SHARP LARGE-TIME ASYMPTOTICS OF THE CORRELATION INTEGRAL IN THE SUSPENDED FLOW OVER A HYPERBOLIC TORAL AUTOMORPHISM

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Abstract. Sharp large-time asymptotic results are obtained for the correlation integral $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}$, where $\phi_t\colon X \to X$ is the suspended flow over a hyperbolic toral automorphism, X is the compact smooth three dimensional manifold in which this flow lives, and f and g are real-valued functions on X. We also obtain sharp asymptotic results for the analogous integral for the discrete time case. A large variety of asymptotic behaviors are found, depending sensitively on the regularity properties of the functions f and g. These results are obtained via sharp asymptotic estimations of discrete and continuous convolutions.

Contents

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1. Introduction

1.1. **A Purely Chaotic Hamiltonian System.** Let $\mathbb{T}^2 = \mathbb{R}^2 /\!\!/ \mathbb{Z}^2$ be the two dimensional torus with typical element $\mathbf{z} = \binom{q}{p} + \mathbb{Z}^2$. Define $\mathfrak{P} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and $\mathcal{P}\colon \mathbb{T}^2 \to \mathbb{T}^2\colon \binom{q}{p} + \mathbb{Z}^2 \mapsto \mathfrak{P}\binom{q}{p} + \mathbb{Z}^2$. This is a *hyperbolic toral automorphism.*

Consider the four dimensional manifold $\mathbb{T}^2 \times \mathbb{R}^2$, with typical element $(\mathbf{z}, \tau, E)$, and equipped with the symplectic form $\tilde{\omega} = dq \wedge dp + d\tau \wedge dE$. Define the Hamiltonian function $\tilde{H}\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}\colon (\mathbf{z}, \tau, E) \mapsto E$. Hamilton's equations of motion are:

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = 0,\ \dot{p} = -\frac{\partial \tilde{H}}{\partial q} = 0,\ \dot{\tau} = \frac{\partial \tilde{H}}{\partial E} = 1,\ \dot{E} = -\frac{\partial \tilde{H}}{\partial \tau} = 0.$$

This gives rise to the flow $\tilde{\Phi}_t\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2\colon (\mathbf{z}, \tau, E) \mapsto (\mathbf{z}, \tau + t, E)$, $t \in \mathbb{R}$. This flow is symplectic since $\tilde{\Phi}_t^* \tilde{\omega} = \tilde{\omega}$; furthermore it is time-reversible via the mapping $\tilde{\mathcal{I}}\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2\colon (\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right) + \mathbb{Z}^2, \tau, E) \mapsto (\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right) + \mathbb{Z}^2, -\tau, E)$. This means that $\tilde{\mathcal{I}} \circ \tilde{\Phi}_t \circ \tilde{\mathcal{I}} = \tilde{\Phi}_{-t}$, for all $t \in \mathbb{R}$, i.e. $\tilde{\mathcal{I}}$ is a *reversor* [11]. Notice also that $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}^{-1}$ and $\tilde{\mathcal{I}}^* \tilde{\omega} = -\tilde{\omega}$.

Now consider the symplectic diffeomorphism $\mathcal{G}\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2\colon (\mathbf{z}, \tau, E) \mapsto (\mathcal{P}(\mathbf{z}), \tau - 1, E)$. This diffeomorphism satisfies the relations $\mathcal{G}^* \tilde{\omega} = \tilde{\omega}$, $\tilde{H} \circ \mathcal{G} = \tilde{H}$, $\mathcal{G} \circ \tilde{\Phi}_t = \tilde{\Phi}_t \circ \mathcal{G}$, and $\mathcal{G}^{-1} \circ \tilde{\mathcal{I}} = \tilde{\mathcal{I}} \circ \mathcal{G}$. Suppose $\mathcal{G}^0$ is the identity mapping, $\mathcal{G}^1 = \mathcal{G}$, and $\mathcal{G}^{n+m} = \mathcal{G}^n \circ \mathcal{G}^m$ for all $m, n \in \mathbb{Z}$. This defines an action of the group $\mathbb{Z}$ on $\mathbb{T}^2 \times \mathbb{R}^2$ by the rule: $n \cdot (\mathbf{z}, \tau, E) = \mathcal{G}^n(\mathbf{z}, \tau, E)$, $n \in \mathbb{Z}$. Let $\mathcal{Y} = \mathbb{Z} \backslash\!\!\backslash (T^2 \times \mathbb{R}^2)$ be the set of orbits of this group action. Since the group action is fixed point free and proper, the orbit space $\mathcal{Y}$ is a smooth manifold. Furthermore the symplectic form $\tilde{\omega}$ and the mappings $\tilde{H}$, $\tilde{\Phi}_t$ and $\tilde{\mathcal{I}}$ all descend to the manifold $\mathcal{Y}$, where they will be denoted by ω, H, Φ_t, and $\mathcal{I}$ respectively; the identities relating the objects with tildes continue to hold without them. If we consider $\mathbf{z}$ to already constitute a local coordinate description of a point of $\mathbb{T}^2$ then we can cover $\mathcal{Y}$ with two local coordinate domains:

$$V_1 = \{\mathbb{Z} \cdot (\mathbf{z}, \tau, E) \in \mathcal{Y} \mid 0 < \tau < 1\}, \qquad V_2 = \{\mathbb{Z} \cdot (\mathbf{z}, \tau, E) \in \mathcal{Y} \mid -\tfrac{1}{2} < \tau < \tfrac{1}{2}\}.$$

These can be considered as the ranges of two parameterizations:

$$\psi_1\colon \mathbb{T}^2 \times (0,1) \times \mathbb{R} \to V_1, \qquad \psi_2\colon \mathbb{T}^2 \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \mathbb{R} \to V_2,$$

defined by the rule: $\psi_j(\mathbf{z},\tau,E) = \mathbb{Z}\cdot(\mathbf{z},\tau,E)$, $j = 1,2$. The overlap mapping relating these two local coordinate systems on $V_1 \cap V_2$ is as follows:

$$\psi_2^{-1}\circ\psi_1 \colon \mathbb{T}^2 \times [(0,\tfrac{1}{2})\cup(\tfrac{1}{2},1)] \times \mathbb{R} \to \mathbb{T}^2 \times [(-\tfrac{1}{2},0)\cup(0,\tfrac{1}{2})] \times \mathbb{R}$$
$$: (\mathbf{z},\tau,E) \mapsto \begin{cases} (\mathbf{z},\tau,E) & 0<\tau<\frac{1}{2}, \\ (\mathcal{P}(\mathbf{z}),\tau-1,E) & \frac{1}{2}<\tau<1, \end{cases}$$

which is obviously a diffeomorphism. The symplectic volume form on $\mathcal{Y}$ is $-\frac{1}{2}\omega\wedge\omega$, which is $dq\wedge d\tau\wedge dp\wedge dE$ in either of these local coordinate systems.

