DISCRETE OPTIMIZATION: PROBLEM SET 5

Problem 1.

(1) The assignment problem: Suppose we have some number of computer programs that we want to run, and we have some number of computers to run them on. Unfortunately, each computer can only run one program at a time, and some of the programs can only run on some of the computers. So for each computer, we're given a list of the programs that it can run. How can we find the best assignment of programs to computers?

(2) Use the method above to find the best assignment for the following situation:

Computer a can run program 1,2,or 5, computer b can run program 1,2,3, or 4, computer c can run program 1 or 2, computer d can run program 1 or 2, computer e can run program 2.

Problem 2. Find a proof of Hall's Theorem (either your own, or researched on the internet, or a blend of these) that you can understand and explain. (I know of at least three different proofs.)

Problem 3. A graph is called *k-regular* if the degree of every vertex is k. Show that a k-regular bipartite graph always has a matching that uses every vertex.

Problem 4. (slightly more difficult than normal) Show that in a connected, $2k$-regular graph (not necessarily bipartite), there is always a set of edges so that every vertex is covered exactly twice. (i.e., there is some collection of cycles that covers the whole graph.) (*First hint:* what do you know about graphs where every vertex has even degree?)

Problem 5. Suppose we have a collection $\{A_1, A_2, \ldots, A_k\}$, where each A_i is a subset of $\{1, 2, \ldots, n\}$. A collection of k numbers $a_1, a_2, \ldots, a_k$ where $a_i \in A_i$ for all $i \leq k$ is called a *system of distinct representatives* for the subsets. Translate Hall's theorem into the language of SDRs. That is, write the necessary and sufficient conditions for when a set system has a SDR, and show how they follow from Hall's theorem.

Problem 6. For the following set systems, find a system of distinct representatives, or explain why it is impossible.

(1) $\{3, 4, 5\}, \{2, 5, 6\}, \{1, 2, 5\}, \{1, 2, 3\}, \{1, 3, 6\}$
(2) $\{1, 2, 3, 4, 5, 6\}, \{1, 3, 4\}, \{1, 4, 7\}, \{2, 3, 5, 6\}, \{3, 4, 7\}, \{1, 3, 4, 7\}, \{1, 3, 7\}$