bob shakes 0 hands (since Alice, Calvin and Cathy shake hands with Bernice). Since Calvin shakes hands with both Alan and Bernice, $h(\text{Calvin}) \geq 2$. Now, $h(\text{Alice}) \neq 1$ since otherwise each of Calvin and Cathy do not shake hands with either Alice or Bob so that no one shakes 3 hands. Thus, Cathy shakes 1 hand, apparently Bernice's. This means that Alice cannot shake 3 hands so that $h(\text{Alice}) = 2$ and $h(\text{Calvin}) = 3$. One now checks that $h(\text{Alan}) = 2$ (he shakes hands with Bernice and Calvin and nobody else).

3. If an integer $m > 1$ divides $2^{29} - 1$, then so does each of m's prime factors. It follows that the smallest integer $m > 1$ that divides $2^{29} - 1$ is a prime. Taking this prime to be q and taking $a = 2$ and $p = 29$ in the information given in the problem, we deduce that $q - 1$ is divisible by 29. Since q is prime, we only need consider odd q. The odd numbers that are one more than a multiple of 29 are $59, 117, 175, 233, \ldots$. One checks directly that $2^{29} - 1$ is not divisible by 59. The numbers 117 and 175 are not prime (the first is divisible by 3 and the second by 5). Finally, one checks directly that $2^{29} - 1$ is divisible by 233, so this is the answer.

4. Let
$$f(x) = (x-1)(x-2)(x-3)\cdots(x-2002)(x-2003) + 1.$$
Since $f(x)$ has degree 2003, the graph of $y = f(x)$ crosses the x-axis at most 2003 times. Observe that $f(k) > 0$ for each $k \in \{1, 2, \ldots, 2003\}$. Also,
$$f(0.5) < 0,\ f(2.5) < 0,\ f(4.5) < 0,\ \ldots,\ f(2002.5) < 0.$$
Therefore, the graph of $y = f(x)$ must cross the x-axis at least once in each of the intervals
$$(0.5, 1), (2, 2.5), (2.5, 3), (4, 4.5), (4.5, 5), \ldots, (2002, 2002.5), (2002.5, 2003).$$
Since this consists of 2003 intervals, we deduce that the graph of $y = f(x)$ crosses the x-axis exactly once in each of the intervals above and nowhere else. It follows that the graph crosses the x-axis 6 times in the interval $[1000, 1005]$.

5. We want to write the number
$$\sqrt{29085} = 170.5432496\ldots$$
in base 26. Since $170 = 6 \cdot 26 + 14$, the first two digits come from converting 6 and 14 to letters, so they are **G** and **O**, respectively. Since $14/26 = 0.53846\ldots$ and $15/26 = 0.57692\ldots$, we obtain that the next digit corresponds to 14 and, hence, is the letter **O**. Next, we use that
$$170.5432496\cdots - \left(6 \cdot 26 + 14 + \frac{14}{26}\right) = 0.004788\ldots.$$
Since $3/26^2 = 0.0044\ldots$ and $4/26^2 = 0.0059\ldots$, the fourth digit corresponds to 3 and, hence, is the letter **D**. Thus, the answer is GOOD.

6. Consider the two circles tangent to each other and a line as shown to the right. Suppose the smaller circle has radius r and the larger circle radius R. Then the shaded triangle is a right triangle with hypotenuse $R+r$ and one leg of length $R-r$. It follows that if the "horizontal distance between these circles" (that is the length of the remaining leg) is h, then



$$h^2 = (R+r)^2 - (R-r)^2 = 4Rr \implies h = 2\sqrt{Rr}.$$

In the problem, let $h(i,j)$ denote the horizontal distance between $\mathcal{C}_i$ and $\mathcal{C}_j$. Let r denote the radius of $\mathcal{C}_2$. Since $h(0,1) = h(0,2) + h(1,2)$, we deduce from the formula for h above that $2 = 2\sqrt{r}+2\sqrt{r}$ so that $r = 1/4$. Let r' denote the radius of $\mathcal{C}_3$. Since $h(0,2) = h(0,3)+h(2,3)$, we deduce from the formula for h above that $1 = 2\sqrt{r'} + \sqrt{r'}$ so that $r' = 1/9$.

7. Let $f(x) = x^{2003} + ax^{10} + 200$ where $a \in \{1, 2, \dots, 200\}$. If z is a complex number with $|z| \ge 1.003$, then

$$|f(z)| \ge |z|^{2003} - a|z|^{10} - 200 = |z|^{10}\big(|z|^{1993} - a\big) - 200 \ge 1.03\big(391.525 - a\big) - 200.$$

For $a \le 197$, this last expression is > 0 so that $f(z) \ne 0$. In other words, $f(x)$ has no roots with absolute value ≥ 1.003 unless $a > 197$. For $a \in \{198, 199, 200\}$, one checks directly that the value of $f(-1.003) > 0$ and $f(-2) < 0$ so that $f(x)$ in fact has a negative real root between -1.003 and -2. The answer, therefore, is 3.

8. One can use that if n is large, then $S_n \approx \log n$ and $S_{2n} \approx \log(2n) = \log n + \log 2$ so that $T_n = S_{2n} - S_n \approx \log 2$. This idea is not a true explanation, though, as we are not given sufficient information in the problem to determine how good an approximation $\log n$ is to S_n. We give a more precise explanation as follows. Let c denote the constant that T_n approaches as n tends to infinity. Let k be a positive integer. We use that

$$1 + T_1 + T_2 + T_4 + T_8 + \cdots + T_{2^{k-1}} = S_{2^k}.$$

We rewrite this as

$$\frac{1 + T_1 + T_2 + T_4 + T_8 + \cdots + T_{2^{k-1}}}{k+1} \cdot \frac{k+1}{k} \cdot \frac{1}{\log 2} = \frac{S_{2^k}}{k \log 2} = \frac{S_{2^k}}{\log\left(2^k\right)}.$$

Observe that the first fraction on the left is simply the average of $1, T_1, T_2, T_4, \dots, T_{2^{k-1}}$ and, as k gets large, this average approaches c. Also, as k gets large, $(k+1)/k$ and the last fraction above approach 1 (the latter by what is given). We deduce that $c/\log 2 = 1$ so that $c = \log 2$.