This Hamiltonian flow can be restricted to any level set of the Hamiltonian function H. Define $X = \{\mathbb{Z}\cdot(\mathbf{z},\tau,E) \in \mathcal{Y} \mid E = 0\}$. This is a compact smooth 3-dimensional manifold, with local coordinates $(\mathbf{z},\tau)$ defined on the coordinate domains $U_j = V_j \cap X$, $j = 1,2$. If $\mathbf{x} \in X$, $v \in T_{\mathbf{x}}\mathcal{Y}$ such that $dH_{\mathbf{x}}(v) \neq 0$, and $v_1, v_2, v_3 \in T_{\mathbf{x}}X$ then

$$\begin{aligned}
(dq\wedge d\tau\wedge dp\wedge dE)(v,v_1,v_2,v_3) &= \det\begin{pmatrix} dq(v) & dq(v_1) & dq(v_2) & dq(v_3)\\ d\tau(v) & d\tau(v_1) & d\tau(v_2) & d\tau(v_3)\\ dp(v) & dp(v_1) & dp(v_2) & dp(v_3)\\ dE(v) & dE(v_1) & dE(v_2) & dE(v_3)\end{pmatrix}\\
&= \det\begin{pmatrix} dq(v) & dq(v_1) & dq(v_2) & dq(v_3)\\ d\tau(v) & d\tau(v_1) & d\tau(v_2) & d\tau(v_3)\\ dp(v) & dp(v_1) & dp(v_2) & dp(v_3)\\ dE(v) & 0 & 0 & 0\end{pmatrix}\\
&= -\,dE(v)\det\begin{pmatrix} dq(v_1) & dq(v_2) & dq(v_3)\\ d\tau(v_1) & d\tau(v_2) & d\tau(v_3)\\ dp(v_1) & dp(v_2) & dp(v_3)\end{pmatrix}\\
&= dE(v)(d\tau\wedge dq\wedge dp)(v_1,v_2,v_3).
\end{aligned}$$

Therefore

$$\frac{(dq\wedge d\tau\wedge dp\wedge dE)(v,v_1,v_2,v_3)}{dE(v)} = (d\tau\wedge dq\wedge dp)(v_1,v_2,v_3).$$

In this way X is naturally equipped with the volume form $d\mathbf{x} = d\tau\wedge dq\wedge dp$, which induces a probability measure on X. Let ϕ_t and $\mathcal{I}$ denote the restriction of the domain and codomain of Φ_t and $\mathcal{I}$ from $\mathcal{Y}$ to X. We have $\phi_t^*(d\tau\wedge dq\wedge dp) = d\tau\wedge dq\wedge dp$ and $\mathcal{I}^*(d\tau\wedge dq\wedge dp) = d\tau\wedge dq\wedge dp$. Also the relation $\mathcal{I}\circ\phi_t\circ\mathcal{I} = \phi_{-t}$ holds, expressing the time-reversible nature of this measure-preserving dynamical system. The Hamiltonian vector field on X is associated with the derivation $\partial/\partial\tau$.

Define $\mathcal{S} = \{\mathbb{Z}\cdot(\mathbf{z},\tau,0) \in X \mid \tau = 0\}$. It is called a *Poincaré section*; it is obviously canonically diffeomorphic to $\mathbb{T}^2$. Flowing for a unit time takes $\mathcal{S}$ into itself, and $\phi_1\colon \mathcal{S}\to\mathcal{S}$ corresponds to $\mathcal{P}\colon \mathbb{T}^2\to\mathbb{T}^2$ via the canonical diffeomorphism $\mathcal{S}\cong\mathbb{T}^2$; hence we may think of $\mathcal{P}$ as the *Poincaré return mapping.* $\{\phi_t\}$ is called the *suspended flow over* the hyperbolic toral automorphism $\mathcal{P}$. The *return time function* on $\mathcal{S}$ is identically equal to 1.

It is easy to see that the dynamical system $(X,\, d\mathbf{x}, \{\phi_t\})$ is ergodic. For if $U\subset X$ is measurable and $\phi_t(U) = U$ for all $t\in\mathbb{R}$ and if $V = U\cap S$ then V is measurable and $\phi_1(V) = V$. Applying the canonical diffeomorphism $\mathcal{S}\cong\mathbb{T}^2$ we may consider V as a measurable subset of $\mathbb{T}^2$ such that $\mathcal{P}(V) = V$. If $h = \chi_V$ then $\hat{h}(\mathbf{m}) = \hat{h}(\mathfrak{P}\mathbf{m})$ for all $\mathbf{m}\in\mathbb{Z}^2$. Since $h\in L^2(\mathbb{T}^2)$ we have $\hat{h}(\mathbf{m}) = 0$ unless $\mathbf{m} = \mathbf{0}$. Thus h must be equal almost everywhere to either 0 or 1. This in turn implies U either has measure 0 or 1.

Thus the entire phase space X is a single ergodic component, justifying our appellation: a purely chaotic Hamiltonian system. It is much more common to find examples of Hamiltonian systems with two degrees of freedom, such as the Hénon-Heiles system, where the compact three dimensional level set of the Hamiltonian is a union of many different ergodic components intertwined in a very complex manner [13].

1.2. **Asymptotics of Covariance.** Suppose $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are random variables on X. If we transport f from time 0 to time t we obtain the random variable $f \circ \phi_{-t}$. Because ϕ_{-t} preserves volume we have $E[f \circ \phi_{-t}] = E[f] = \int_X f(\mathbf{x})\, d\mathbf{x}$. The covariance of $f \circ \phi_{-t}$ and g is

$$\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x} - \int_X f(\mathbf{x})\, d\mathbf{x} \int_X g(\mathbf{x}')\, d\mathbf{x}'.$$

The Birkhoff-Khinchin Ergodic Theorem [6] implies that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x}\, dt = \int_X f(\mathbf{x})\, d\mathbf{x} \int_X g(\mathbf{x}')\, d\mathbf{x}',$$

but this allows $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x}$ to oscillate endlessly about the value $E[f]E[g]$ in such a way that the above covariance does not tend to 0 as $t \to \infty$. In fact the suspended flow we have defined is a well-known example where this happens, i.e. the flow fails to be *mixing*. For mixing flows it is common to inquire about the precise *rate of mixing*, i.e. sharp estimates of how the covariance decays to zero as $t \to \infty$. However, even when the flow fails to be mixing, as in the case we are considering, it is a natural question to understand the first few terms of the large-time asymptotics of $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x}$.

