NONABELIAN RELATIVE COHOMOLOGY AND BUNDLES

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§1 Cohomology of Coverings and Bundle Interpretations. A good introductory reference for sheaf cohomology in the abelian case is Forster []. Information on relative cohomology in the abelian case can be found in Komatsu []. A discussion of 0th and 1st cohomology (not relative) in the nonabelian case is in Babbitt Varadarajan []. We will discuss relative cohomology in the nonabelian case. We will not be using any heavy machinery from algebraic geometry/topology although one of our aims is to make it possible to do so. Our main emphasis will be on precise definitions and geometric interpretations.

We will specify categories using a quotient notation, where the numerator indicates the objects in the category and the denominator indicates the arrows, e.g. Groups/group homomorphisms. The rule of composition of arrows will usually be implicitly understood. All functors will be covariant. In all categorical concepts and terminology we follow MacLane []. If X is a topological space, and $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ is an open covering of X, then we will use the shorthand notation: $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Suppose $\mathcal{A}$ is a concrete category, i.e. it is equipped with a "forgetful functor" $\mathcal{A} \to$ Sets/mappings. We will not introduce any notation for this functor, but we interpret the expression $x \in A$, where A is an object of $\mathcal{A}$, to mean that x is a member of the underlying set associated to the object A by the forgetful functor. Similarly, we interpret the expression $f(x)$, where $f\colon A \to B$ is an arrow in $\mathcal{A}$ and $x \in A$, to denote the image of x under the function associated to the arrow f by the forgetful functor.

Definition. Suppose X is a topological space. A *sheaf* $\mathcal{F}$ on X with values in the concrete category $\mathcal{A}$ is a functor $\mathcal{F}$: Open sets of X/reverse inclusions $\to \mathcal{A}$ satisfying the following collation condition: for every open set $U \subset X$, for every open covering $\{V_i\}_{i\in\mathcal{I}}$ of U, and for every family $\{f_i \in \mathcal{F}(V_i)\}$ the pairwise matching condition $\mathcal{F}(V_i \supset V_{ij})(f_i) = \mathcal{F}(V_j \supset V_{ij})(f_j)$ for all i and j implies that there exists a unique $f \in \mathcal{F}(U)$ such that $\mathcal{F}(U \supset V_i)(f) = f_i$ for all i.

Important examples include the following.

1. The sheaf $\mathcal{O}_X$ of complex-valued holomorphic functions on a complex manifold X. For every open subset U of X, $\mathcal{O}_X(U)$ denotes the $\mathbb{C}$-algebra of all complex-valued functions defined and holomorphic on U, and whenever $U \supset V$ are open subsets of X, $\mathcal{O}_X(U \supset V)$ denotes the $\mathbb{C}$-algebra homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which restricts a function defined on U to the subset V. If $X = \mathbb{C}$ (resp. $X = \overline{\mathbb{C}}$) then we will write $\mathcal{O}_X = \mathcal{O}$ (resp. $\mathcal{O}_X = \overline{\mathcal{O}}$).
2. The sheaf C_X^∞ of complex-valued smooth functions defined on subsets of a smooth manifold X.
3. The sheaf $GL(n, \mathcal{O}_X)$ of groups. This sheaf associates to every open subset U of the complex manifold X the group of $n \times n$ matrices with entries in the ring $\mathcal{O}_X(U)$ which are invertible in the collection of marices of the same type. This group is isomorphic to the group of holomorphic maps from U into the complex Lie group $GL(n, \mathbb{C})$.

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Let X be a topological space and S a closed subset. Suppose $\mathcal{I} \supset \mathcal{I}'$ are index sets, and $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ is an open covering of X, and $\mathcal{U}' = \{U_i\}_{i\in\mathcal{I}'}$ is an open covering of $X \setminus S$. In this situation we will say $(\mathcal{U}, \mathcal{U}')$ is an open covering of $(X, X \setminus S)$. Let $\mathcal{G}$ denote a sheaf of groups on X.

Definition 1. $C^0(\mathcal{U}, \mathcal{G})$ is defined to be the group of all families $\{f_i\}_{i\in\mathcal{I}}$ such that $f_i \in \mathcal{G}(U_i)$ for all $i \in \mathcal{I}$. Elements of this group are called 0-*cochains* associated with the covering $\mathcal{U}$. Define $C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$ to be the subgroup of $C^0(\mathcal{U}, \mathcal{G})$ consisting of those families $\{f_i\}_{i\in\mathcal{I}}$ such that $f_i = 1$ for all $i \in \mathcal{I}'$. Such families are called *relative* 0-cochains.

Definition 2. $Z^1(\mathcal{U}, \mathcal{G})$ is defined to be the pointed set of all families $\{g_{ij}\}_{(i,j)\in\mathcal{I}^2}$ such that $g_{ij} \in \mathcal{G}(U_{ij})$ and

$$g_{ii} = 1$$
$$\mathcal{G}(U_{ij} \supset U_{ijk})(g_{ij}) = \mathcal{G}(U_{ik} \supset U_{ijk})(g_{ik}) \cdot \mathcal{G}(U_{kj} \supset U_{ijk})(g_{kj})$$

for all $(i, j, k) \in \mathcal{I}^3$. Elements of $Z^1(\mathcal{U}, \mathcal{G})$ are called 1-cocycles associated to the covering $\mathcal{U}$. The distinguished element of $Z^1(\mathcal{U}, \mathcal{G})$ is the "unit cocycle", i.e. $g_{ij} = 1$ for all $(i, j) \in \mathcal{I}^2$. Define $Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ to be the subset of $Z^1(\mathcal{U}, \mathcal{G})$ consisting of those cocycles $\{g_{ij}\}_{(i,j)\in\mathcal{I}^2}$ such that $g_{ij} = 1$ for all $(i, j) \in (\mathcal{I}')^2$. These families are called *relative* 1-cocycles.

Definition 3. If $g = \{g_{ij}\}_{(i,j)\in\mathcal{I}^2} \in Z^1(\mathcal{U}, \mathcal{G})$ and $f = \{f_i\}_{i\in\mathcal{I}} \in C^0(\mathcal{U}, \mathcal{G})$, then define g^f to be the family $\{(g^f)_{ij}\}_{(i,j)\in\mathcal{I}^2}$, where

$$(g^f)_{ij} = \mathcal{G}(U_i \supset U_{ij})(f_i)^{-1} \cdot g_{ij} \cdot \mathcal{G}(U_j \supset U_{ij})(f_j)$$

is in $\mathcal{G}(U_{ij})$ for all $(i, j) \in \mathcal{I}^2$.

Fact. *The map $(g, f) \mapsto g^f$ defines a right action of the group $C^0(\mathcal{U}, \mathcal{G})$ on the set $Z^1(\mathcal{U}, \mathcal{G})$, as well as a right action of the subgroup $C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$ on the subset $Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$.*

Definition 4. $H^0(\mathcal{U}, \mathcal{G})$ is defined to be the the group $\{f \in C^0(\mathcal{U}, \mathcal{G}) \mid 1^f = 1\}$, where 1 denotes the trivial 1-cocycle. This group is called the 0-*cohomology* group associated to the covering $\mathcal{U}$. Define $H^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$ to be equal to $H^0(\mathcal{U}, \mathcal{G}) \cap C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$. It is called the *relative* 0-cohomology group.

