Comprehensive Exam, Summer 2003

for

MATH 788F & MATH 788G

1. Prove Perron's theorem that if $a_0, a_1, \ldots, a_n$ are integers with $a_n = 1$, $a_0 \neq 0$, and
$$|a_{n-1}| > |a_n| + |a_{n-2}| + |a_{n-3}| + \cdots + |a_1| + |a_0|,$$
then $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is irreducible. You may use Rouché's theorem as stated below, but otherwise your argument should be self-contained.

   Rouché's Theorem *Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let*
$$\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}.$$
   If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counted to their multiplicities) inside the circle $\mathcal{C}$ (i.e., in the interior of the region bounded by $\mathcal{C}$).

2. Determine whether each of the following is Eisenstein. Justify your answers.

   (a) $x^2 - 3x - 2$ (b) $x^3 + 4x^2 + x - 1$

3. Prove Kroncecker's theorem that if $f(x)$ is an irreducible monic polynomial with all of its roots on the unit circle (i.e., all of its roots have absolute value 1), then $f(x)$ is a cyclotomic polynomial.

4. Let n be a positive integer, and set
$$S_n = \left\{ f(x) = \sum_{j=0}^{n} \varepsilon_j x^j : \varepsilon_j \in \{0,1\} \text{ for each } j \text{ and } \varepsilon_0 = 1 \right\}.$$
Let $g(x)$ be an arbitrary polynomial in $\mathbb{Z}[x]$. Prove there are $\leq 2^n/(n+1)$ polynomials in S_n divisible by $g(x)^2$.

5. (a) Prove that $x^{255} - 1$ factors modulo 2 as a product of distinct irreducible polynomials each of degree ≤ 8.

   (b) It follows from part (a) that each irreducible factor of $x^{255} - 1$ modulo 2 has degree in $S = \{1, 2, 3, \ldots, 8\}$. Which numbers in S actually occur as the degree of some irreducible factor of $x^{255} - 1$ modulo 2? Justify your answer.

6. (a) Prove that the non-reciprocal part of $1 + x + x^2 + x^4 + x^8 + x^{16} + x^{32}$ is irreducible.

   (b) Prove that the non-reciprocal part of $1 + x + x^2 + x^4 + x^8 + \cdots + x^{2^n}$ is irreducible for every integer $n \geq 2$.

7. We showed that if n is a positive integer and $a_1, a_2, \ldots, a_n$ are distinct integers, then
$$(x - a_1)(x - a_2)\cdots(x - a_n) - 1$$
is irreducible. Determine with justification whether or not there exists an N such that if n is an integer $\geq N$ and $a_1, a_2, \ldots, a_n$ are *rational* numbers, then $(x - a_1)(x - a_2)\cdots(x - a_n) - 1$ is irreducible over the rationals.

8. For m a positive integer, define

$$f_m(x) = (54x^2 + 189)x^m + (146x^6 - 54).$$

It is *not* the case that $f_m(x)$ is irreducible for every positive integer m.

(a) Using Newton polygons, determine information about the factorization of $f_m(x)$. (This is vague. For the purposes of part (c) below and partial credit, in the event you do not complete this problem, you will want to be rather precise here.)

(b) Prove that there is an absolute constant C (independent of m) such that if $f_m(\alpha) = 0$, then $|\alpha| \leq C$.

(c) Explain how parts (a) and (b) imply that there are at most finitely many positive integers m for which $f_m(x)$ is reducible.