The purpose of this paper is to characterize, in as much detail as we can, the large-time asymptotics of $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x}$ under various hypotheses about f and g. In particular we are interested in the case where both f and g are smooth functions. See section 6 for our main results. We will call our problem the *decay of correlation* problem, although in our case neither the covariance, nor the correlation, of $f \circ \phi_{-t}$ and g actually decays to 0 as $t \to \infty$. With a similar abuse of terminology we will call $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\, d\mathbf{x}$ the *correlation integral*.

We will see that it is not practical to give simple hypotheses on f and g and then extract explicit asymptotics of the correlation integral under those hypotheses. The variety of possible asymptotic behaviors is much too great for this. We will show this by example, more or less. Thus by placing very restrictive hypotheses on f and g we will succeed in extracting sharp asymptotics on the correlation integral, but the nature of these hypotheses show that very complicated asymptotic behaviors are easily possible by taking linear combinations of different fs (and/or gs). Thus if the reader comes to this paper with a particular f and g in mind and desires to know the large-time asymptotics of the corresponding correlation integral, our results are unlikely to be of help. However our results do show very clearly how to construct a multiplicity of functions f and g with a correlation integral possessing a particular large-time asymptotic behavior.

1.3. **A Linear Partial Differential Equation.** If $F(\mathbf{x},t) = f(\phi_{-t}(\mathbf{x}))$ for all $\mathbf{x} \in X$ and $t \in \mathbb{R}$ then F satisfies the initial value problem

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \tau} = 0, \qquad F(\mathbf{x},0) = f(\mathbf{x}).$$

By $\frac{\partial F}{\partial \tau}$ we mean the application of the derivation $\frac{\partial}{\partial \tau}$, associated to a smooth vector field on X, to the function F. The solutions of this linear partial differential equation (on the compact 3 dimensional spatial domain X) become highly oscillatory as $t \to \infty$. Hence there will be an increasing amount of cancellation in the integral $\int_X F(\mathbf{x},t)g(\mathbf{x})\,d\mathbf{x}$ as $t \to \infty$. Thus our problem is to understand in detail the large-time behavior of the solution $F(\mathbf{x},t)$ of this initial value problem in a distributional sense. Thus it is conceptually related to work on other evolution equations which have oscillatory solutions, such as linear dispersive equations [7]. However the methods we employ are totally different from any of those used in work on evolution equations, since the mechanism inducing oscillations in the above is quite distinct from the phenomenon of dispersion.

1.4. **Physical Motivations: Protein Folding.** One of the motivations for studying rates of mixing comes from the field of statistical mechanics. Hamiltonian systems of a time-reversible nature are pervasive in the classical mechanics of particles subject to various forces of interaction [14]. In that subject the symplectic phase space $\mathcal{Y}$ contains coordinates for the positions and momenta of all the particles in the system. The Hamiltonian function H represents the total energy of the system, and is conserved on trajectories $\Phi_t(\mathbf{y})$ of Hamilton's equations of motion. Let X represent a smooth compact level set of H, with its natural volume element (constructed as above), normalized to be a probability measure. This probability measure, denoted by $d\mathbf{x}$, represents a maximal ignorance of the phase point of the system subject only to the constraint that the total energy has a given fixed value. The flow Φ_t takes X into itself, and when restricted to X is denoted by ϕ_t, which is measure-preserving. The measure-preserving time-reversal mapping $\mathcal{I}$ simply negates the momenta of all the particles.

In this setting it becomes paradoxical that trajectories almost never exhibit time-irreversible behaviors as $t \to \infty$. Take for example the problem of protein folding [17]. The system would include the protein atoms as well as many water molecules, all confined in a finite region of space. Let $U \subset X$ correspond to proteins with small radius of gyration, which would include those with the protein folded; the subset U has probability strictly between 0 and 1. By the Poincaré Recurrence Theorem almost every trajectory which starts in $X \setminus U$ must revisit $X \setminus U$ infinitely often as $t \to \infty$. Thus almost no trajectory starts with an unfolded protein at $t = 0$ and then stays folded for all sufficiently large time. However the time interval between two successive visits to $X \setminus U$ might be extremely large, which helps to reconcile this result with experimental observations.

Furthermore time-reversibility gives rise to an additional "paradox". Suppose $T > 0$ is larger than needed for a particular protein to fold (as determined by experiment). Let $W_T = (X \setminus U) \cap \phi_{-T}(U)$, i.e. initial conditions which represent unfolded proteins but which become "folded" after time T. One might intuitively think that W_T has a probability near 1, or at least larger than 1/2. But the set $W'_T = U \cap \phi_{-T}(X \setminus U)$ of initial conditions which start out "folded" and end up unfolded at time T, which is clearly disjoint from W_T, has the same probability as that of W_T. This is because $\mathcal{I}(U) = U$ and (using time-reversibility $\phi_{-T}\circ\mathcal{I} = \mathcal{I}\circ\phi_T$) we have $(\mathcal{I} \circ \phi_T)(W_T) = W'_T$. Thus both W_T and W'_T must have probabilities less than or equal to 1/2.

How is it then that in practice that we see proteins folding, and not the reverse? The error in the above discussion which leads to a paradoxical conclusion is the assumption that the probability of W_T should be larger than 1/2 since in experiments we see proteins folding in time T, not folded proteins unfolding in time T. The probability measure on X need not coincide with the one appropriate for such experiments. In the usual experiments, involving many copies of the same protein in an aqueous solution, the procedure is to force them all (somehow) to be unfolded at $t = 0$ and then to stand back (i.e. remove the force) and see what happens. The many proteins in the experiment at $t = 0$ constitute a random sample from a probability space, but this space is not X; it is more like $X \setminus U$. Let $f = \chi_{X\setminus U}$ and $g = \chi_U$ be characteristic (i.e. indicator) functions of the sets $X \setminus U$ and U respectively. Let $F(\mathbf{x}) = f(\mathbf{x})/\int_X f(\mathbf{x}')\,d\mathbf{x}'$ be the initial probability density of the folding experiment. Since $\chi_{\phi_{-T}(U)} = g \circ \phi_T$, the probability of W_T is

$$\int_X f(\mathbf{x})g(\phi_T(\mathbf{x}))\,d\mathbf{x} = \int_X f(\phi_{-T}(\mathbf{x}'))g(\mathbf{x}')\,d\mathbf{x}'.$$

The *conditional* probability of W_T given $X \setminus U$ is $\int_X F(\phi_{-T}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}$. The experimental evidence suggests that this conditional probability is very close to 1. According to the ergodic theorem for large T the integral $\int_X F(\phi_{-T}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}$ should oscillate around a quantity, which (under an ergodicity assumption on the flow $\{\phi_t\}$) can be identified as $\int_X g(\mathbf{x})\,d\mathbf{x}$, which is the probability of U. Thus the above experimentally motivated assertion is easily seen to be possible provided there is some sort of approximate mixing in the flow (at least ergodicity and asymptotically small amplitude oscillations), and the probability of U is close to 1. In this way we see how the study of the large-time behavior of the correlation integral can be motivated.

