Complex Analysis Solutions

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Page 108, 8a) We have from (15) that $|\sin^2 z|^2 = \sin^2 x + \sinh^2 y$. But $\sinh^2 y \geq 0$ since it is a square, so

$$|\sin^2 z|^2 = \sin^2 x + \sinh^2 y \geq \sin^2 x.$$

Then taking square roots yields $|\sin z| \geq \sqrt{\sin^2 x} = |\sin x|$.

11 We have, letting $z = x + iy$, $\bar{z} = x - iy$,

$$\begin{aligned}
\sin \bar{z} &= \frac{1}{2i}(e^{i\bar{z}} - e^{-i\bar{z}}) \\
&= \frac{1}{2i}(e^{y+ix} - e^{-y-ix}) \\
&= \frac{1}{2i}\left(e^{y}(\cos x + i \sin x) - e^{-y}(\cos x - i \sin x)\right) \\
&= \frac{1}{2i}\left((e^{y} - e^{-y}) \cos x + i(e^{y} + e^{-y}) \sin x\right) \\
&= \frac{1}{2}(e^{y} + e^{-y}) \sin x - \frac{i}{2}(e^{y} - e^{-y}) \cos x \\
&= \cosh y \sin x - i \sinh y \cos x
\end{aligned}$$

Thus, we let $u(x,y) = \cosh y \sin x$ and $v(x,y) = -\sinh y \cos x$ and take the partial derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cosh y \cos x \\
\frac{\partial u}{\partial y} &= \sinh y \sin x \\
\frac{\partial v}{\partial x} &= \sinh y \sin x \\
\frac{\partial v}{\partial y} &= -\cosh y \cos x.
\end{aligned}$$

The Cauchy-Riemann equations say that we must have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ everywhere, which is clearly not true. Hence, $\sin \bar{z}$ is not analytic.

In a similar manner, $\cos \bar{z} = \cosh y \cos x + i \sinh y \sin x$, and by letting $u(x, y) = \cosh y \cos x$ and $v(x, y) = \sinh y \sin x$,

$$\begin{aligned}
u_x &= -\cosh y \sin x \\
u_y &= \sinh y \cos x \\
v_x &= \sinh y \cos x \\
v_y &= \cosh y \sin x
\end{aligned}$$

which again do not satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$.

Page 111, 3 The identities 6 and 9 from section 34 are $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ and $\cos^2 z + \sin^2 z = 1$. Note that

$$\begin{aligned}
-i \sin(iz) &= \sinh z \\
\cos(iz) &= \cosh z
\end{aligned}$$

so substituting these into 6 and 9 yield,

$$1 = \cos^2(iz) + \sin^2(iz) = (\cosh z)^2 + \left(-\frac{\sinh z}{i}\right)^2 = \cosh^2 z - \sinh^2 z,$$

and

$$\begin{aligned}
\cosh(z_1 + z_2) &= \cos(iz_1 + iz_2) \\
&= \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 \\
&= \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2.
\end{aligned}$$

5 We have

$$\begin{aligned}
|\cosh^2 z|^2 = \left|\frac{e^{iz} + e^{-iz}}{2}\right|^2 &= \frac{1}{2}\left|e^{-y+ix} + e^{y-ix}\right|^2 \\
&= \frac{1}{4}\left|e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)\right|^2 \\
&= \frac{1}{4}\left|(e^{y} + e^{-y}) \cos x - i(e^{y} - e^{-y}) \sin x\right|^2 \\
&= \left|\cosh y \cos x - i \sinh y \sin x\right|^2 \\
&= \cosh^2 y \cos^2 x + \sinh^2 y \sin^2 x \\
&= \cosh^2 y \cos^2 x + \sinh^2 y(1 - \cos^2 x) \\
&= \sinh^2 y + \cos^2 x(\cosh^2 y - \sinh^2 y) \\
&= \sinh^2 y + \cos^2 x.
\end{aligned}$$

Page 114, 1b We use the identity $\tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z}$ (equation (4) of section 36). Then, for $z = 1+i$,

$$\begin{aligned}
\tan^{-1}(1+i) &= \frac{i}{2} \ln \frac{i+(1+i)}{i-(1+i)} \\
&= \frac{i}{2} \ln \frac{1+2i}{-1} \\
&= \frac{i}{2} \ln(-1-2i) \\
&= \frac{i}{2} \ln \left(-\sqrt{5} e^{i \tan^{-1}(2) + 2\pi i n}\right) \\
&= \frac{i}{2} \left(\frac{1}{2} \ln 5 + i \tan^{-1} 2 + i(2n+1)\pi\right) \\
&= -(2n+1)\frac{\pi}{2} - \frac{1}{2} \tan^{-1} 2 + i \frac{\ln 5}{4}
\end{aligned}$$

where n is any integer.

