Clearly explain what you are doing during every step of any computations. You must justify all your answers.

Problem 1 Determine if the set $G = \{x^2 - y,\ y^3 - 1\}$ in $\mathbb{Q}[x,y]$ is a Gröbner basis for the lexicographic ordering.

Problem 2 Find a Gröbner basis for the ideal $I = \langle x^2 - y,\ xy - x\rangle \subseteq \mathbb{Q}[x,y]$ for the lexicographic ordering.

Problem 3 The set

$$G = \{x + 2y^3 + z^4 - z^2 + z,\ y^2 - 2yz,\ 3yz,\ z^4 + z\}$$

in $\mathbb{Q}[x,y]$ is a Gröbner basis using the lexicographic ordering (you do not need to check this). Find a *reduced* Gröbner basis for $\langle G\rangle$.

Problem 4 The set

$$G = \{x^2 - y^4,\ xy - y^3,\ y^5 - 2y\}$$

in $\mathbb{Q}[x,y]$ is a Gröbner basis using the lexicographic ordering (you do not need to check this). Determine if $f = x^2y + 3y$ is an element of the ideal $I = \langle G\rangle$.

Problem 5 Find all real solutions $(x,y,z) \in \mathbb{R}^3$ of the following system:

$$\begin{aligned} xyz + 2y - 4x &= 4 \\ y^2z^2 - 4y^2 + yz &= 2 \\ z^4 - 3z^2 &= 4\ . \end{aligned}$$

(Hint: you do not need to compute a Gröbner basis!)

Problem 6 Consider the ideal $I = \langle G\rangle \subseteq \mathbb{Q}[x,y,z]$ where

$$G = \{x^2 + x + 2z - 1,\ xz - x + 2z - 2,\ y - z,\ 2z^2 - z - 1\}$$

is a Gröbner basis using the lexicographic ordering (you do not need to check this). Find $I \cap \mathbb{Q}[y,z]$ and $I \cap \mathbb{Q}[z]$.

Problem 7 Consider the ideal $I = \langle G\rangle \subseteq \mathbb{R}[x,y,z]$ where

$$G = \{xy + x - z,\ xz - x + y + z - 1,\ y^2 + yz + z^2 - 1\}$$

is a Gröbner basis using the lexicographic ordering (you do not need to check this). Describe the set of all partial solutions $(y,z) = (a_2,a_3)$ that extend to full solutions $(a_1,a_2,a_3) \in \mathbb{V}(I)$.

Problem 8 Determine all singular points of $\mathbb{V}(x^3 - 3xy + y^3) \subseteq \mathbb{R}^2$.