The particular Hamiltonian system considered in this paper is a drastic simplification of those arising from systems of many particles, a simplification which nevertheless retains (we hope) most of the truly essential formal properties. In our case, although the underlying dynamics is not mixing, we wish to quantify the rates of the relaxation involved and relate those rates to properties of the observable quantities represented by the functions f and g. We do not however believe this system exhibits generic behavior because the return time function is constant on a Poincaré section, an obviously nongeneric feature. Nevertheless this special case can be understood explicitly whereas the behavior in the more generic case of a nonconstant return time function seems to be much more difficult.

1.5. **Related Work.** There is another Hamiltonian system restricted to a compact level set of the Hamiltonian where the (time-reversible) flow is in fact mixing, and where the rate of mixing has been characterized. This is the case of geodesic flow in the unit (co)tangent bundle of a compact surface of constant negative curvature [21]. Sharp exponential decay of correlation results can use the formalism of [20]. Exponential decay of correlations has been proved for (exact) contact Anosov flows [15], and for other Anosov flows with certain (rather nongeneric) regularity properties [8]. It is not difficult to show that the flow $\{\phi_t\}$ we study, although being an Anosov flow, is not covered in these studies. Other low dimensional continuous time dynamical systems preserve many of the features we have emphasized as important. For example, billiard flow in a compact planar region can be formulated as a Hamiltonian (possibly not complete) flow on a (noncompact) symplectic manifold [11]. Apparently billiard flow in a "stadium" can only be expected to have polynomial (rather than exponential) decay of correlations [16].

1.6. Summary of the Rest of the Paper.

- Section 2 examines a series of reductions in complexity of the decay of correlations problem.
- Section 3 gives a variety of explicit examples of discrete convolutions and their decay rates.
- Section 4 proves sharp asymptotic decay estimates for a class of discrete convolutions.
- Section 5 states results on the sharp rate of decay of the discrete time correlation
- Section 6 studies sharp asymptotic results of the continuous time correlation integral.

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2. Reductions of the Problem

2.1. From Continuous Time to Discrete Time. The study of the large-time asymptotic behavior of the correlation integral $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}$ in the case of a suspended flow over the hyperbolic toral automorphism $\mathcal{P}$ can be reduced to the corresponding problem for the discrete time dynamical system $\mathcal{P}$. As we defined in the introduction we will use $\mathcal{P}\colon \mathbb{T}^2 \to \mathbb{T}^2\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \mathbb{Z}^2 \mapsto \mathfrak{P}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \mathbb{Z}^2$ as our hyperbolic toral automorphism, where $\mathfrak{P} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Suppose $t = n + s$, $n \in \mathbb{Z}$, $n \geq 0$, $0 \leq s < 1$. We may think of both n and s as being functions of t. Then

$$
\begin{aligned}
\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x} &= \int_s^1 \int_{\mathbb{T}^2} (f \circ \phi_{-s})(\phi_{-n}(\mathbf{z},\tau))g(\mathbf{z},\tau)\,d\mathbf{z}\,d\tau \\
&\quad + \int_0^s \int_{\mathbb{T}^2} (f \circ \phi_{1-s})(\phi_{-n-1}(\mathbf{z},\tau))g(\mathbf{z},\tau)\,d\mathbf{z}\,d\tau \\
&= \int_s^1 \left[\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}), \tau - s)g(\mathbf{z},\tau)\,d\mathbf{z}\right] d\tau \\
&\quad + \int_0^s \left[\int_{\mathbb{T}^2} f(\mathcal{P}^{-n-1}(\mathbf{z}), 1 + \tau - s)g(\mathbf{z},\tau)\,d\mathbf{z}\right] d\tau.
\end{aligned}
$$

In the above we denote by $f(\mathbf{z},\tau)$ the quantity $(f \circ \psi_1)(\mathbf{z},\tau,0)$. If we regard s as being fixed and allow $n \to \infty$ we see that we must discover the large n asymptotics of the integral $\int_{\mathbb{T}^2} \tilde{f}(\mathcal{P}^{-n}(\mathbf{z}))\tilde{g}(\mathbf{z})\,d\mathbf{z}$, where $\tilde{f}, \tilde{g}\colon \mathbb{T}^2 \to \mathbb{R}$ are suitable functions. This is called the *Discrete Time Problem.* As is well-known (see the next subsection) the leading-order term of the large n asymptotics of $\int_{\mathbb{T}^2} \tilde{f}(\mathcal{P}^{-n}(\mathbf{z}))\tilde{g}(\mathbf{z})\,d\mathbf{z}$ is $\int_{\mathbb{T}^2} \tilde{f}(\mathbf{z})\,d\mathbf{z} \int_{\mathbb{T}^2} \tilde{g}(\mathbf{z}')\,d\mathbf{z}'$. So for example, for fixed s and τ, the leading-order large

n asymptotics of $\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}),\tau-s)g(\mathbf{z},\tau)\,d\mathbf{z}$ should be

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}),\tau-s)\,d\mathbf{z}\int_{\mathbb{T}^2} g(\mathbf{z}',\tau)\,d\mathbf{z}' = \int_{\mathbb{T}^2} f(\mathbf{z},\tau-s)\,d\mathbf{z}\int_{\mathbb{T}^2} g(\mathbf{z}',\tau)\,d\mathbf{z}'$$

Define $\Lambda_f(t)=\int_{\mathbb{T}^2} f(\mathbf{z},s)\,d\mathbf{z}$ for $t=n+s$, $0\le s<1$ and $n\in\mathbb{Z}$. Let $\Lambda_g(t)$ be defined similarly. Then the leading-order term of the large-time asymptotics of $\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}$ should be

$$\begin{aligned}&\int_s^1 \Lambda_f(\tau-s)\Lambda_g(\tau)\,d\tau + \int_0^s \Lambda_f(1+\tau-s)\Lambda_g(\tau)\,d\tau\\ &\quad= \int_s^1 \Lambda_f(\tau-s-n)\Lambda_g(\tau)\,d\tau + \int_0^s \Lambda_f(\tau-s-n)\Lambda_g(\tau)\,d\tau\\ &\quad= \int_0^1 \Lambda_f(\tau-t)\Lambda_g(\tau)\,d\tau.\end{aligned}$$

This is a periodic function of t with period 1. Therefore we have

$$\begin{aligned}&\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x} - \int_0^1 \Lambda_f(\tau-t)\Lambda_g(\tau)\,d\tau\\ &\quad= \int_s^1 \left[\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}),\tau-s)g(\mathbf{z},\tau)\,d\mathbf{z} - \Lambda_f(\tau-s)\Lambda_g(\tau)\right]d\tau\\ &\qquad+ \int_0^s \left[\int_{\mathbb{T}^2} f(\mathcal{P}^{-n-1}(\mathbf{z}),1+\tau-s)g(\mathbf{z},\tau)\,d\mathbf{z} - \Lambda_f(1+\tau-s)\Lambda_g(\tau)\right]d\tau.\end{aligned}$$

We are interested in the sharp decay rates of this quantity as $t\to\infty$, and if possible we desire to extract explicit asymptotic expressions for this quantity. These should arise from the corresponding estimates and expressions in the discrete time problem.