1c We use the identity $\cosh^{-1} z = \ln\left(z + (z^2-1)^{1/2}\right)$ (equation (9) of section 36). Then, for $z = -1$,

$$\begin{aligned}
\cosh^{-1}(-1) &= \ln\left((-1) + ((-1)^2 - 1)^{1/2}\right) \\
&= \ln\left((-1) + (0)^{1/2}\right) \\
&= \ln(e^{i\pi + 2\pi i n}) \\
&= i\pi(2n+1).
\end{aligned}$$

Page 121, 3 Suppose $m \neq n$. Then $m - n \neq 0$ and the integral becomes

$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \frac{1}{i(m-n)} e^{i(m-n)\theta}\Big|_0^{2\pi} \frac{1}{i(m-n)}(e^{i(m-n)2\pi} - 1) = \frac{1}{i(m-n)}(1-1) = 0.$$

If $n = m$, then $m - n = 0$ and the integral becomes

$$\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \int_0^{2\pi} e^0\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

Page 126, 5 Write $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$ so

$$w(t) = f(z(t)) = u(x(t), y(t)) + iv(x(t), y(t)).$$

Then, using the chain rule from ordinary calculus,

$$\begin{aligned}
w'(t) &= \frac{d}{dt} u(x(t), y(t)) + i \frac{d}{dt} v(x(t), y(t)) \\
&= (u_x x' + u_y y') + i(v_x x' + v_y y').
\end{aligned}$$

From the Cauchy Riemann equations, we rewrite the imaginary part to be

$$\begin{aligned} w'(t) &= (u_x x' + u_y y') + i(-u_y x' + u_x y') \\ &= u_y(y' - ix') + u_x(x' + iy') \\ &= -iu_y(x' + iy') + u_x(x' + iy') \\ &= (x' + iy')(u_x - iu_y) \\ &= (x' + iy')(u_x + iv_x) \\ &= z'(t) f'(z(t)) \end{aligned}$$

where in the last line, we use the first theorem of section 22.

Page 135, 4 We let $z(t) = t + it^3, -1 \le t \le 1$ parameterize C. Then $dz = (1 + 3it^2)dt$ and

$$\begin{aligned} \int_C f(z)\,dz &= \int_{-1}^{1} f(t + it^3)(1 + 3it^2)\,dt \\ &= \int_{-1}^{0} f(t + it^3)(1 + 3it^2)\,dt + \int_0^1 f(t + it^3)(1 + 3it^2)\,dt \\ &= \int_{-1}^{0} (1 + 3it^2)\,dt + \int_0^1 4t^3(1 + 3it^2)\,dt \\ &= (t + it^3)\Big|_{-1}^{0} + (t^4 + 2it^6)\Big|_0^1 \\ &= (0 - (-1 - i)) + (1 + 2i) \\ &= 2 + 3i. \end{aligned}$$

Page 140, 2 We have

$$\begin{aligned} \left| \int_C \frac{dz}{z^4} \right| &\le \max_{z \in C} \frac{1}{|z|^4} \cdot \text{length}(C) \\ &= \frac{1}{(\min_{z\in C} |z|)^4} \cdot \text{length}(C). \end{aligned}$$

Since C is a line segment, length$(C) = |1 - i| = \sqrt{2}$. Since the midpoint of C is closest to the origin, i.e. $|z|$ is a minimum at the midpoint, we have

$$\min_{z\in C} |z| = \left| \frac{1+i}{2} \right| = \frac{1}{2}|1 + i| = \frac{1}{\sqrt{2}}.$$

Hence, from above,

$$\left| \int_C \frac{dz}{z^4} \right| \le \frac{1}{(\min_{z\in C} |z|)^4} \cdot \text{length}(C) = \frac{1}{(1/\sqrt{2})^4}\sqrt{2} = 4\sqrt{2}.$$

Page 149, 3 An antiderivative of $f(z) = (z - z_0)^{n-1}$ in the domain $\mathbb{C} \setminus \{z_0\}$, where n is a nonzero integer, is $F(z) = \frac{1}{n}(z - z_0)^n$ since $F'(z) = f(z)$. Hence, since C is a closed contour, the integral $\int_C f(z)\,dz = 0$ from the theorem.