Comprehensive Exam, Summer 2004

for

MATH 788F & MATH 788G

1. Prove Lemma 1 and the theorem below. For the proof of the theorem, you may use Lemma 2 without proving it.

   Lemma 1. *Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x)$ has exactly 1 root α (with multiplicity 1) such that $|\alpha| \geq 1$. Then $f(x)$ is irreducible.*

   Lemma 2. *Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counting multiplicity) inside the circle $\mathcal{C}$ (i.e., in the interior of the region bounded by $\mathcal{C}$).*

   Theorem. *If $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $n \geq 1$ is such that $a_n = 1$, $a_0 \neq 0$, and $|a_{n-1}| > 1 + |a_{n-2}| + |a_{n-3}| + \cdots + |a_0|$, then $f(x)$ is irreducible.*

2. Determine whether each of the polynomials

$$x^3 + 3x + 2 \qquad \text{and} \qquad x^4 - 2x - 1$$

   is Eisenstein. For each one that is Eisenstein, indicate all primes p and all integers $k \in \{0, 1, \ldots, p-1\}$ for which $f(x+k)$ is in Eisenstein form with respect to p. Be sure to show work justifying your answers.

3. (a) Let n be a positive integer, and let $f(x)$ be defined by

$$\int_0^x (1 + t + t^2 + \cdots + t^n)\, dt = x f(x).$$

   So

$$f(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^n}{n+1}.$$

   Prove that if $n+1$ is a prime, then $f(x)$ is irreducible over the rationals. (Hint: If you are having difficulty seeing what to do, consider specific values of n with $n+1$ prime. Multiply by an appropriate number to make the problem into a question about irreducibility over the integers.)

   (b) Let n and k be positive integers, and let $f(x)$ be defined by

$$\int_0^x (t^{k-1} + t^k + t^{k+1} + \cdots + t^{k-1+n})\, dt = x^k f(x).$$

   So

$$f(x) = \frac{1}{k} + \frac{x}{k+1} + \frac{x^2}{k+2} + \cdots + \frac{x^n}{k+n}.$$

   Prove that if there is a prime $p > n$ for which $p | k(k+n)$ and $p^2 \nmid k(k+n)$, then $f(x)$ is irreducible over the rationals. (Hint: Note that if $k = 1$, part (b) implies part (a). If you explain this and do part (b), then you do not need to do part (a). However, solving part (a) first may give you a hint as how to approach part (b).)

4. (a) It is known that if $x \geq 11$, then there are at least two distinct primes in the interval $(x/2, x]$. Using this, prove that if $n \geq 10$, then the polynomial

$$f(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^n}{n+1}$$

   is irreducible over the rationals. (Hint: Use Newton polygons.)

   (b) Using Newton polygons, justify that if

$$f(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6}$$

is reducible, then it has $-2/5$ as a root. Furthermore, use this information to show that $f(x)$ is in fact irreducible. (Note: This is to clarify what is *not* acceptable here. You can use Newton polygons quickly and deduce that if $f(x)$ is reducible, then it has a rational root. Then you could apply the rational root test to seemingly finish off the problem. This would involve checking 48 possible rational numbers, though half of them could be eliminated immediately. The problem, however, is to use Newton polygons to deduce that if $f(x)$ is reducible, then $-2/5$ is a root. You will not get credit for using the rational root test to get at what rational roots $f(x)$ has. The rational root test should *not* be used in this problem.)

5. Prove Kronecker's theorem stated below.

   Theorem. *If $f(x) \in \mathbb{Z}[x]$ is monic, is irreducible, and has all its roots on $\{z : |z| = 1\}$, then $f(x)$ is a cyclotomic polynomial.*

6. How many incongruent irreducible factors of the cyclotomic polynomial $\Phi_{2004}(x)$ are there modulo 3 and what are their degrees? Be specific and show the appropriate work.

7. Let
$$f(x) = x^{2004} + x^7 + x^4 + x + 1.$$
   Then $\tilde{f}(x) = x^{2004} + x^{2003} + x^{2000} + x^{1997} + 1$. It follows that
$$\begin{aligned}(x^7 + x^6 + x^3 + 1)f(x) - x^7\tilde{f}(x) &= x^{14} + x^{13} + x^{11} + 2x^{10} + x^8 + 3x^7 + x^6 + 2x^4 + x^3 + x + 1 \\ &= (x^4 + x^3 + x^2 + x + 1)(x^4 - x^2 + 1)(x^6 + x^3 + 1).\end{aligned}$$
   Also,
$$\begin{aligned}f(x)\tilde{f}(x) = x^{4008} &+ x^{4007} + x^{4004} + x^{4001} + x^{2011} + x^{2010} + x^{2008} + 2x^{2007} + x^{2005} \\ &+ 5x^{2004} + x^{2003} + 2x^{2001} + x^{2000} + x^{1998} + x^{1997} + x^7 + x^4 + x + 1.\end{aligned}$$
   Using this information, do the following.

   (a) Prove that $f(x)$ has no reciprocal factors.

   (b) Prove that $f(x)$ is irreducible. (There is a general theorem that classifies the $0, 1$-polynomials with *five* nonzero terms that have a reducible non-reciprocal part. If you happen to know this theorem, do not use it. The point of the problem is to see if you can show that the non-reciprocal part of $f(x)$ is irreducible as we did in class. Other results about $0, 1$-polynomials done in class can be used.)