# Math 374
# Practice Test 3

1) Let $A = \{1,2,3,4\}$, $B = \{a, \{b\}, \{a, \{b\}\}\}$

a) List the set $A \cup B$.

$\boxed{A \cup B = \{1,2,3,4, a, \{b\}, \{a, \{b\}\}\}}$

b) what is the cardinality of $A \times B$ ?

$4 \cdot 3 = \boxed{12}$

c) what is the cardinality of $P(B)$ ?

$2^3 = \boxed{8}$

d) what is the cardinality of $P(P(B))$ ?

$2^{2^3} = 2^8 = \boxed{256}$

e) For each of the following determine if the statement is true or false.

i) $\{a, \{b\}\} \subseteq B$ T

ii) $\{a, \{b\}\} \in B$ T

iii) $\{1,2\} \in A$ F

iv) $\{1,2\} \in P(A)$ T

v) $\{a,b\} \subseteq B$ F

vi) $\emptyset \subseteq A \cap B$ T

vii) $\emptyset \in P(A) \cap P(B)$ T

viii) $\{\{1,2\}, \{b\}\} \subseteq A \cup B$ F

2) Liscence plates in Utah come in two types: Standard plates and vanity plates. Standard plates have three distinct letters, followed by 3 digits, where the first digit must be nonzero. Vanity plates use two letters (not neccesarily distinct) followed by 3 digits, where again the first digit must not be zero.
How many different Liscence plates are there?

Option 1: Standard Plate — $P(26,3)\cdot 900$

Step 1: Letters. ($P(26,3)$ ways)
Step 2: First digit (9 choices)
Step 2: Second digit (10)
Step 3: 3rd digit 10

Option 2: Vanity Plate $\to 26^2\cdot 900$

Step 1: First letter 26 ways
Step 2: Second letter 26 ways
Step 3: Number 900 ways.

$\boxed{\text{So total of } P(26,3)\cdot 900 + 26^2\cdot 900 \text{ options}}$

3) A standard set of pool balls is a collection of 15 balls, numbered (unsurprisingly) 1–15. Balls 1–8 are solids, and balls 9–15 are stripes.

a) How many collections of 3 balls are there?

$$\boxed{\binom{15}{3}}$$

b) How many collections of 5 balls have 3 stripes and 2 solids?

Step 1: choose stripes ($\binom{7}{3}$ ways)
Step 2: choose solids ($\binom{8}{2}$ ways)

$\boxed{\binom{7}{3}\cdot\binom{8}{2} \text{ collections}}$

c) How many 5-ball collections are there with at least one solid?

(# 5 ball collections) = (# No Solids) + (# at least one Solid)

$\binom{15}{5}$ = ~~(# No Solids)~~ $\binom{7}{5}$ + (# at least one Solid)

$$\boxed{(\#\text{ at least one solid}) = \binom{15}{5} - \binom{7}{5}}$$

4) We surveyed 150 college students about how they commute to school. 83 said they sometimes drive, 97 said they sometimes bike, and 28 said they sometimes ride a skateboard. 53 bike and drive, 14 drive and skateboard, and 7 bike and skateboard. 2 students said they sometimes drive, sometimes bike, and sometimes ride a skateboard. How many of the surveyed students use <u>none</u> of these 3 modes of transport?

A = Drive   B = Bike   C = Skateboard.

<u>Want</u> $|(A\cup B\cup C)^c|$. This is $|S| - |A\cup B\cup C|$.

$$|A\cup B\cup C| = 83 + 97 + 28 - 53 - 14 - 7 + 2 \quad \text{(by I-e)}$$

$$= 136$$

So $|(A\cup B\cup C)^c| = 150 - 136 = \boxed{14}$

5) Show that in any collection of 51 different numbers, bigger than 0 and less than 100, that two of the numbers add to 100.

Let $\{50\}$ be one pigeonhole, and the sets

$\{1,99\}, \{2,98\}, \{3,97\}, \dots \{48,52\}, \{49,51\}$

be the others. There are $\boxed{\text{fifty}}$ of these.

Assign our $\boxed{51}$ numbers (pigeons) to the sets that contain them.

By pigeonhole, some set gets both of its members, which add to 100.

6) A dealership has 8 identical brand new cars to put on display. 3 are blue, 2 are red, 2 are green, and one is white. How many different ways can the dealership park them in a line?

Same as arranging the letters
BBBRRGGW into distinct words.

There are $\boxed{\dfrac{\overset{\#\text{letters}}{8!}}{\underset{\#B\text{'s}}{3!}\,\underset{\#R\text{'s}}{2!}\,\underset{\#G\text{'s}}{2!}\,\underset{\#W\text{'s}}{1!}}}$ ways to do this.

7) I've come across 21 bitcoins, and decided to distribute them to 5 of my friends.

a) How many ways can I do this? I need to tell each of my friends how many they get. "Stars and bars" with 21 stars, and 4 bars (separators). There are

$$\binom{21+4}{4} = \boxed{\binom{25}{4}} \text{ ways to do this.}$$

b) How many ways can I do this if I promise each of my 5 friends that they get at least 1 bitcoin?

Give each person one, then you have 16 left. The same process as a) says there are

$$\boxed{\binom{20}{4}} \text{ ways to distribute the remaining coins}$$

8) Expand $(2x-y)^4$ using the binomial Theorem

$$= \binom{4}{0}(2x)^4(-y)^0 + \binom{4}{1}(2x)^3(-y) + \binom{4}{2}(2x)^2(-y)^2 + \binom{4}{3}(2x)(-y)^3 + \binom{4}{4}(2x)^0(-y$$

$$= 1\cdot 16x^4 + 4\cdot 8\cdot(-1)\,x^3y + 6\cdot 4\, x^2y^2 + 4\cdot 2(-1)\,xy^3 + 1\cdot y^4$$

$$= \boxed{16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4}$$

9) What is the coefficient of $a^5b^6$ in $\left(2a - \frac{b}{2}\right)^{11}$ ?

This is $i=6$ in the expansion. term is

$$\binom{11}{6}(2a)^5\left(-\frac{b}{2}\right)^6$$

$$= \binom{11}{6}\cdot 2^5\cdot\left(-\frac{1}{2}\right)^6 a^5b^6$$

$$= \binom{11}{6}\cdot\frac{1}{2}\, a^5b^6$$

<u>Coefficient</u> is $\boxed{\binom{11}{6}/2}$