Therefore we intend to study the discrete time problem in considerable detail in sections 3, 4, and 5. The results we obtain on the discrete time problem are of interest independent of the connection to the continuous time problem. The parallelism between the discrete time and the continuous time results is striking and worth making explicit.

However, we will soon (see section 2.3) find another and more direct method to obtain the desired asymptotic error estimates for the continuous time correlation integral, more direct than the approach through the discrete time results.

2.2. **Fourier Series and Discrete Convolutions.** The hyperbolic toral automorphism $\mathcal{P}$ is completely understood by means of Fourier series:

$$f(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{f}(\mathbf{m})e^{2\pi i\mathbf{m}\cdot\mathbf{z}}, \qquad \hat{f}(\mathbf{m}) = \int_{\mathbb{T}^2} e^{-2\pi i\mathbf{m}\cdot\mathbf{z}} f(\mathbf{z})\,d\mathbf{z},$$

since by the symmetry and invertibility of $\mathfrak{P}$ we have

$$\begin{aligned}f(\mathcal{P}^{-n}(\mathbf{z})) &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{f}(\mathbf{m})e^{2\pi i\mathbf{m}\cdot\mathcal{P}^{-n}(\mathbf{z})} = \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{f}(\mathbf{m})e^{2\pi i\mathfrak{P}^{-n}\mathbf{m}\cdot\mathbf{z}}\\ &= \sum_{\mathbf{m}'\in\mathbb{Z}^2} \hat{f}(\mathfrak{P}^n\mathbf{m}')e^{2\pi i\mathbf{m}'\cdot\mathbf{z}}.\end{aligned}$$

Therefore the discrete time version of the correlation integral is

$$\begin{aligned}\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\,d\mathbf{z} &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{f}(\mathfrak{P}^n\mathbf{m}) \int_{\mathbb{T}^2} e^{2\pi i\mathbf{m}\cdot\mathbf{z}} g(\mathbf{z})\,d\mathbf{z} \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{f}(\mathfrak{P}^n\mathbf{m})\hat{g}(-\mathbf{m}) \\ &= \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + \sum_{\mathbf{m}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \hat{f}(\mathfrak{P}^n\mathbf{m})\hat{g}(-\mathbf{m}).\end{aligned}$$

The first term on the right $\hat{f}(\mathbf{0})\hat{g}(\mathbf{0})$ is the leading order term of the large n asymptotics of the correlation integral. Since f and g are real-valued we have the relation $\overline{\hat{f}(\mathbf{m})} = \hat{f}(-\mathbf{m})$ for all $\mathbf{m} \in \mathbb{Z}^2$, and similarly for g. An explicit second term of the large n asymptotics must be extracted from the second term on the right, and will obviously depend on the behavior of $\hat{f}(\mathbf{m})$ and $\hat{g}(-\mathbf{m}) = \overline{\hat{g}(\mathbf{m})}$ on orbits $\mathcal{O}(\mathbf{m}_0) = \{\mathfrak{P}^k\mathbf{m}_0 \mid k \in \mathbb{Z}\}$, $\mathbf{m}_0 \neq \mathbf{0}$.

It is helpful at this stage to study the nature of the orbits $\mathcal{O}(\mathbf{m}_0)$. Obviously $\mathcal{O}(\mathbf{0}) = \{\mathbf{0}\}$. All the other orbits are countably infinite sets. Recall that $\mathfrak{P}\mathbf{v}_\pm = \mathbf{v}_\pm\lambda_\pm$, where $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$ and $\mathbf{v}_\pm = \begin{pmatrix} 1 \\ \frac{\pm\sqrt{5}}{2}\end{pmatrix}$. For each $\mathbf{m} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ we have $\mathbf{m} = \mathbf{v}_+\alpha_+ + \mathbf{v}_-\alpha_-$ for some real constants α_+, α_-. Hence $\mathfrak{P}^k\mathbf{m} = \mathbf{v}_+\lambda_+^k\alpha_+ + \mathbf{v}_-\lambda_-^k\alpha_-$. Since $\lambda_- = \lambda_+^{-1}$ we have that the quantity $\alpha_+\alpha_-$ is fixed on each orbit. A short calculation shows that $\alpha_+\alpha_-$ is proportional to $-n_1^2 + n_1n_2 + n_2^2$, where $\mathbf{m} = \binom{n_1}{n_2}$. Thus each infinite orbit lies on a branch of the hyperbola $-n_1^2 + n_1n_2 + n_2^2 = c$, for an appropriate choice of $c \in \mathbb{Z} \setminus \{0\}$. However, a given hyperbola might contain several orbits. In order to choose a standard representative for each orbit note that since $\mathfrak{P}$ is linear it maps lines through the origin in $\mathbb{R}^2$ into themselves. Consider a line with the equation $x = \mu y$. It gets mapped into a line with the equation $x = \frac{(2\mu+1)}{(\mu+1)}y$. There are two fixed lines $\mathcal{L}_\pm$, with $\mu = \frac{1\pm\sqrt{5}}{2}$; these are the asymptotic lines of all the hyperbolae. If $\frac{1-\sqrt{5}}{2} < \mu < \frac{1+\sqrt{5}}{2}$ then $\frac{(2\mu+1)}{(\mu+1)}$ is an increasing function of μ and always $\frac{(2\mu+1)}{(\mu+1)} > \mu$. Thus lines in the family $\mathcal{F}_1$, defined by the inequality $\frac{1-\sqrt{5}}{2} < \mu < \frac{1+\sqrt{5}}{2}$, are rotated clockwise by the action of $\mathfrak{P}$ and stay in this family. Since the line with $\mu = 0$ is rotated into the line with $\mu = 1$, each orbit in the quadrant $\frac{1-\sqrt{5}}{2} < \frac{n_1}{n_2} < \frac{1+\sqrt{5}}{2}$, $n_2 > 0$, must intersect the sector $S_1 = \{\binom{n_1}{n_2} \mid n_2 > 0, 0 \le n_1 < n_2\}$, which corresponds to $0 \le \mu < 1$, and therefore has exactly one representative in that sector. See Figure 1.