Fact. *$H^0(\mathcal{U}, \mathcal{G}) \cong \mathcal{G}(X)$, and $H^0(\mathcal{U}', \mathcal{G}) \cong \mathcal{G}(X \setminus S)$. If X is a complex manifold, $\mathcal{G} = GL(n, \mathcal{O}_X)$, and S is a closed subset of X with empty interior, then $H^0(\mathcal{U}, \mathcal{U}', \mathcal{G}) \cong \{1\}$. In particular, these groups are independent of the coverings $\mathcal{U}$ and $\mathcal{U}'$.*

Definition 5. Define $H^1(\mathcal{U}, \mathcal{G})$ to be the pointed set of all orbits in $Z^1(\mathcal{U}, \mathcal{G})$ of the right action of the group $C^0(\mathcal{U}, \mathcal{G})$. It is called the 1-*cohomology* set associated to the covering $\mathcal{U}$. The distinguished element of $H^1(\mathcal{U}, \mathcal{G})$ is the orbit of the unit cocycle. Define $H^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ to be the pointed set of all orbits in $Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ of the right action of the group $C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$. It is called the *relative* 1-cohomology set associated to the covering $(\mathcal{U}, \mathcal{U}')$ of $(X, X \setminus S)$. So in symbols

$$H^1(\mathcal{U}, \mathcal{G}) = Z^1(\mathcal{U}, \mathcal{G})/C^0(\mathcal{U}, \mathcal{G})$$
$$H^1(\mathcal{U}, \mathcal{U}', \mathcal{G}) = Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})/C^0(\mathcal{U}, \mathcal{U}', \mathcal{G}).$$

Remark. The definitions of ordinary 0-cochains, 1-cocycles, 0-cohomology, and 1-cohomology are contained in the definitions of the corresponding relative concepts as a special case: take $S = X$ and $\mathcal{I}' = \emptyset$.

Now we will examine the geometric interpretations of these constructions involving holomorphic principal bundles over a complex manifold X. Since the correspondence between $H^1(X, GL(n, \mathcal{O}_X))$ (defined in the next section) and the pointed set of isomorphism classes in the category of principal fibre bundles over X with structure group $GL(n, \mathbb{C})$/bundle isomorphisms is well-known, we will explain how things must be modified to give a geometric interpretation to the relative 1-cohomology set $H^1(X, X \setminus S, \mathcal{G})$. By the above remark our explanation will include the usual case as well. This section will address an interpretation of cohomology associated to a open covering $(\mathcal{U}, \mathcal{U}')$ of $(X, X \setminus S)$; the general case will be completed in the next section.

Recall a *groupoid* is a category all of whose arrows are isomorphisms. *Groupoid homomorphisms* are simply functors. If Z^1 is a set equipped with a right action of a group C^0, then there is a naturally associated groupoid defined as follows.

(1) The objects are the elements of the set Z^1.
(2) The arrows $f\colon g_2 \to g_1$, where $g_1, g_2 \in Z^1$ and $f \in C^0$, are triples (g_2, f, g_1) where $g_2 = g_1^f$.
(3) The composition of an arrow $h\colon g_3 \to g_2$ with $f\colon g_2 \to g_1$ is defined to be $fh\colon g_3 \to g_1$, i.e. (g_3, fh, g_1), since $g_3 = g_2^h = (g_1^f)^h = g_1^{(fh)}$.

Suppose X is a complex manifold, $S \subset X$ is a closed subset, and $\mathcal{G} = GL(n, \mathcal{O}_X)$. Suppose $\mathcal{I}' \subset \mathcal{I}$ are index sets, $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$, $\mathcal{U}' = \{U_i\}_{i\in\mathcal{I}'}$, and $(\mathcal{U}, \mathcal{U}')$ is an open covering of $(X, X \setminus S)$ as above. Denote by $Z = Z(\mathcal{U}, \mathcal{U}')$ the groupoid associated (as in the previous paragraph) to the set $Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ equipped with the right action of the group $C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$ given in Definition 3. We will examine the relation between this groupoid and the following "geometric" groupoid, denoted by $B = B(\mathcal{U}, \mathcal{U}')$, defined as follows.

(1) The objects are triples $(\rho\colon E \to X, \mathcal{B}, \sigma)$, where $\rho\colon E \to X$ is a principal fibre bundle with structure group $GL(n, \mathbb{C})$, $\mathcal{B} = \{\sigma_i\}_{i\in\mathcal{I}}$ is an holomorphically compatible atlas of local trivializations of the bundle $\rho\colon E \to X$ defined on preimages of sets in the covering $\mathcal{U}$, i.e. $\sigma_i\colon \rho^{-1}(U_i) \to U_i \times GL(n, \mathbb{C})$ for all $i \in \mathcal{I}$, and $\sigma\colon \rho^{-1}(X \setminus S) \to (X \setminus S) \times GL(n, \mathbb{C})$ is a (distinguished) trivialization holomorphically compatible with (but not necessarily a member of) the atlas $\mathcal{B}$ such that for all $i \in \mathcal{I}'$ we have $\sigma|_{\rho^{-1}(U_i)} = \sigma_i$.
(2) The arrows are isomorphisms of principal fibre bundles over X with structure group $GL(n, \mathbb{C})$ which over $X \setminus S$ are given by the identity map with respect to the two distinguished trivializations.

We will show that these two categories (groupoids) are *equivalent*: we will exhibit an adjoint pair $(\mathcal{E}, \mathcal{T})$ of functors, and a unit and counit $1, \phi$ of the adjunction which will constitute an *equivalence of categories*. We will suppress in our notation the dependence of everything on the covering $(\mathcal{U}, \mathcal{U}')$.

First we will define the functor $\mathcal{E}\colon Z \to B$. Suppose $g = \{g_{ij}\}_{(i,j)\in\mathcal{I}^2} \in Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G}) \subset Z^1(\mathcal{U}, \mathcal{G})$. Then it is well-known that this data can be used to define an holomorphic principal bundle $p\colon E(g) \to X$ with structure group $GL(n, \mathbb{C})$, together with an atlas $\mathcal{A}$ of local trivializations defined on preimages of sets in the covering $\mathcal{U}$, as follows. The total space $E(g)$ is a complex manifold obtained by gluing:

$$E(g) = \left[\coprod_{i\in\mathcal{I}} U_i \times GL(n, \mathbb{C})\right] \Big/ \sim,$$

where $(x, A) \sim (y, B)$, $x \in U_i$, $y \in U_j$, $A, B \in GL(n, \mathbb{C})$ if and only if $x = y \in U_{ij}$ and $A = g_{ij}(x) \cdot B$. The conditions on g which insure that it is a 1-cocycle imply that $\sim$ is an equivalence relation. The projection map p is defined by mapping each $\sim$ equivalence class of ordered pairs onto the uniquely defined first component of the pair. If $j \in \mathcal{I}$ then the map

$$U_j \times GL(n, \mathbb{C}) \hookrightarrow \coprod_{i\in\mathcal{I}} U_i \times GL(n, \mathbb{C}) \to E(g)$$

is an injection whose image is $p^{-1}(U_j)$. Let $\tau_j\colon p^{-1}(U_j) \to U_j \times GL(n, \mathbb{C})$ denote the inverse of this map. Define $\mathcal{A} = \{\tau_j\}_{j\in\mathcal{I}}$. This collection of maps will function as an atlas defining the analytic manifold structure of $E(g)$ since the overlap map of two charts

$$\tau_j^{-1} \circ \tau_i\colon U_{ij} \times GL(n, \mathbb{C}) \to p^{-1}(U_{ij}) \to U_{ij} \times GL(n, \mathbb{C})\colon (x, B) \mapsto (x, g_{ij}(x) \cdot B)$$

is analytic. $\mathcal{A}$ also is an atlas of holomorphic local trivializations of the bundle $p\colon E(g) \to X$, and the family $g = \{g_{ij}\}_{(i,j)\in\mathcal{I}^2}$ constitute the transition functions. A canonical trivialization $\tau\colon p^{-1}(X \setminus S) \to (X \setminus S) \times GL(n, \mathbb{C})$ can be defined by the rule $\tau([(x, A)]) = \tau_i([(x, A)])$ whenever $i \in \mathcal{I}'$ and $x \in U_i$. This map is well-defined since $g_{ij}(x) = 1$ whenever $x \in U_{ij}$ and both i and j are in $\mathcal{I}'$. τ is clearly bijective and holomorphically compatible with the atlas $\mathcal{A} = \{\tau_i\}_{i\in\mathcal{I}}$. The functor $\mathcal{E}$ therefore associates to the object g of the groupoid Z the object $(p\colon E(g) \to X, \mathcal{A}, \tau)$ of the groupoid B.

