Due May 8 at 11:30 am.

The Rules:

- You are **allowed** to use your notes from class.
- You are **allowed** to use resources posted on the course website.
- You are **allowed** to use the course textbook.
- You are **allowed** to use Macaulay2.
- You **cannot** discuss the exam with your classmates or any other person.
- You **cannot** use other textbooks or online resources to help solve problems.

If a problem below is denoted "(By hand!)", then you must do all the calculations without Macaulay2. You may check your answers with Macaulay2, but you will not get full credit unless you write out all the details. In the remaining problems, you can use Macaulay2.

Problem 1 Rewrite the polynomial

$$x^6y^6z + 2x^6y^3z^4 + 3x^6y^6 + 4x^8y^2z^3$$

using each of the *lex*, *grlex*, and *grevlex* orders.

Problem 2 (By hand!) Compute $\gcd(x^4 + 2x^3 - 4x^2 - 2x + 3,\ x^4 + 3x^3 + x^2 - 3x - 2)$.

Problem 3 (By hand!) Consider the following polynomials:

$$f = x^3y - 2x^2y^2 + xy^3 \qquad \begin{aligned} g_1 &= x^2y - xy \\ g_2 &= xy^2 - y^3 \\ g_3 &= y^2 - 2y \end{aligned}$$

Compute the remainder of f on division by the ordered set $G = (g_1,\ g_2,\ g_3)$ using the division algorithm and the *lex* order.

Problem 4 (By hand!) Find the reduced Gröbner basis for the ideal $I = \langle x^2 - y^2,\ xy - 1\rangle \subseteq \mathbb{Q}[x, y]$ for the *lex* order.

Problem 5 Consider the ideal

$$I = \langle x^4y - 3x^3z^2 + 4,\ x^2y^3 - xy + 4z,\ x^3 - z\rangle$$

in $k[x, y, z]$. Determine whether $x^3y - y^2 - 2z$ is contained in I.

Problem 6 Consider the variety $V \subset \mathbb{C}^3$ given by the parametrization

$$\begin{aligned} x &= t^2 - t \\ y &= t^3 \\ z &= t^2 \ . \end{aligned}$$

where t is a parameter. Find generators for an ideal I such that $V = \mathbf{V}(I)$.

Problem 7 Consider the ideal $I \subseteq \mathbb{C}[x, y, z]$ where

$$I = \langle x^2y^2 - x^2z^2 - yz,\ y^2 + z^2 - 1 \rangle\ .$$

Describe the set of all partial solutions $(y, z) = (a_2, a_3)$ that *do not* extend to full solutions $(x, y, z) = (a_1, a_2, a_3) \in \mathbf{V}(I)$.

Problem 8 Determine all singular points of the real plane curve

$$\mathbf{V}((y^2 - x^2)(x - 1)(2x - 3) - 4(x^2 + y^2 - 2x)^2).$$

Problem 9 Consider the ideals

$$I = \langle x^2, y \rangle \text{ and } J = \langle x - y \rangle$$

in $\mathbb{Q}[x, y]$. Find $I \cap J$, $I : J$ and $I : J^\infty$.

Problem 10 Suppose $I, J \subseteq k[x_1, \ldots, x_n]$ are ideals. Prove that $(I : J)J \subseteq I$.

Problem 11 Consider the complex variety $X \subseteq \mathbb{C}^3$ defined by the equations:

$$x^2y^2 - x^2y + xy^3 - xy^2 = 0$$
$$x^3y - xy^3 = 0\ .$$

Determine the minimal decomposition of X into irreducible components.

Problem 12 Find an example of each of the following:

(1) Ideals $I, J \subseteq \mathbb{C}[x, y]$ such that $I \neq J$, but $\mathbf{V}(I) = \mathbf{V}(J)$.
(2) An ideal $I \subseteq \mathbb{R}[x, y]$ such that $\mathbf{V}(I) = \emptyset$, but $I \neq \mathbb{R}[x, y]$.
(3) Radical ideals $I, J \subseteq \mathbb{C}[x, y]$ such that $I + J$ is not radical.
(4) Ideals $I, J \subseteq \mathbb{C}[x, y]$ such that $IJ \neq I \cap J$.