Let $\mathcal{F}_2$ denote the family of lines in $\mathbb{R}^2$ through the origin which are not in $\mathcal{F}_1 \cup \{\mathcal{L}_+, \mathcal{L}_-\}$. Consider the matrix $\mathfrak{R} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1\end{pmatrix}$. We have $\mathfrak{R}(\mathbf{v}_+, \mathbf{v}_-) = (\mathbf{v}_+, -\mathbf{v}_-)$ and $\mathfrak{P} = \mathfrak{R}\mathfrak{P}\mathfrak{R}$. The action of $\mathfrak{P}$ on the lines in $\mathcal{F}_2$ can be understood if we note that the action of $\mathfrak{R}$ (i.e. reflection about the line $\mathcal{L}_+$) takes $\mathcal{F}_2$ bijectively onto $\mathcal{F}_1$. Thus the lines in $\mathcal{F}_2$ are rotated counterclockwise by the action of $\mathfrak{P}$, and stay within the family $\mathcal{F}_2$. Also since the line with $\mu = -1$ is rotated into the line with $\mu = \pm\infty$, each orbit in the quadrant $\frac{2}{1-\sqrt{5}} < \frac{n_2}{n_1} < \frac{2}{1+\sqrt{5}}$, $n_1 > 0$, must intersect the sector $S_2 = \{\binom{n_1}{n_2} \mid n_1 > 0, -n_1 < n_2 \le 0\}$, which corresponds to $-\infty \le \mu < -1$, and also has exactly one representative in that sector.

Since for a real-valued f the values of $\hat{f}$ on $\mathcal{O}(-\mathbf{m}_0)$ are determined by the values of $\hat{f}$ on $\mathcal{O}(\mathbf{m}_0)$ we may index the distinct sets $\mathcal{O}(\mathbf{m}_0) \cup \mathcal{O}(-\mathbf{m}_0)$ by $\mathbf{m}_0 \in S_1 \cup S_2$.

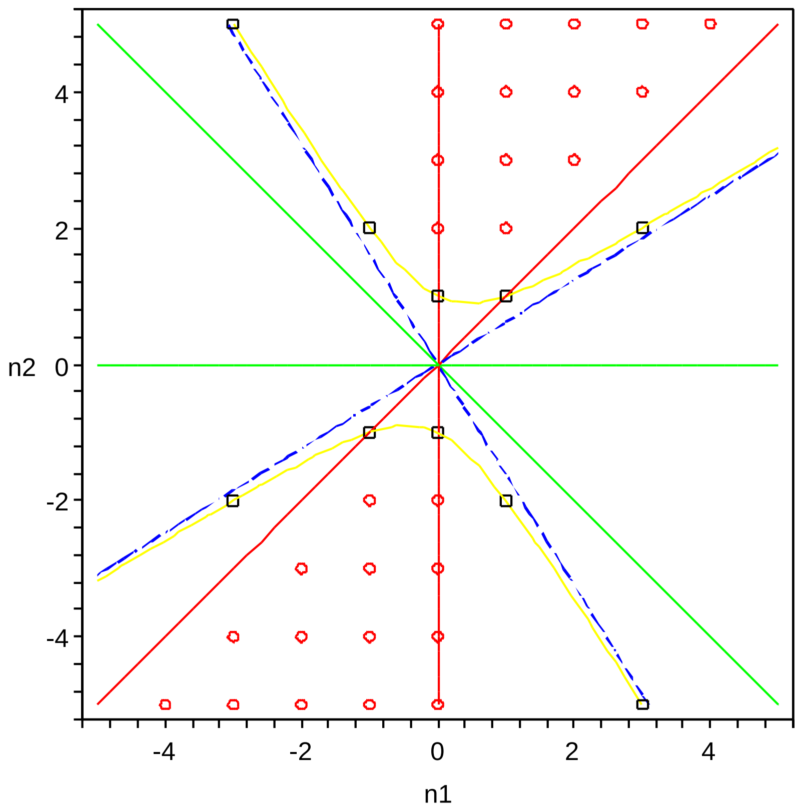

FIGURE 1. $\mathcal{O}(\mathbf{m}_0)\cup\mathcal{O}(-\mathbf{m}_0)$, where $\mathbf{m}_0 = \binom{0}{1}$, is pictured as black squares. The yellow hyperbola has the equation $-n_1^2+n_1n_2+n_2^2 = 1$. The red lines bound the sector S_1 and its negative. Representatives of orbits passing through S_1 or its negative are shown as red circles. The green lines bound the sector S_2 and its negative. Lines spanned by $\mathbf{v}_+ = \binom{1+\sqrt{5}}{2}$ and $\mathbf{v}_- = \binom{1-\sqrt{5}}{2}$ are shown as blue dashed lines.

Hence

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\,d\mathbf{z}$$
$$= \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + \sum_{\mathbf{m}_0\in S_1\cup S_2}\sum_{\mathbf{m}\in\mathcal{O}(\mathbf{m}_0)}\left[\hat{f}(\mathfrak{P}^n\mathbf{m})\hat{g}(-\mathbf{m}) + \hat{f}(-\mathfrak{P}^n\mathbf{m})\hat{g}(\mathbf{m})\right].$$

Each $\mathbf{m} \in \mathcal{O}(\mathbf{m}_0)$ can be expressed as $\mathbf{m} = \mathfrak{P}^{-k}\mathbf{m}_0$ for some $k \in \mathbb{Z}$. Thus we have

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\,d\mathbf{z} - \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) = \sum_{\mathbf{m}_0 \in S_1 \cup S_2} \sum_{k=-\infty}^{\infty} \left[\hat{f}(\mathfrak{P}^{n-k}\mathbf{m}_0)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0)} + \overline{\hat{f}(\mathfrak{P}^{n-k}\mathbf{m}_0)}\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0) \right].$$

Notice that if we define

$$a_k = \hat{f}(\mathfrak{P}^k\mathbf{m}_0), \qquad b_k = \overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0)},$$

for all $k \in \mathbb{Z}$ then we may write

$$\sum_{k=-\infty}^{\infty} \hat{f}(\mathfrak{P}^{n-k}\mathbf{m}_0)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0)} = \sum_{k=-\infty}^{\infty} a_{n-k}b_k,$$

which is a discrete convolution. Thus the second term of the large n asymptotics of the discrete time correlation integral must be extracted from a countable sum of discrete convolutions, each complex conjugate pair of which is uncoupled from the others. Suppose for simplicity that $\mathbf{m}_0 \in S_1 \cup S_2$ is fixed and that $\hat{f}(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}^2 \setminus [\mathcal{O}(\mathbf{m}_0) \cup \mathcal{O}(-\mathbf{m}_0) \cup \{\mathbf{0}\}]$. This assumption eliminates all of the discrete convolutions except for one (complex conjugate pair). Thus the discrete time problem can be reduced under these strong hypotheses to the problem of extracting the large n asymptotics of the single discrete convolution $2\Re \sum_{k=-\infty}^{\infty} a_{n-k}b_k$.

