SOLUTIONS TO TEAM PROBLEMS
FEBRUARY, 2002

Answers:

1. $76°$
2. $0.89879\ldots$ and $-0.99939\ldots$
3. "point" or "origin" or $(0,0)$
4. 2 and 3
5. 1502
6. $\dfrac{\sqrt{6}}{2}$ or $\dfrac{\sqrt{3}}{\sqrt{2}}$ or $\sqrt{\dfrac{3}{2}}$
7. 13860
8. 22

1. From the given information, $\angle DCE = \angle DEC = x$. Hence, $\angle ADC = 2x$. Since A, B, C, and D all lie on the same circle, the angles $\angle ADC$ and $\angle ABC$ are supplementary angles. Therefore, $2x = \angle ADC = 180° - \angle ABC = 180° - 28° = 152°$, so $x = 76°$.

2. The numbers 2^3 and 2^{15} are congruent modulo 360 (that is, their difference is divisible by 360). Hence, the value of $\sin\big((2^n)°\big)$ will be the same as $\sin\big((2^{n-12})°\big)$ for $n \geq 15$. In other words, $\sin\big((2^n)°\big)$ only takes on different values for $0 \leq n \leq 14$. The maximum value of $\sin\big((2^n)°\big)$ occurs when $n = 6$ and it is $0.89879\ldots$. The minimum value occurs when $n = 13$ and it is $-0.99939\ldots$.

3. The graph is a point (the origin). Set $\alpha = (-1+\sqrt{-3})/2$ and $\beta = (-1-\sqrt{-3})/2$ (the roots of $x^2+x+1=0$). Then $x^2+xy+y^2 = (x-\alpha y)(x-\beta y)$. It follows that $x^2+xy+y^2=0$ if and only if $x = \alpha y$ or $x = \beta y$. Since α and β are imaginary, either of these happens for real x and y only in the case that $x = y = 0$.

4. The problem is equivalent to finding the first two digits after the decimal in the decimal expansion of $10^{11111110}/2003$. The number 11111110 has a remainder of 10 when divided by 2002. Hence, there is an integer k such that $11111110 = 2002k + 10$. Since $10^{2002}-1$ is divisible by 2003 and $10^{2002}-1$ is a factor of $10^{2002k}-1$, 2003 divides $10^{2002k}-1$. Thus, $10^{2002k}-1 = 2003m$ for some integer m. We deduce
$$\frac{10^{11111110}}{2003} = \frac{10^{2002k+10}}{2003} = \frac{(10^{2002k}-1)\cdot 10^{10}}{2003} + \frac{10^{10}}{2003} = 10^{10}m + 4992511.23315\ldots.$$
Hence, the 11111111^{th} and 11111112^{th} digits after the decimal in the decimal expansion of $1/2003$ are 2 and 3, respectively.

5. Let $f(x) = a_{2002}x^{2002} + a_{2001}x^{2001} + \cdots + a_1x + a_0$ be the expansion of
$$\big(x-2^2\big)\big(x-2^{2^2}\big)\cdots\big(x-2^{2^{1001}}\big)\big(x+2^{2^{1002}}\big)\big(x+2^{2^{1003}}\big)\cdots\big(x+2^{2^{2002}}\big).$$
Observe that each a_j can be expressed as a sum of distinct terms of the form $\varepsilon 2^w$ where w is a sum of $n-j$ distinct powers of 2 (appearing in the exponents of the product above)

and $\varepsilon = \pm 1$. For any integer $m > 0$,

$$2^m > 2^{m-1} + 2^{m-2} + \cdots + 2 + 1.$$

It follows that the sign of a_j is determined only by the sign of the term $\varepsilon 2^w$ in a_j with w as large as possible. The table to the right indicates the term in a_j with w as large as possible. The sign of each term and the value of w are given. The first 1001 rows correspond to positive terms. The next 1002 rows alternate in sign so that they include 501 positive terms and 501 negative terms. The total number of positive coefficients is therefore 1502.

j	w	sign
2002	0	$+$
2001	2^{2002}	$+$
2000	$2^{2002} + 2^{2001}$	$+$
$\vdots$	$\vdots$	$\vdots$
1001	$2^{2002} + 2^{2001} + \cdots + 2^{1002}$	$+$
1000	$2^{2002} + 2^{2001} + \cdots + 2^{1002} + 2^{1001}$	$-$
999	$2^{2002} + 2^{2001} + \cdots + 2^{1001} + 2^{1000}$	$+$
998	$2^{2002} + 2^{2001} + \cdots + 2^{1000} + 2^{999}$	$-$
$\vdots$	$\vdots$	$\vdots$
1	$2^{2002} + 2^{2001} + \cdots + 2^3 + 2^2$	$+$
0	$2^{2002} + 2^{2001} + \cdots + 2^2 + 2$	$-$

