DISCRETE OPTIMIZATION: PROBLEM SET 1

Problem 1. Prove that if G is a simple graph with at least two vertices, then G must have at least two vertices with the same degree.

Problem 2. Prove that a connected graph where every vertex has even degree has an Eulerian tour.

Problem 3. The *complete* graph K_n is the simple graph on n vertices where every possible edge is included. In this problem, we color the edge with two colors (say red and blue). Three vertices form a *monochromatic triangle* if the edges connecting them are all assigned the same color.

(1) Show that there is a way to color K_5 so that there are *no* monochromatic triangles.
(2) Prove that no matter how you color K_6, there is at least one monochromatic triangle. (Preferably this proof is *not* done by listing all 32,768 ways of coloring and pointing to a monochromatic triangle in each one...)

Problem 4. A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either *in* D, or is a neighbor of a vertex in D. Finding a small dominating set has important applications in networks, such as monitoring or controlling with lowest cost. The smallest number of vertices possible in a dominating set is called the *domination number* of the graph.

(1) What is the domination number of K_n?
(2) What is the domination number of P_n, the graph that is a path with n vertices?
(3) Describe an algorithm that finds a relatively small, but not necessarily minimum size, dominating set for a graph. (You don't need to *prove* that the set is small, but justify it somewhat)