Now suppose $f = \{f_i\}_{i\in\mathcal{I}} \in C^0(\mathcal{U},\mathcal{U}',\mathcal{G})$, so that $f\colon g^f \to g$ is an arrow in the groupoid Z. Then $g^f \in Z^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ determines a principal bundle $\tilde{p}\colon E(g^f) \to X$, an atlas $\tilde{\mathcal{A}} = \{\tilde{\tau}_i\}_{i\in\mathcal{I}}$, and a distinguished trivialization $\tilde{\tau}$, in the above described manner. But it follows from the definition of g^f that the maps $U_i \times GL(n,\mathbb{C}) \to U_i \times GL(n,\mathbb{C})\colon (x,A) \mapsto (x, f_i(x)A)$ for all $i \in \mathcal{I}$ fit together

$$\begin{array}{ccccc}
(x,A) \in & U_{ij} \times GL(n,\mathbb{C}) & \longrightarrow & U_{ij} \times GL(n,\mathbb{C}) & \ni (x, f_j(x)A) \\
\downarrow & \uparrow{\scriptstyle \tilde{\tau}_j} & & \uparrow{\scriptstyle \tau_j} & \downarrow \\
[(x,A)] \in & \tilde{p}^{-1}(U_{ij}) & \xrightarrow{\tilde{f}} & p^{-1}(U_{ij}) & \ni [(x, f_j(x)A)] \\
\downarrow & \downarrow{\scriptstyle \tilde{\tau}_i} & & \downarrow{\scriptstyle \tau_i} & \downarrow \\
(x, (g^f)_{ij}(x)A) \in & U_{ij} \times GL(n,\mathbb{C}) & \longrightarrow & U_{ij} \times GL(n,\mathbb{C}) & \ni (x, g_{ij}(x)f_j(x)A) = (x, f_i(x)(g^f)_{ij}(x)A)
\end{array}$$

to yield an isomorphism of complex manifolds $\tilde{f}\colon E(g^f) \to E(g)$ which is also an isomorphism of principal bundles, i.e. $p \circ \tilde{f} = \tilde{p}$. Since $f_i(x) = 1$ for all $x \in U_i$ and all $i \in \mathcal{I}'$, this isomorphism has the additional property that it is the identity in the trivializations $(\tilde{\tau}, \tau)$. Thus the functor $\mathcal{E}$ associates to the arrow $f\colon g^f \to g$ in the groupoid Z the arrow $\tilde{f}\colon E(g^f) \to E(g)$ in the groupoid B. This association is functorial.

Now we will define the functor $\mathcal{T}\colon B \to Z$. Let $(\rho\colon E \to X, \mathcal{B}, \sigma)$ be an object in the groupoid B. Then the overlap $\sigma_j^{-1} \circ \sigma_i$ between two local trivializations will necessarily be expressible in terms of transition functions $g = \{g_{ij}\}_{(i,j)\in\mathcal{I}^2}$. A family of transition functions will necessarily satisfy the conditions defining a 1-cocycle. But since each of the trivializations σ_i for $i \in \mathcal{I}'$ is a restriction of the distinguished trivialization σ, we have $g_{ij} = 1$ whenever $i, j \in \mathcal{I}'$, and thus $g \in Z^1(\mathcal{U},\mathcal{U}',\mathcal{G})$. This g is the object in the groupoid Z that the functor $\mathcal{T}$ associates to $(\rho\colon E \to X, \mathcal{B}, \sigma)$.

Now suppose $(\rho_2\colon E_2 \to X, \mathcal{B}_2 = \{\sigma_{2i}\}_{i\in\mathcal{I}}, \sigma_2)$ and $(\rho_1\colon E_1 \to X, \mathcal{B}_1 = \{\sigma_{1i}\}_{i\in\mathcal{I}}, \sigma_1)$ are objects in the groupoid B and $\hat{f}\colon E_2 \to E_1$ is an arrow between them in that same goupoid. Then for every $i \in \mathcal{I}$ there is an holomorphic mapping $f_i\colon U_i \to GL(n,\mathbb{C})$ such that $[\sigma_{1i} \circ \hat{f} \circ \sigma_{2i}^{-1}](x,A) = (x, f_i(x)A)$ for every $(x,A) \in U_i \times GL(n,\mathbb{C})$. Thus $f = \{f_i\}_{i\in\mathcal{I}} \in C^0(\mathcal{U},\mathcal{G})$. In the distinguished trivializations we have $[\sigma_1 \circ \hat{f} \circ \sigma_2^{-1}](x,A) = (x,A)$ for every $(x,A) \in U_i \times GL(n,\mathbb{C})$. Since for all $i \in \mathcal{I}'$ we have that σ_{ki} is a restriction of σ_k, $k = 1,2$, it follows that $f_i = 1$ for such i. Therefore $f \in C^0(\mathcal{U},\mathcal{U}',\mathcal{G})$. If $g_1 = \{g_{1ij}\}_{(i,j)\in\mathcal{I}^2}, g_2 = \{g_{2ij}\}_{(i,j)\in\mathcal{I}^2} \in Z^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ denote the families of transition functions of the bundles ρ_1, ρ_2 respectively, then we have that $f_i(x)g_{2ij}(x) = g_{1ij}(x)f_j(x)$ for all $x \in U_{ij}$. Thus $g_2 = g_1^f$. So (g_2, f, g_1) is the arrow in the groupoid Z which the functor $\mathcal{T}$ associates to $\hat{f}$. This association is functorial.

It is clear that $\mathcal{T}\mathcal{E}$ is equal to the identity functor on Z. We will now describe a natural isomorphism ϕ from $\mathcal{E}\mathcal{T}$ to the identity functor on B. To each object $(\rho\colon E \to X, \mathcal{B}, \sigma)$ of the groupoid B we define the isomorphism $\phi_{(\rho,\mathcal{B},\sigma)}\colon E(g) \to E$, where $g = T(\rho,\mathcal{B},\sigma)$, as follows. Since the bundles $(\rho\colon E \to X, \mathcal{B})$ and $(p\colon E(g) \to X, \mathcal{A})$ have the same transition functions, namely g, we define $\phi_{(\rho,\mathcal{B},\sigma)}$ so that for all $i \in \mathcal{I}$ its local presentation in the trivializations (τ_i, σ_i) is the identity map. Clearly we have $\phi \circ \rho = p$. Since for all $i \in \mathcal{I}'$ the trivializations τ_i, σ_i are restrictions of the distinguished trivializations τ, σ respectively, it follows that the local presentation of $\phi_{(\rho,\mathcal{B},\sigma)}$ in the distinguished trivializations (τ, σ) is also the identity map. So $\phi_{(\rho,\mathcal{B},\sigma)}$ determines an isomorphism in B. To see that ϕ is a *natural* isomorphism suppose that $(\rho_2\colon E_2 \to X, \mathcal{B}_2, \sigma_2)$ and $(\rho_1\colon E_1 \to X, \mathcal{B}_1, \sigma_1)$ are objects in the groupoid B and $\hat{f}\colon E_2 \to E_1$ is an arrow between them. Let $f \in C^0(\mathcal{U},\mathcal{U}',\mathcal{G})$ be such that $\mathcal{T}$ associates (g_2, f, g_1) to $\hat{f}$. Let $\tilde{f}\colon E(g_2) \to E(g_1)$ be the arrow that $\mathcal{E}$ associates to (g_2, f, g_1). It follows immediately that $\phi_1 \circ \hat{f} = \tilde{f} \circ \phi_2$, where $\phi_k = \phi_{(\rho_k,\mathcal{B}_k,\sigma_k)}$, $k = 1,2$. Thus we have proved the following.

