Math 780/784 Comps: August, 2006

General Instructions to Fill this Space on the Test: You have 4 hours to complete this test. No calculators, computers, books or notes should be used during the test. The first four problems on this test are associated with your Math 780 course and the remaining four problems are related to Math 784. Answer as many of these problems as you can in the time allotted. The goal is to show what you understand of the material taught in each of these classes, so be sure to spend some time on problems from both courses. If you do not know how to do a problem but have some knowledge of the material that you think the problem is testing, feel free to explain this and indicate what you do know. If you think you have only given a partial answer to the problem posed, then you might want to clarify this as well ... that you do not feel that the answer is adequate. If you think your answer is wrong and you really want to share the wrong answer with us, then indicate that you think it is wrong. All of these say something about what you understand of the material, most of which is probably better than a wrong answer that you apparently think is correct (though indicating that you think a calculation like $1 + 1 = 2$ is wrong may well be considered worse).

The Problems

1. Determine the remainder when 2006^{2006} is divided by 520.

2. For coprime non-zero integers a and b, show that
$$\frac{\overline{a}_b}{b} + \frac{\overline{b}_a}{a} - \frac{1}{ab} \in \mathbb{Z},$$
where $\overline{m}_n$ stands for the multiplicative inverse of m modulo n.

3. Suppose $p \equiv 1 \pmod 8$ is prime. Show that $\left(\dfrac{p-1}{2}\right)!$ is a quadratic residue modulo p.

4. Let $\sigma(n)$ be the sum-of-divisors function. Show that there exists is a constant c such that
$$\sum_{n \le x} \frac{\sigma(n)}{n^2} = \zeta(2)\log x + c + o(1)$$
as x tends to infinity.

5. Observe that
$$13 = \frac{4^3 + 1^3}{4 + 1} \quad \text{and} \quad 13^2 = \frac{15^3 + 8^3}{15 + 8}.$$
Prove that for each positive integer n, there exist relatively prime integers a and b (not necessarily positive) such that
$$13^n = \frac{a^3 + b^3}{a + b}.$$
(Hint: Note that $a^2 - ab + b^2 = (a - \alpha b)(a - \beta b)$ where α and β are the two roots of $x^2 - x + 1$.)

6. Prove the theorem below. You should begin by considering $\beta \in \mathbb{Q}(\alpha)$ and using that there are $N(x)$ and $D(x)$ in $\mathbb{Q}[x]$ such that $\deg D(x) \le n-1$, $D(\alpha) \neq 0$, and $\beta = N(\alpha)/D(\alpha)$. You do not need to prove such $N(x)$ and $D(x)$ exist. Note that the theorem is asserting that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ forms a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, so this is what you are trying to prove. In other words, don't use that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ as this would be circular reasoning. You should also not deduce the theorem below as a consequence of a more general or more difficult theorem.

   Theorem: *Let α be an algebraic number with minimal polynomial $f(x)$ of degree n. Every element of $\mathbb{Q}(\alpha)$ can be expressed uniquely in the form $g(\alpha)$ where $g(x) \in \mathbb{Q}[x]$ with $\deg g(x) \le n-1$.*

7. Let R be the ring of integers in an algebraic number field $\mathbb{Q}(\alpha)$. In this problem, $N(x)$ is used to denote the norm of $x \in \mathbb{Q}(\alpha)$ over $\mathbb{Q}$. For each part, clearly indicate whether you believe the given statement is true or false. In this context, true means true for all choices of the variables satisfying the given conditions and false means false for some choice of these variables. If you believe the statement is true, provide a proof. If you believe the statement is false, provide a counter example. Note that one of the parts is true and one is false. This is information that is meant to help you, but saying something like, "This part is true because I know one of the parts is true and the other part is false" is NOT an acceptable justification. A correct justification of one part should be independent of the other part.

   (a) If a and b are in R and $N(a)$ and $N(b)$ are relatively prime in $\mathbb{Z}$, then the ideal (a, b) equals the ideal (1).

   (b) If a and b are in R and $N(a)$ and $N(b)$ are *not* relatively prime in $\mathbb{Z}$, then the ideal (a, b) does *not* equal the ideal (1).

8. Let R be the ring of algebraic integers in $\mathbb{Q}(\sqrt{-26})$. The class number for the field $\mathbb{Q}(\sqrt{-26})$ is 6. You may use this information on the class number without proving it.

   (a) Let A, B and C be ideals in R with A and B principal, $B \neq (0)$ and $A = BC$. Using the definition of the product of two ideals, prove that C is principal.

   (b) Let A and C be ideals in R with A principal. Suppose $C^5 = A$. Using (a) and the definition of class numbers (but no lemmas from class), prove that C is principal.

   (c) Show that $y^2 + 26 = x^{17}$ has finitely many solutions in integers x and y. (Note that I am not asking for the solutions.)