Comprehensive Exam, Summer 2004

for

MATH 780 & 782

1. Find the remainder when $2003!$ is divided by 2011, that is find the number $r \in \{0, 1, \ldots, 2010\}$ such that $2003! \equiv r \pmod{2011}$. (Hint: You can use that 2011 is a prime number). Explain.

2. Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$ and $q = 2p + 1$ is a prime number too. Show that $2^p - 1$ is not a prime number. (Hint: Show that 2 is a square modulo a certain prime to prove $q|(2^p - 1)$.)

3. Let $a > 2$ be an integer.

   (i) Let $n_k > 1$ be an integer such that $n_k|(a^{n_k} - 1)$. Show that if $n_{k+1} = a^{n_k} - 1$ then $n_{k+1}|(a^{n_{k+1}} - 1)$.

   (ii) Show that there exist infinitely many positive integers n such that $n|(a^n - 1)$.

   (iii) Let p be an odd prime. Show that there exists a prime q with $q < p$ such that $q|\mathrm{ord}_p(2)$.

   (iv) Show that there is no integer $n > 1$ such that $n|(2^n - 1)$. (Hint: Let p be the least prime divisor of n and use (iii).)

4. Let
$$\varphi(x) = \int_0^x \left(\frac{1}{2} - \{t\} \right) dt.$$

   (i) Show that $\varphi(x + 1) = \varphi(x)$ for all x, and $0 \le \varphi(x) \le \frac{1}{8}$ for all x.

   (ii) Apply the Euler-Maclaurin summation formula in the form
$$\sum_{a<n\le b} f(n) = \int_a^b f(t)dt + \left(\frac{1}{2} - \{t\} \right) f(t)\Big|_a^b - \varphi(t)f'(t)\Big|_a^b + \int_a^b \varphi(t)f''(t)dt$$
   to $f(t) = \ln t$ to show that there exists a positive constant $c \in (e^{\frac{7}{8}}, e)$ such that for every positive integer n we have:
$$n! = c\sqrt{n}\left(\frac{n}{e}\right)^n e^{\theta_n} \quad \text{with} \quad 0 < \theta_n < \frac{1}{8n}.$$

5. Let a and m be integers with $m \ge 1$ and $\gcd(a, m) = 1$. We had a function $w(L(s, \chi))$ in class that we showed satisfied
$$w(L(s, \chi)) = \sum_p \frac{\chi(p)}{p^s} + E \quad \text{where } \mathrm{Re}(s) > 1 \text{ and } |E| \le 1.$$
   For a non-principal character χ, we also showed that $w(L(s, \chi))$ remains bounded as $s \to 1^+$. Using this, prove Dirichlet's theorem that there are infinitely many primes $p \equiv a \pmod m$.

6. Given that
$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \quad \text{for } s = \sigma + it \text{ where } \sigma > 1,$$
   explain how the Prime Number Theorem follows from the Wiener-Ikahara Theorem stated below. Your argument should explain why the hypotheses of the Wiener-Ikahara Theorem hold, what $A(x)$ and $f(s)$ are, and the connection with the Prime Number Theorem.

   Theorem. *Let $A(x)$ be a non-negative, non-decreasing function of x, defined for $x \in [0, \infty)$. Suppose that the integral $\int_0^\infty A(x)e^{-xs}\,dx$ converges for $\sigma > 1$ to a function $f(s)$ which is analytic for $\sigma \ge 1$ except for a simple pole at $s = 1$ with residue 1. Then $\lim_{x\to\infty} e^{-x}A(x) = 1$.*

7. Let α be a real number. Show that

$$\sum_{p \le x} (\log p)^\alpha \sim x\,(\log x)^{\alpha-1}$$

by rewriting the sum as a Riemann-Stieltjes integral and using the Prime Number Theorem. (Comments: You may need to break the problem up into cases depending on the sign or size of α. Also, if you do what I intend, you will end up with an error term that is an integral which needs to be approximated. You can handle the integral by breaking up the limits of integration to get a sum of two integrals that are more manageable.)

8. Complete this row of the *Dirichlet* character table modulo 16. Give a brief explanation for each table entry you make.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
χ	0			i				-1							0	

9. In this problem, n denotes a positive integer and p_n denotes the n^{th} prime. For $n > 1$, define q_n to be the largest prime divisor of n. For example, if $n = 6$, then $p_n = 13$ and $q_n = 3$.

(a) Explain why the Prime Number Theorem is equivalent to $p_n \sim n \log n$.
(b) Assuming the first sum converges below, justify each equality or inequality (three total) in the following:

$$\sum_{n=2}^{\infty} \frac{1}{nq_n} = \sum_{n=1}^{\infty} \frac{1}{p_n^2} \prod_{j=1}^{n} \left(1 - \frac{1}{p_j}\right)^{-1} \le \sum_{n=1}^{\infty} \frac{1}{p_n^2} \prod_{j=1}^{n} \left(1 - \frac{1}{j+1}\right)^{-1} = \sum_{n=1}^{\infty} \frac{n+1}{p_n^2}.$$

(c) Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{nq_n}$ converge? Justify your answer.