In order to understand these asymptotics we introduce the complex parameter ω and consider the Laurent series

$$(*) \qquad \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{n-k}b_k \right) \omega^n = \left(\sum_{l=-\infty}^{\infty} a_l \omega^l \right) \left(\sum_{k=-\infty}^{\infty} b_k \omega^k \right).$$

We will call this the *generating function* of the convolution. Exactly how we will employ the generating function to study the large n asymptotics of the discrete convolution will start becoming clear in section 3.2.

2.3. **Continuous Time Convolutions.** Now we return to the notation and assumptions of section 2.1 and combine the main results of the last two sections to obtain the following:

$$\begin{aligned}
&\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x} - \int_0^1 \Lambda_f(\tau - t)\Lambda_g(\tau)\,d\tau \\
&\quad = \int_s^1 \left\{ \sum_{\mathbf{m}_0 \in S_1 \cup S_2} \sum_{k=-\infty}^{\infty} 2\Re \left[\hat{f}(\mathfrak{P}^{n-k}\mathbf{m}_0; \tau - s)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0; \tau)} \right] \right\} d\tau \\
&\qquad + \int_0^s \left\{ \sum_{\mathbf{m}_0 \in S_1 \cup S_2} \sum_{k=-\infty}^{\infty} 2\Re \left[\hat{f}(\mathfrak{P}^{n+1-k}\mathbf{m}_0; 1 + \tau - s)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0; \tau)} \right] \right\} d\tau \\
&\quad = 2\Re \sum_{\mathbf{m}_0 \in S_1 \cup S_2} \sum_{k=-\infty}^{\infty} \left\{ \int_s^1 \hat{f}(\mathfrak{P}^{n-k}\mathbf{m}_0; \tau - s)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0; \tau)}\,d\tau \right. \\
&\qquad\qquad \left. + \int_0^s \hat{f}(\mathfrak{P}^{n+1-k}\mathbf{m}_0; 1 + \tau - s)\overline{\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0; \tau)}\,d\tau \right\}.
\end{aligned}$$

In the above we are denoting by $\hat{f}(\mathbf{m};\tau)$ the quantity $(f\circ\psi_1)(\cdot,\tau,0)\hat{\ }(\mathbf{m})$. Now if $f\colon X\to\mathbb{R}$ is continuous then in terms of the $(\mathbf{z},\tau)$ coordinates on X we have $f(\mathbf{z},1)=f(\mathcal{P}(\mathbf{z}),0)$ for all $\mathbf{z}\in\mathbb{T}^2$. In terms of Fourier series this becomes $\hat{f}(\mathbf{m};1)=\hat{f}(\mathfrak{P}^{-1}\mathbf{m};0)$ for all $\mathbf{m}\in\mathbb{Z}^2$. This relation suggests the introduction of the following transform of f:

$$(\mathfrak{F}_{\mathbf{m}}f)(u)=\hat{f}(\mathfrak{P}^{-k}\mathbf{m};u-k),\qquad u\in\mathbb{R},k\in\mathbb{Z},k\le u<k+1.$$

Notice that $(\mathfrak{F}_{\mathbf{0}}f)(u)=\hat{f}(\mathfrak{P}^{-k}\mathbf{0};u-k)=\hat{f}(\mathbf{0};u-k)=\Lambda_f(u-k)=\Lambda_f(u)$. It is not difficult to show that

$$(\mathfrak{F}_{\mathbf{m}}f)(u)=(f\circ\psi_2)(\cdot,u-k,0)\hat{\ }(\mathfrak{P}^{-k}\mathbf{m}),\qquad k\in\mathbb{Z},k-\tfrac{1}{2}<u<k+\tfrac{1}{2}.$$

Thus when f is smooth the transform $\mathfrak{F}_{\mathbf{m}}f$ is also smooth for each $\mathbf{m}\in\mathbb{Z}^2$.

This allows us to rewrite our main expression as follows:

$$\begin{aligned}
&\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}-\int_0^1\Lambda_f(\tau-t)\Lambda_g(\tau)\,d\tau\\
&\quad=2\Re\sum_{\mathbf{m}_0\in S_1\cup S_2}\sum_{k=-\infty}^{\infty}\Big\{\int_s^1(\mathfrak{F}_{\mathbf{m}_0}f)(k-n+\tau-s)\overline{(\mathfrak{F}_{\mathbf{m}_0}g)(k+\tau)}\,d\tau\\
&\qquad\qquad+\int_0^s(\mathfrak{F}_{\mathbf{m}_0}f)(k-n-1+1+\tau-s)\overline{(\mathfrak{F}_{\mathbf{m}_0}g)(k+\tau)}\,d\tau\Big\}\\
&\quad=2\Re\sum_{\mathbf{m}_0\in S_1\cup S_2}\sum_{k=-\infty}^{\infty}\int_0^1(\mathfrak{F}_{\mathbf{m}_0}f)(k+\tau-n-s)\overline{(\mathfrak{F}_{\mathbf{m}_0}g)(k+\tau)}\,d\tau\\
&\quad=2\Re\sum_{\mathbf{m}_0\in S_1\cup S_2}\int_{-\infty}^{\infty}(\mathfrak{F}_{\mathbf{m}_0}f)(u-t)\overline{(\mathfrak{F}_{\mathbf{m}_0}g)(u)}\,du.
\end{aligned}$$

If $h\colon\mathbb{R}\to\mathbb{R}$ then define $h^{\checkmark}(u)=h(-u)$ for all $u\in\mathbb{R}$. Thus we get the remarkable and important relation:

$$\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}-\int_0^1\Lambda_f(\tau-t)\Lambda_g(\tau)\,d\tau=2\Re\sum_{\mathbf{m}_0\in S_1\cup S_2}[(\mathfrak{F}_{\mathbf{m}_0}f)^{\checkmark}*\overline{(\mathfrak{F}_{\mathbf{m}_0}g)}](t).$$

Again under the assumption that $\mathbf{m}_0\in S_1\cup S_2$ is fixed and that $\mathfrak{F}_{\mathbf{m}}f\equiv 0$ for all $\mathbf{m}\in(S_1\cup S_2)\setminus\{\mathbf{0},\mathbf{m}_0\}$ we have

$$\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x}-\int_0^1\Lambda_f(\tau-t)\Lambda_g(\tau)\,d\tau=2\Re[(\mathfrak{F}_{\mathbf{m}_0}f)^{\checkmark}*\overline{(\mathfrak{F}_{\mathbf{m}_0}g)}](t),$$

i.e. we must extract the large argument asymptotics of a single real variable convolution of two functions.

