Team Problems January, 2003

Instructions: Answer as many of the problems below as you can. At the end of the time allotted, turn in a list of your answers. Your answers should be expressed in simplest form. Exact answers are required on all problems unless specified otherwise.

1. Let

$$A = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}}}}}}}, \quad B = 10^{10^{100}}, \quad C = 2^{2^{2^{2^2}}}, \quad \text{and} \quad D = \sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\sqrt{3}}}}}}}}.$$

   Order these from least to greatest. Express your answer in the form $A < B < C < D$. (Note that exponentiation is done from the top down. For example, $2^{3^4} = 2^{81}$.)

2. Three married couples, Alan and Alice, Bernice and Bob, and Calvin and Cathy, attend a dinner. They shake hands. No one shakes hands with him-or-herself. No one shakes hands with his-or-her spouse. And no two people shake hands more than once. Bob doesn't like Calvin, so the two of them don't shake hands with each other. On the other hand, Alan does shake hands with Calvin. After everyone finishes shaking hands, Alan asks everyone else, including his wife, how many hands they shook. Each person gives him a different answer. Assuming everyone has given a correct response to Alan, determine how many hands each person shook.

3. If p is a prime, a an integer > 1, and q a prime dividing $a^p - 1$, then it is known that either p divides $q-1$ or q divides $a-1$. What is the smallest integer > 1 dividing $2^{29} - 1$?

4. How many times does the graph of

$$y = (x-1)(x-2)(x-3)\cdots(x-2002)(x-2003) + 1$$

   cross the x-axis in the interval $[1000, 1005]$?

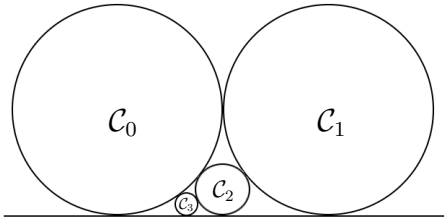
Solutions are located at the website `http://www.math.sc.edu/~filaseta/contests/contests.html`

5. The base 10 number $\sqrt{2}$ can be converted to base 2 as $1.0110101\ldots$. This means that

$$\sqrt{2} = 1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots .$$

The base 2 digits of the base 10 number $\sqrt{2}$ are, therefore, $1, 0, 1, 1, 0, 1, 0, 1, \ldots$. Suppose we wish to convert a base 10 number to base 26. Let the digit 0 be represented by A, the digit 1 be represented by B, the digit 2 be represented by C, and so on. In other words, the digits in base 26 are to be represented by the alphabet in its usual order. What are the first four letters (digits) when the base 10 number $\sqrt{29085}$ is converted to base 26?

6. Two circles $\mathcal{C}_0$ and $\mathcal{C}_1$, each with radius 1, are tangent to each other and to line ℓ. Circle $\mathcal{C}_2$ is tangent to $\mathcal{C}_0$, $\mathcal{C}_1$, and to ℓ as shown. Circle $\mathcal{C}_3$ is tangent to $\mathcal{C}_1$, $\mathcal{C}_2$, and ℓ as shown. What is the radius of $\mathcal{C}_3$?



7. For how many $a \in \{1, 2, \ldots, 200\}$ does the polynomial

$$x^{2003} + ax^{10} + 200$$

have a root (real or imaginary) with absolute value ≥ 1.003?

8. For n a positive integer, define

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

and

$$T_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}.$$

It is known that $S_n / \log n$ approaches 1 as n tends to infinity. Here, the base of the logarithm is the number e (which you do not need to have prior knowledge of for this problem). It is also known that T_n approaches a constant as n tends to infinity. What is that constant?