6. The number $a + b\mathrm{i}$ is a root of $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. The root of $x^5 - 1$ with positive real and imaginary parts is $\cos(72°) + \mathrm{i}\sin(72°)$. So $a = \cos(72°)$ and $b = \sin(72°)$. From the given information,

$$\cos(72°)\cos(75°) + \sin(72°)s = \cos(3°).$$

This uniquely determines s and from the identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$ with $A = 75°$ and $B = 72°$, we see that $s = \sin(75°)$. Hence,

$$\begin{aligned} r + s &= \cos(75°) + \sin(75°) \\ &= \cos(45°)\cos(30°) - \sin(45°)\sin(30°) + \sin(45°)\cos(30°) + \sin(30°)\cos(45°) = \frac{\sqrt{6}}{2}. \end{aligned}$$

7. The equation $x^2 - 2y^2 = 1$ implies that x is odd. Write $x = 2n+1$. Then

$$2y^2 = (2n+1)^2 - 1 = 4n^2 + 4n = 4n(n+1).$$

Since y has all of its prime factors ≤ 11, so does $n(n+1)$. The information in the problem implies now that

$$y^2 \leq 2n(n+1) \leq 2 \cdot 9800 \cdot 9801 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2.$$

Hence, $y \leq 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 13860$. With $y = 13860$ and $x = 2n+1 = 2 \cdot 9800 + 1 = 19601$, it is easy to see that $x^2 - 2y^2 = 1$ is in fact satisfied. Hence, the answer is 13860.

8. We use numbers $\theta_1, \theta_2, \ldots$ to represent parts of years so that each $\theta_j \in [0,1)$. We will use $\varepsilon_1, \varepsilon_2, \ldots$ to represent numbers that are either 0 or 1. At some point, Tammy has lived $4 + \theta_1$ years, John has lived $j + \theta_2$ years, and Martha has lived $m + \theta_3$ years where j and m are integers satisfying $j = 3m$. At another time, say $a + \theta_4$ years later where a is an

integer, Martha is twice as old as Tammy and John is five times as old as Tammy. Since Tammy will have lived $4 + a + \theta_1 + \theta_4$ years, she will be $4 + a + \varepsilon_1$ years old (where, as noted earlier, $\varepsilon_j \in \{0, 1\}$). Similarly, John will be $j + a + \varepsilon_2$ years old and Martha will be $m + a + \varepsilon_3$ years old. Hence,

$$(*) \qquad m + a + \varepsilon_3 = 2(4 + a + \varepsilon_1) \qquad \text{and} \qquad j + a + \varepsilon_2 = 5(4 + a + \varepsilon_1).$$

At yet another time, say $b + \theta_5$ years after Tammy lived $4 + \theta_1$ years, we obtain Tammy is $T = 4+b+\epsilon_4$ years old and $j+b+\varepsilon_5 = 2(m+b+\varepsilon_6)$. This last equation and $j = 3m$ imply $b = j - 2m + \varepsilon_5 - 2\varepsilon_6 = m + \varepsilon_5 - 2\varepsilon_6$. From the first equation in $(*)$, $m = a + 8 + 2\varepsilon_1 - \varepsilon_3$. From the second equation in $(*)$ and $j = 3m$, we obtain $3m = 4a+20+5\varepsilon_1-\varepsilon_2$. Therefore,

$$m = 4m - 3m = 4(a + 8 + 2\varepsilon_1 - \varepsilon_3) - (4a + 20 + 5\varepsilon_1 - \varepsilon_2) = 12 + 3\varepsilon_1 - 4\varepsilon_3 + \varepsilon_2.$$

Combining the above, we deduce $T = 16 + 3\varepsilon_1 - 4\varepsilon_3 + \varepsilon_2 + \varepsilon_5 - 2\varepsilon_6 + \epsilon_4$. Since each $\varepsilon_j \in \{0, 1\}$, we obtain that $T \leq 16+3+1+1+1 = 22$. One checks that $t = 4.6$, $j = 48.6$, $m = 16$, $a = 6.5$, and $b = 17.5$ lead to the conditions in the problem holding with $T = 22$, so 22 is the answer.