Fact. *Suppose X is a complex manifold, $S \subset X$ is a closed subset, and $\mathcal{G} = GL(n, \mathcal{O}_X)$. Suppose $\mathcal{I}' \subset \mathcal{I}$ are index sets, $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$, $\mathcal{U}' = \{U_i\}_{i\in\mathcal{I}'}$, and $(\mathcal{U},\mathcal{U}')$ is an open covering of $(X, X \setminus S)$. Then the groupoids $Z(\mathcal{U},\mathcal{U}')$ and $B(\mathcal{U},\mathcal{U}')$ are equivalent as categories.*

In Definition 5 we described $H^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ as the pointed set of all orbits in $Z^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ under the right action of $C^0(\mathcal{U},\mathcal{U}',\mathcal{G})$. Another way of saying this is that $H^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ is the set of all isomorphism classes of objects in the groupoid $Z(\mathcal{U},\mathcal{U}')$ (in any category isomorphism is an equivalence relation). Since the categories $Z(\mathcal{U},\mathcal{U}')$ and $B(\mathcal{U},\mathcal{U}')$ are equivalent there is a one-to-one correspondence between the sets of their isomorphism classes. Therefore we have the following.

Corollary. *There is a one-to-one correspondence between $H^1(\mathcal{U},\mathcal{U}',\mathcal{G})$ and the pointed set of isomorphism classes in the groupoid $B(\mathcal{U},\mathcal{U}')$.*

§2 Inductive Limits and Bundle Interpretations. Now suppose $\mathcal{J}$ is another index set and $\mathcal{V} = \{V_j\}_{j\in\mathcal{J}}$ is another open covering of X.

Definition. We say a map $\alpha\colon \mathcal{J} \to \mathcal{I}$ is a *refinement inclusion from* $\mathcal{U}$ *to* $\mathcal{V}$ if for every $j \in \mathcal{J}$ we have $V_j \subset U_{\alpha(j)}$. If there exists a refinement inclusion from $\mathcal{U}$ to $\mathcal{V}$ then we say the covering $\mathcal{V}$ is a *refinement* of the covering $\mathcal{U}$.

Thus we obtain the category: Open coverings of X/refinement inclusions.

Definition. If $\mathcal{J}'$ is a subset of $\mathcal{J}$ such that $\mathcal{V}' = \{V_j\}_{j\in\mathcal{J}'}$ is an open covering of $X \setminus S$, then a map $\alpha\colon \mathcal{J} \to \mathcal{I}$ is said to be a *refinement inclusion from* $(\mathcal{U},\mathcal{U}')$ *to* $(\mathcal{V},\mathcal{V}')$ if α is a refinement inclusion from $\mathcal{U}$ to $\mathcal{V}$ and $\alpha(\mathcal{J}') \subset \mathcal{I}'$. If there exists a refinement inclusion from $(\mathcal{U},\mathcal{U}')$ to $(\mathcal{V},\mathcal{V}')$, then we say $(\mathcal{V},\mathcal{V}')$ is a *refinement* of $(\mathcal{U},\mathcal{U}')$.

Thus we obtain the category: Open coverings of $(X, X \setminus S)$/refinement inclusions. Suppose again that $\mathcal{G}$ is any sheaf of groups on X. To reduce writing we will suppress from our notation the dependence on the sheaf $\mathcal{G}$, which will remain fixed in our discussion.

Fact.

(1) *C^0 is a functor*

$$\frac{\text{Open coverings of } X}{\text{refinement inclusions}} \to \frac{\text{Groups}}{\text{group homomorphisms}}$$

provided whenever $\alpha\colon \mathcal{J} \to \mathcal{I}$ is a refinement inclusion from $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ to $\mathcal{V} = \{V_j\}_{j\in\mathcal{J}}$ and for all $f \in C^0\mathcal{U}$ and for all $j \in \mathcal{J}$ we define

$$(C^0\alpha)(f)_j = \mathcal{G}(U_{\alpha(j)} \supset V_j)(f_{\alpha(j)}).$$

The exact same rule causes C^0 to be a functor

$$\frac{\text{Open coverings of } (X, X \setminus S)}{\text{refinement inclusions}} \to \frac{\text{Groups}}{\text{group homomorphisms}}.$$

(2) *Z^1 is a functor*

$$\frac{\text{Open coverings of } X}{\text{refinement inclusions}} \to \frac{\text{Pointed sets}}{\text{point preserving maps}}$$

provided whenever $\alpha\colon \mathcal{J} \to \mathcal{I}$ is a refinement inclusion from $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ to $\mathcal{V} = \{V_j\}_{j\in\mathcal{J}}$ and for all $g \in Z^1\mathcal{U}$ and for all $(j,k) \in \mathcal{J}^2$ we define

$$(Z^1\alpha)(g)_{jk} = \mathcal{G}(U_{\alpha(j)\alpha(k)} \supset V_{jk})(g_{\alpha(j)\alpha(k)}).$$

The exact same rule causes Z^1 to be a functor

$$\frac{\text{Open coverings of } (X, X \setminus S)}{\text{refinement inclusions}} \to \frac{\text{Pointed sets}}{\text{point preserving maps}}.$$

(3) *Suppose α is a refinement inclusion from $\mathcal{U}$ to $\mathcal{V}$, $g \in Z^1\mathcal{U}$ and $f \in C^0\mathcal{U}$. Then*

$$(Z^1\alpha)(g^f) = (Z^1\alpha)(g)^{(C^0\alpha)(f)}.$$

In particular this relationship holds when $g \in Z^1(\mathcal{U},\mathcal{U}')$ and $f \in C^0(\mathcal{U},\mathcal{U}')$.

Proof. Simple exercise. □

Remark. If we make use of the groupoid $Z(\mathcal{U},\mathcal{U}')$ that we defined in the previous section then we can give a reinterpretation of the above fact. The association $(\mathcal{U},\mathcal{U}')\mapsto Z(\mathcal{U},\mathcal{U}')$ is the object map of a functor

$$Z\colon \frac{\text{Open coverings of } (X, X\setminus S)}{\text{refinement inclusions}} \to \frac{\text{Groupoids}}{\text{groupoid homomorphisms}}.$$

If α is a refinement inclusion from $(\mathcal{U},\mathcal{U}')$ to $(\mathcal{V},\mathcal{V}')$ then define $Z\alpha\colon Z(\mathcal{U},\mathcal{U}')\to Z(\mathcal{V},\mathcal{V}')$ to be the groupoid homomorphism, taking the arrow (g^f, f, g) into $((Z^1\alpha)(g)^{(C^0\alpha)(f)}, (C^0\alpha)(f), (Z^1\alpha)(g))$. Thus the content of (3) above is that this association is indeed a groupoid homomorphism.

Proposition. *Adopt the notation of the previous Fact.*

(1) *The map $Z^1\alpha\colon Z^1\mathcal{U}\to Z^1\mathcal{V}$ takes orbits to orbits, and hence induces a point-preserving map, denoted by $H^1\alpha$, from $H^1\mathcal{U}$ to $H^1\mathcal{V}$. Thus H^1 is a functor*

$$\frac{\text{Open coverings of } X}{\text{refinement inclusions}} \to \frac{\text{Pointed sets}}{\text{point preserving maps}}$$

In exactly the same way, H^1 is a functor

$$\frac{\text{Open coverings of } (X, X\setminus S)}{\text{refinement inclusions}} \to \frac{\text{Pointed sets}}{\text{point preserving maps}}.$$

(2) *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then the map $H^1\alpha$ is injective and independent of the refinement inclusion α from $\mathcal{U}$ to $\mathcal{V}$. Similarly, if $(\mathcal{V},\mathcal{V}')$ is a refinement of $(\mathcal{U},\mathcal{U}')$, then the map $H^1\alpha$ is injective and independent of the refinement inclusion α from $(\mathcal{U},\mathcal{U}')$ to $(\mathcal{V},\mathcal{V}')$.*

Proof. (1) is obvious from part (3) of the above Fact. We will prove (2) in the relative case, following the proof of the usual case in Babbitt Varadarajan [], page 111. First we will show that the map $H^1\alpha$ is independent of the refinement inclusion α. Suppose $\beta\colon \mathcal{J}\to\mathcal{I}$ is another refinement inclusion from $(\mathcal{U},\mathcal{U}')$ to $(\mathcal{V},\mathcal{V}')$, i.e. $V_j\subset U_{\beta(j)}$ for all $j\in\mathcal{J}$ and $\beta(\mathcal{J}')\subset\mathcal{I}'$. Suppose $g\in Z^1(\mathcal{U},\mathcal{U}')$. We must show that the orbit of $(Z^1\alpha)(g)$ and the orbit of $(Z^1\beta)(g)$ under the action of $C^0(\mathcal{V},\mathcal{V}')$ coincide. This will happen if and only if there is an $f\in C^0(\mathcal{V},\mathcal{V}')$ such that $(Z^1\alpha)(g)^f = (Z^1\beta)(g)$. For all $j\in\mathcal{J}$ define $f_j = \mathcal{G}(U_{\alpha(j)\beta(j)}\supset V_j)(g_{\alpha(j)\beta(j)})$. Now compute:

$$\begin{aligned}
[(Z^1\alpha)(g)^f]_{jk} &= \mathcal{G}(V_j\supset V_{jk})(f_j)^{-1}\cdot(Z^1\alpha)(g)_{jk}\cdot\mathcal{G}(V_k\supset V_{jk})(f_k)\\
&= \mathcal{G}(U_{\alpha(j)\beta(j)}\supset V_{jk})(g_{\alpha(j)\beta(j)})^{-1}\cdot\mathcal{G}(U_{\alpha(j)\alpha(k)}\supset V_{jk})(g_{\alpha(j)\alpha(k)})\cdot\mathcal{G}(U_{\alpha(k)\beta(k)}\supset V_{jk})(g_{\alpha(k)\beta(k)})\\
&= \mathcal{G}(U_{\beta(j)\beta(k)}\supset V_{jk})(g_{\beta(j)\beta(k)})\\
&= (Z^1\beta)(g)_{jk}.
\end{aligned}$$

Furthermore, if $j\in\mathcal{J}'$ then $\alpha(j)\in\mathcal{I}'$ and $\beta(j)\in\mathcal{I}'$ and therefore $g_{\alpha(j)\beta(j)}=1$. Thus $f\in C^0(\mathcal{V},\mathcal{V}')$.

Now we will show that the map $H^1\alpha\colon H^1(\mathcal{U},\mathcal{U}')\to H^1(\mathcal{V},\mathcal{V}')$ is injective. So suppose $g,\tilde g\in Z^1(\mathcal{U},\mathcal{U}')$ are such that the image under $H^1\alpha$ of the orbit of g and of the orbit of $\tilde g$ coincide, i.e. there is an $f\in C^0(\mathcal{V},\mathcal{V}')$ such that $(Z^1\alpha)(\tilde g) = (Z^1\alpha)(g)^f$. We want to find an $h\in C^0(\mathcal{U},\mathcal{U}')$ such that $\tilde g = g^h$ and $f = (C^0\alpha)(h)$. Let $i\in\mathcal{I}$ and $j\in\mathcal{J}$ and define $W_{i,j} = U_i\cap V_j$ and

$$h_{i,j} = \mathcal{G}(U_{i\alpha(j)}\supset W_{i,j})(g_{i\alpha(j)})\cdot\mathcal{G}(V_j\supset W_{i,j})(f_j)\cdot\mathcal{G}(U_{\alpha(j)i}\supset W_{i,j})(\tilde g_{\alpha(j)i}).$$

If $k\in\mathcal{J}$ then define $W_{i,jk} = U_i\cap V_{jk}$. First we will show that $\mathcal{G}(W_{i,k}\supset W_{i,jk})(h_{i,k}) = \mathcal{G}(W_{i,j}\supset W_{i,jk})(h_{i,j})$. If we restrict the known relation $(Z^1\alpha)(\tilde g) = (Z^1\alpha)(g)^f$ to $W_{i,jk}$ and use the cocycle conditions we get

$$\begin{aligned}
\mathcal{G}(W_{i,k}\supset W_{i,jk})(h_{i,k}) &= \mathcal{G}(U_{i\alpha(k)}\supset W_{i,jk})(g_{i\alpha(k)})\cdot\mathcal{G}(V_j\supset W_{i,jk})(f_k)\cdot\mathcal{G}(U_{\alpha(k)i}\supset W_{i,jk})(\tilde g_{\alpha(k)i})\\
&= \mathcal{G}(U_{i\alpha(k)}\supset W_{i,jk})(g_{i\alpha(k)})\cdot\mathcal{G}(U_{\alpha(k)\alpha(j)}\supset W_{i,jk})(g_{\alpha(k)\alpha(j)})\cdot\mathcal{G}(V_j\supset W_{i,jk})(f_j)\\
&\qquad\cdot\mathcal{G}(U_{\alpha(j)\alpha(k)}\supset W_{i,jk})(\tilde g_{\alpha(j)\alpha(k)})\cdot\mathcal{G}(U_{\alpha(k)i}\supset W_{i,jk})(\tilde g_{\alpha(k)i})\\
&= \mathcal{G}(U_{i\alpha(j)}\supset W_{i,jk})(g_{i\alpha(j)})\cdot\mathcal{G}(V_j\supset W_{i,jk})(f_j)\cdot\mathcal{G}(U_{\alpha(j)i}\supset W_{i,jk})(\tilde g_{\alpha(j)i})\\
&= \mathcal{G}(W_{i,j}\supset W_{i,jk})(h_{i,j}).
\end{aligned}$$

Since $U_i = \cup_{j\in\mathcal{J}} W_{i,j}$ we have by the collation condition for the sheaf $\mathcal{G}$ that there exists $h_i \in \mathcal{G}(U_i)$ such that $\mathcal{G}(U_i \supset W_{i,j})(h_i) = h_{i,j}$ for all $j \in \mathcal{J}$. Note also that if $i \in \mathcal{I}'$ then $U_i = \cup_{j\in\mathcal{J}'} W_{i,j}$, $h_{i,j} = 1$ for all $j \in \mathcal{J}'$ (since $\alpha(j) \in \mathcal{I}'$), and thus $h_i = 1$. Therefore $h \in C^0(\mathcal{U}, \mathcal{U}')$. To show that $f = (C^0\alpha)(h)$ note that for all $j \in \mathcal{J}'$ we have $V_j \subset U_{\alpha(j)}$ and hence $V_j = W_{\alpha(j),j}$. Thus for all $j \in \mathcal{J}$ we have

$$(C^0\alpha)(h)_j = \mathcal{G}(U_{\alpha(j)} \supset V_j)(h_{\alpha(j)}) = \mathcal{G}(U_{\alpha(j)} \supset W_{\alpha(j),j})(h_{\alpha(j)}) = h_{\alpha(j),j} = f_j.$$

Finally it only remains to show that $\tilde{g} = g^h$, i.e. for all $(i,l) \in \mathcal{I}^2$ we have $\mathcal{G}(U_i \supset U_{il})(h_i) \cdot \tilde{g}_{il} = g_{il} \cdot \mathcal{G}(U_l \supset U_{il})(h_l)$. Let $(i,l) \in \mathcal{I}^2$ be given. Since $U_{il} = \cup_{j\in\mathcal{J}} W_{il,j}$, where $W_{il,j} = U_{il} \cap V_j$, it suffices by the collation property to show that for all $j \in \mathcal{J}$ the above equality holds when restricted to $W_{il,j}$.

$$\begin{aligned}
&\mathcal{G}(U_i \supset W_{il,j})(h_i) \cdot \mathcal{G}(U_{il} \supset W_{il,j})(\tilde{g}_{il}) \\
&\quad = \mathcal{G}(W_{i,j} \supset W_{il,j})(h_{i,j}) \cdot \mathcal{G}(U_{il} \supset W_{il,j})(\tilde{g}_{il}) \\
&\quad = \mathcal{G}(U_{i\alpha(j)} \supset W_{il,j})(g_{i\alpha(j)}) \cdot \mathcal{G}(V_j \supset W_{il,j})(f_j) \cdot \mathcal{G}(U_{\alpha(j)i} \supset W_{il,j})(\tilde{g}_{\alpha(j)i}) \cdot \mathcal{G}(U_{il} \supset W_{il,j})(\tilde{g}_{il}) \\
&\quad = \mathcal{G}(U_{il} \supset W_{il,j})(g_{il}) \cdot \mathcal{G}(U_{l\alpha(j)} \supset W_{il,j})(g_{l\alpha(j)}) \cdot \mathcal{G}(V_j \supset W_{il,j})(f_j) \cdot \mathcal{G}(U_{\alpha(j)l} \supset W_{il,j})(\tilde{g}_{\alpha(j)l}) \\
&\quad = \mathcal{G}(U_{il} \supset W_{il,j})(g_{il}) \cdot \mathcal{G}(W_{l,j} \supset W_{il,j})(h_{l,j}) \\
&\quad = \mathcal{G}(U_{il} \supset W_{il,j})(g_{il}) \cdot \mathcal{G}(U_l \supset W_{il,j})(h_l). \quad \square
\end{aligned}$$

Definition 6. We define $H^1(X, \mathcal{G})$ to be the inductive limit of the functor $H^1(\cdot, \mathcal{G})$. More concretely,

$$H^1(X, \mathcal{G}) = \coprod_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G}) \Big/ \sim,$$

where the disjoint union is over all open coverings $\mathcal{U}$ of X, and $h \sim \tilde{h}$, $h \in H^1(\mathcal{U}, \mathcal{G})$, $\tilde{h} \in H^1(\tilde{\mathcal{U}}, \mathcal{G})$ if and only if there exists an open covering $\mathcal{V}$ which is a refinement of both $\mathcal{U}$ and $\tilde{\mathcal{U}}$ such that $H^1(\alpha, \mathcal{G})(h) = H^1(\tilde{\alpha}, \mathcal{G})(\tilde{h})$, where α (resp. $\tilde{\alpha}$) is any refinement inclusion from $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) to $\mathcal{V}$. Using the exactly analogous construction, we define $H^1(X, X \setminus S, \mathcal{G})$ to be the inductive limit of the functor $H^1(\cdot, \cdot, \mathcal{G})$.

Fact. *Suppose X is a complex manifold, $\mathcal{G} = GL(n, \mathcal{O}_X)$, $\mathcal{U}$ is an open covering of X, and $g \in Z^1(\mathcal{U}, \mathcal{G})$. Suppose also that $\mathcal{V}$ is a refinement of $\mathcal{U}$, $\alpha\colon \mathcal{J} \to \mathcal{I}$ is a refinement inclusion from $\mathcal{U}$ to $\mathcal{V}$, and let $h = Z^1(\alpha, \mathcal{G})(g)$. Then the assignments $(x, A) \in V_j \times GL(n, \mathbb{C}) \mapsto (x, A) \in U_{\alpha(j)} \times GL(n, \mathbb{C})$, for $j \in \mathcal{J}$, fit together to form an isomorphism $\phi_\alpha\colon E(\mathcal{V}, h) \to E(\mathcal{U}, g)$ of principal bundles over X.*

Thus, in the situation of the previous fact, we see that an element of $H^1(X, \mathcal{G})$ has a similar geometric interpretation as elements of $H^1(\mathcal{U}, \mathcal{G})$, except now a broader type of isomorphisms of principal bundles is allowed. If $\rho\colon E \to X$ is any principal fibre bundle over X with structure group $GL(n, \mathbb{C})$, then any atlas of local trivializations of this bundle will be over some open covering $\mathcal{U}$ of X, and the family of transition functions of this atlas determines a 1-cocycle $g \in Z^1(\mathcal{U}, \mathcal{G})$. It is a well-known geometric fact that there is an isomorphism $\phi\colon E \to E(\mathcal{U}, g)$ of principal fibre bundles, i.e. $p \circ \phi = \rho$. So up to such an isomorphism, every principal fibre bundle over X with structure group $GL(n, \mathbb{C})$ is accounted for in $H^1(X, \mathcal{G})$. Finally, suppose $p\colon E(\mathcal{U}, g) \to X$ and $\tilde{p}\colon E(\tilde{\mathcal{U}}, \tilde{g}) \to X$ are two isomorphic principal fibre bundles with structure group $GL(n, \mathbb{C})$. Define $\mathcal{J} = \mathcal{I} \times \tilde{\mathcal{I}}$ and $\mathcal{V} = \{V_{(i,\tilde{i})} = U_i \cap \tilde{U}_{\tilde{i}}\}_{(i,\tilde{i})\in\mathcal{J}}$. Define α (resp. $\tilde{a}$) to be the refinement inclusion from $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) to $\mathcal{V}$ given by the mapping $\mathcal{J} \to \mathcal{I}\colon (i, \tilde{i}) \mapsto i$ (resp. $\mathcal{J} \to \tilde{\mathcal{I}}\colon (i, \tilde{i}) \mapsto \tilde{i}$). Define $h = Z^1(\alpha, \mathcal{G})(g)$ and $\tilde{h} = Z^1(\tilde{\alpha}, \mathcal{G})(g)$. By the previous Fact we have isomorphisms $\phi_\alpha\colon E(\mathcal{V}, h) \to E(\mathcal{U}, g)$ and $\phi_{\tilde{\alpha}}\colon E(\mathcal{V}, \tilde{h}) \to E(\tilde{\mathcal{U}}, \tilde{g})$ of principal fibre bundles. Thus we have that $p\colon E(\mathcal{V}, h) \to X$ and $\tilde{p}\colon E(\mathcal{V}, \tilde{h}) \to X$ are two isomorphic principal fibre bundles. Thus there exists $f \in C^0(\mathcal{V}, \mathcal{G})$ such that $\tilde{h} = h^f$, i.e. h and $\tilde{h}$ determine the same element of $H^1(\mathcal{V}, \mathcal{G})$. Thus we have proved the following.

Fact. *Suppose X is a complex manifold, and $\mathcal{G} = GL(n, \mathcal{O}_X)$. Then there is a point-preserving one-to-one correspondence between $H^1(X, \mathcal{G})$ and the set of isomorphism classes in the category Holomorphic principal fibre bundles over X with structure group $GL(n, \mathbb{C})$/bundle maps.*

Suppose S is a closed subset of X as before. A principal fibre bundle over X is said to be *trivial over* $X \setminus S$ if it possesses an atlas of local trivializations containing a trivialization over $X \setminus S$. We will define a new category called, for want of a better name, Hyperbundles on X, supported on S/hyperbundle maps. The objects of this category will be pairs $(p\colon E \to X, \tau)$, where $p\colon E \to X$ is a holomorphic principal fibre bundle on X with structure group $GL(n, \mathbb{C})$ and $\tau\colon p^{-1}(X \setminus S) \to (X \setminus S) \times GL(n, \mathbb{C})$ is a distinguished local trivialization contained in the atlas of the bundle p. Hyperbundle maps will be ordinary bundle maps which are the identity over $X \setminus S$, as computed using the distinguished local trivializations.

Fact. *Suppose X is a complex manifold, S is a closed subset of X, and $\mathcal{G} = GL(n, \mathcal{O}_X)$. Then there is a point-preserving one-to-one correspondence between $H^1(X, X \setminus S, \mathcal{G})$ and the set of isomorphism classes in the category Hyperbundles on X, supported on S/hyperbundle maps.*

3 "Long Exact Sequence" of Non-Abelian Relative Cohomology. Now suppose $(\mathcal{U}, \mathcal{U}')$ is an open covering of $(X, X \setminus S)$, and consider the following "commutative diagram":

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^0(\mathcal{U},\mathcal{U}',\mathcal{G}) & \xrightarrow{\alpha_1} & H^0(\mathcal{U},\mathcal{G}) & \xrightarrow{\alpha_2} & H^0(\mathcal{U}',\mathcal{G}) & & \\
 & & {\scriptstyle a_1}\downarrow & & {\scriptstyle a_2}\downarrow & & {\scriptstyle a_3}\downarrow & & \\
1 & \longrightarrow & C^0(\mathcal{U},\mathcal{U}',\mathcal{G}) & \xrightarrow{\beta_1} & C^0(\mathcal{U},\mathcal{G}) & \xrightarrow{\beta_2} & C^0(\mathcal{U}',\mathcal{G}) & \longrightarrow & 1 \\
 & & {\scriptstyle b_1}\downarrow & & {\scriptstyle b_2}\downarrow & & {\scriptstyle b_3}\downarrow & & \\
1 & \longrightarrow & Z^1(\mathcal{U},\mathcal{U}',\mathcal{G}) & \xrightarrow{\gamma_1} & Z^1(\mathcal{U},\mathcal{G}) & \xrightarrow{\gamma_2} & Z^1(\mathcal{U}',\mathcal{G}) & & \\
 & & {\scriptstyle c_1}\downarrow & & {\scriptstyle c_2}\downarrow & & {\scriptstyle c_3}\downarrow & & \\
 & & H^1(\mathcal{U},\mathcal{U}',\mathcal{G}) & \xrightarrow{\delta_1} & H^1(\mathcal{U},\mathcal{G}) & \xrightarrow{\delta_2} & H^1(\mathcal{U}',\mathcal{G}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

All the maps above and including the βs and the 0-cochains are group homomorphisms. All the maps below that point are point-preserving maps. The a_is are subgroup inclusions. β_1 is a normal subgroup inclusion. β_2 retains only the portion of the 0-cochain on $\mathcal{U}'$. The α_is are just the restrictions of the β_is. All the sequences of group homomorphisms are exact at every interior node, and the two rectangles formed by group homomorphisms commute.

The b_is map 0-cochains f to 1^f, so that the vertical sequences are "exact" at the 0-cochain nodes, i.e. the image of a_i is the inverse image under b_i of 1. γ_1 is an inclusion. It has the property that for all $g \in Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ and for all $f \in C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$, $\gamma_1(g^f) = \gamma_1(g)^{\beta_1(f)}$. γ_2 retains only that portion of a 1-cocyle which pertains only to the covering $\mathcal{U}'$. It has the property that for all $g \in Z^1(\mathcal{U}, \mathcal{G})$ and for all $f \in C^0(\mathcal{U}, \mathcal{G})$, $\gamma_2(g^f) = \gamma_2(g)^{\beta_2(f)}$. If X is a Riemann surface then γ_2 is surjective, although we will not need to use this fact. In general the horizontal sequence of the γ_is and the 1-cocycles is "exact" at the two interior nodes. The two rectangles between the β_is and the γ_is commute.

The maps c_i map a 1-cocycle into its orbit under the action of the 0-cochains. The c_is are surjective by definition. The image of b_i is the inverse image under c_i of 1. Consequently, the vertical sequences are "exact" at every interior node. Because of the above described compatibility of the γ_is with the group actions, the γ_is map orbits into orbits, and hence induce the maps δ_1, δ_2 between orbit spaces, such that the two rectangles between the γ_is and the δ_is commute. $\delta_2 \circ \delta_1$ is trivial. A "diagram chase" verifies our geometric intuition by proving that the image of δ_1 coincides with the inverse image under δ_2 of 1.

Theorem. *Suppose $f \in H^0(\mathcal{U}', \mathcal{G})$ and $\phi, \psi \in \beta_2^{-1}(f)$ (always nonempty). Then $1^\phi, 1^\psi \in Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$, $\phi^{-1}\psi \in C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$, and $1^\psi = (1^\phi)^{\phi^{-1}\psi}$. Thus there is a well-defined map*

$$K\colon H^0(\mathcal{U}', \mathcal{G}) \to H^1(\mathcal{U}, \mathcal{U}', \mathcal{G})\colon f \mapsto c_1(1^\phi), \qquad \textit{where } \beta_2(\phi) = f.$$

This map has the following properties.

(1) *The image of α_2 is equal to the inverse image under K of 1. More generally, the inverse image of every point in the image of K is a right coset of the image of α_2 in $H^0(\mathcal{U}', \mathcal{G})$.*
(2) *The image of K in $H^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ is equal to the inverse image under δ_1 of 1.*

Proof. (1) Let G be the image of α_2 in $H^0(\mathcal{U}', \mathcal{G})$. Let $f \in H^0(\mathcal{U}', \mathcal{G})$ and $g = K(f)$. We want to show that $Gf = K^{-1}(g)$. First we claim: $K(Gf) = \{g\}$. Let $h \in H^0(\mathcal{U}, \mathcal{G})$; we will show $K(\alpha_2(h)f) = g$. If $\phi \in C^0(\mathcal{U}, \mathcal{G})$ is such that $\beta_2(\phi) = f$, then $\beta_2(h\phi) = \beta_2(h)\beta_2(\phi) = \alpha_2(h)f$. So $K(\alpha_2(h)f) = c_1(1^{h\phi}) = c_1((1^h)^\phi) = c_1(1^\phi) = g$.

Now suppose $f_1 \in H^0(\mathcal{U}', \mathcal{G})$ and $g = K(f_1)$. We claim that $f_1 \in Gf$. Let $\phi_1 \in C^0(\mathcal{U}, \mathcal{G})$ be such that $\beta_2(\phi_1) = f_1$. Since $c_1(1^\phi) = c_1(1^{\phi_1})$, there exists $\psi \in C^0(\mathcal{U}, \mathcal{U}', \mathcal{G})$ such that $1^{\phi_1} = (1^\phi)^\psi$. So $1^{\phi_1\psi^{-1}\phi^{-1}} = 1$, i.e. $\phi_1\psi^{-1}\phi^{-1} \in H^0(\mathcal{U}, \mathcal{G})$. Also $\alpha_2(\phi_1\psi^{-1}\phi^{-1}) = \beta_2(\phi_1\psi^{-1}\phi^{-1}) = \beta_2(\phi_1)\beta_2(\psi^{-1})\beta_2(\phi^{-1}) = f_1 \cdot 1 \cdot f^{-1} \in G$.

(2) First we show $\delta_1(K(f)) = 1$ for all $f \in H^0(\mathcal{U}', \mathcal{G})$. Let $\phi \in C^0(\mathcal{U}, \mathcal{G})$ be such that $\beta_2(\phi) = f$. Then $\delta_1(c_1(1^\phi)) = c_2(\gamma_1(1^\phi)) = c_2(1^\phi) = 1$. Next suppose $\delta_1(g) = 1$ for some $g \in H^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$. We want to show that $g = K(f)$ for some $f \in H^0(\mathcal{U}', \mathcal{G})$. There exists $h \in Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G})$ such that $g = c_1(h)$. Therefore $c_2(h) = 1$, i.e. there exists $\phi \in C^0(\mathcal{U}, \mathcal{G})$ such that $h = 1^\phi$. But $1^{\beta_2(\phi)} = \gamma_2(1^\phi) = \gamma_2(h) = 1$. Therefore $f = \beta_2(\phi) \in H^0(\mathcal{U}', \mathcal{G})$, and $g = c_1(1^\phi) = K(f)$. □

Corollary. *Suppose X is a Riemann surface, S is a closed subset with empty interior, and $\mathcal{G} = GL(n, \mathcal{O}_X)$. Then the map K defined in the previous Theorem establishes a point-preserving bijection between the pointed set $\mathcal{G}(X)\backslash\mathcal{G}(X \setminus S)$ and the subset of $H^1(X, X \setminus S, \mathcal{G})$ consisting of those hyperbundles which are trivializable over X as ordinary bundles. If X is noncompact, $\mathcal{G}(X)\backslash\mathcal{G}(X \setminus S) \cong H^1(X, X \setminus S, \mathcal{G})$.*

4 Relative Cohomology as a Sheaf. Suppose X is a connected Riemann surface, and $U_1 \supset U_2$ are open subsets. Suppose S is a closed subset of X with empty interior. Suppose $(\mathcal{U}_1, \mathcal{U}_1')$ is a covering of $(U_1, U_1 \setminus S)$. Then intersecting every subset in these coverings with U_2 produces an associated covering $(\mathcal{U}_2, \mathcal{U}_2')$ of $(U_2, U_2 \setminus S)$. Let $\mathcal{G}_{U_i}$ denote the restriction of the sheaf $\mathcal{G}$ to the sets U_i, where $\mathcal{G} = GL(n, \mathcal{O}_X)$. We also get restriction maps

$$\mu\colon C^0(\mathcal{U}_1, \mathcal{U}_1', \mathcal{G}_{U_1}) \to C^0(\mathcal{U}_2, \mathcal{U}_2', \mathcal{G}_{U_2}), \qquad \nu\colon Z^1(\mathcal{U}_1, \mathcal{U}_1', \mathcal{G}_{U_1}) \to Z^1(\mathcal{U}_2, \mathcal{U}_2', \mathcal{G}_{U_2}).$$

Furthermore, if $f \in C^0(\mathcal{U}_1, \mathcal{U}_1', \mathcal{G}_{U_1})$ and $g \in Z^1(\mathcal{U}_1, \mathcal{U}_1', \mathcal{G}_{U_1})$, then $\gamma(g^f) = \gamma(g)^{\beta(f)}$. Thus γ maps orbits to orbits, and hence defines a map

$$\rho\colon H^1(\mathcal{U}_1, \mathcal{U}_1', \mathcal{G}_{U_1}) \to H^1(\mathcal{U}_2, \mathcal{U}_2', \mathcal{G}_{U_2}).$$

In a similar way we get restrictions

$$\lambda\colon H^0(\mathcal{U}_1, \mathcal{G}_{U_1}) \to H^0(\mathcal{U}_2, \mathcal{G}_{U_2}), \qquad \lambda'\colon H^0(\mathcal{U}_1', \mathcal{G}_{U_1 \setminus S}) \to H^0(\mathcal{U}_2', \mathcal{G}_{U_2 \setminus S}).$$

If U_1 is noncompact, then the map K induces a bijection $\mathcal{G}(U_i)\backslash\mathcal{G}(U_i \setminus S) \cong H^1(\mathcal{U}_i, \mathcal{U}_i', \mathcal{G}_{U_1})$, $i = 1, 2$. Via these bijections, the restriction map ρ is induced by the restriction λ' of coset representatives to U_2. Extending this idea to the inductive limits we are led to the following.

Theorem. *$H^1(\cdot, \cdot \setminus S, \mathcal{G}.)$ determines a functor from Open subsets of X/reverse inclusions to Pointed sets/point-preserving maps. Furthermore, this functor is in fact a sheaf of pointed sets.*

Proof. The functoriality is trivial. The collation property follows from a use of the mapping K as follows. Let $U \subset X$ be an open subset, and suppose $\{U_i\}_{i \in \mathcal{I}''}$ is an open covering of U. If any of the sets U_i are compact, then they are closed, and hence are either X or $\emptyset$. The collation property is trivially satisfied for any covering such that $U_i = X$ for some i. Furthermore, if the collation condition is satisfied by every covering of U by nonempty sets then it is satisfied by all coverings. So assume $U_i \neq X$ and is nonempty for all i. Since U_i is a noncompact Riemann surface, we have $H^1(U_i, U_i \setminus S, \mathcal{G}_{U_i}) \cong \mathcal{G}(U_i)\backslash\mathcal{G}(U_i \setminus S)$. So, suppose for each $i \in \mathcal{I}$ we are given $m_i \in \mathcal{G}(U_i \setminus S)$ satisfying the pairwise matching property: for every $(i,j) \in \mathcal{I}^2$ there exists $g_{ij} \in \mathcal{G}(U_i \cap U_j)$ such that $m_i|_{U_i \cap U_j} = g_{ij} m_j|_{U_i \cap U_j}$. If $U_i \cap S = \emptyset$ then we may assume $m_i = 1$.

Suppose $k \notin \mathcal{I}''$ and let $\mathcal{I} = \mathcal{I}'' \cup \{k\}$. Define $U_k = U \setminus S$ and $\mathcal{I}' = \{i \in \mathcal{I}'' \mid U_i \cap S = \emptyset\} \cup \{k\}$. Set $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ and $\mathcal{U}' = \{U_i\}_{i \in \mathcal{I}'}$. Clearly $(\mathcal{U}, \mathcal{U}')$ is an open covering of $(U, U \setminus S)$. Define $m_k = 1$ on U_k. Using the rule $g_{ij} = m_i m_j^{-1}$ on $U_i \cap U_j$ for all $(i,j) \in \mathcal{I}^2$ we can extend g_{ij} to a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{U}', \mathcal{G}|_U)$. $\square$

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