We earlier introduced the generating function of a discrete convolution as a tool for analyzing the large n asymptotics of the discrete convolution. In a similar way the large t asymptotics of a continuous convolution may be analyzed using the Fourier transform, because of the well-known equality of the Fourier transform of a convolution with the point-wise product of the two Fourier transforms. Now it will be helpful to point out how the generating function arises naturally from the

Fourier transform. If $h\colon \mathbb{R}\to\mathbb{R}$ is in the Schwartz class then we may write

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-iwt}h(t)\,dt &= \sum_{n=-\infty}^{\infty}\int_0^1 e^{-iw(n+s)}h(n+s)\,ds\\ &= \int_0^1 e^{-iws}\left[\sum_{n=-\infty}^{\infty}(e^{-iw})^n h(n+s)\right]ds.\end{aligned}$$

Thus the Fourier transform of h at $w\in\mathbb{R}$ can be expressed in terms of an integral of the generating functions of the sequences $\{h(n+s)\}_{n\in\mathbb{Z}}$ evaluated at $\omega = e^{-iw}$. If $|h(t)|\le Ce^{-\epsilon|t|}$ for positive constants C and ϵ and for all $t\in\mathbb{R}$ then the Fourier transform of h extends to an analytic function in the strip $|\Im w|<\epsilon$, and for each $0\le s<1$ the generating function of the sequence $n\mapsto h(n+s)$ is analytic in the annulus $e^{-\epsilon}<|\omega|<e^{\epsilon}$.

2.4. **A complete orthonormal set in $L^2(X)$.** The transforms $\mathfrak{F}_{\mathbf{m}}f\colon\mathbb{R}\to\mathbb{C}$ of a function $f\colon X\to\mathbb{C}$, where $\mathbf{m}\in S_1\cup(-S_1)\cup S_2\cup(-S_2)\cup\{\mathbf{0}\}$, can be inverted to recover the function f. We have

$$\begin{aligned}f(\mathbf{z},\tau) &= \sum_{\mathbf{m}\in\mathbb{Z}^2}\hat f(\mathbf{m};\tau)e^{2\pi i\mathbf{m}\cdot\mathbf{z}}\\ &= \hat f(\mathbf{0};\tau)+\sum_{\mathbf{m}_0\in S_1\cup S_2}\sum_{\mathbf{m}\in\mathcal{O}(\mathbf{m}_0)}\left[\hat f(\mathbf{m};\tau)e^{2\pi i\mathbf{m}\cdot\mathbf{z}}+\hat f(-\mathbf{m};\tau)e^{-2\pi i\mathbf{m}\cdot\mathbf{z}}\right]\\ &= (\mathfrak{F}_{\mathbf{0}}f)(\tau)+\sum_{\mathbf{m}_0\in S_1\cup S_2}\sum_{k=-\infty}^{\infty}\Big[(\mathfrak{F}_{\mathbf{m}_0}f)(k+\tau)e^{2\pi i\mathfrak{P}^{-k}\mathbf{m}_0\cdot\mathbf{z}}\\ &\qquad\qquad + (\mathfrak{F}_{-\mathbf{m}_0}f)(k+\tau)e^{-2\pi i\mathfrak{P}^{-k}\mathbf{m}_0\cdot\mathbf{z}}\Big]\end{aligned}$$

A short calculation shows that the mapping of f into its set of transforms $\mathfrak{F}_{\mathbf{m}}f$ is unitary, i.e.

$$\begin{aligned}\int_X \overline{f(\mathbf{x})}g(\mathbf{x})\,d\mathbf{x} &= \int_0^1 \overline{(\mathfrak{F}_{\mathbf{0}}f)(\tau)}(\mathfrak{F}_{\mathbf{0}}g)(\tau)\,d\tau\\ &\quad+\sum_{\mathbf{m}\in S_1\cup S_2\cup(-S_1)\cup(-S_2)}\int_{-\infty}^{\infty}\overline{(\mathfrak{F}_{\mathbf{m}}f)(u)}(\mathfrak{F}_{\mathbf{m}}g)(u)\,du.\end{aligned}$$

It is also clear that $[\mathfrak{F}_{\mathbf{m}}(f\circ\phi_t)](u) = (\mathfrak{F}_{\mathbf{m}}f)(u+t)$ and $\overline{(\mathfrak{F}_{\mathbf{m}}f)(u)} = (\mathfrak{F}_{-\mathbf{m}}\overline{f})(u)$, for all $\mathbf{m},u,t$. These identities can be used to give a short derivation of our main expression of the correlation integral in terms of convolutions of the transforms.

We may use this technology to give a complete orthonormal set for $L^2(X)$, where each function in this set determines an explicit smooth function on X. The previous displayed formula shows that $L^2(X)$ decomposes as a Hilbert space direct sum of $L^2([0,1))$ with infinitely many copies of $L^2(\mathbb{R})$, one for each $\mathbf{m}\in S_1\cup S_2\cup(-S_1)\cup(-S_2)$. In each of these Hilbert spaces there are well-known complete orthonormal sets: $\{e^{2\pi in\tau}\}_{n\in\mathbb{Z}}$ for $L^2([0,1))$, and $\{c_ne^{-u^2/2}H_n(u)\}_{n\ge0}$ for $L^2(\mathbb{R})$, where $c_n^{-2} = 2^nn!\sqrt{\pi}$ and $H_n(u) = (-1)^ne^{u^2}(\frac{d}{du})^n[e^{-u^2}]$ are the Hermite polynomials. Thus a complete orthonormal subset of $L^2(X)$ is indexed by pairs $(\mathbf{m},n)$, where $\mathbf{m}\in\{\mathbf{0}\}\cup S_1\cup S_2\cup(-S_1)\cup(-S_2)$ and $n\in\mathbb{Z}$ ($n\ge0$ if $\mathbf{m}\ne\mathbf{0}$). The basis function of

$(\mathbf{z},\tau)$ corresponding to $(\mathbf{0},n)$ is $e^{2\pi i n\tau}$. The basis function of $(\mathbf{z},\tau)$ corresponding to $(\mathbf{m},n)$, where $\mathbf{m}\in S_1\cup S_2\cup(-S_1)\cup(-S_2)$ and $n\geq 0$ is

$$c_n\sum_{k=-\infty}^{\infty} e^{-(k+\tau)^2/2}H_n(k+\tau)e^{2\pi i\mathfrak{P}^{-k}\mathbf{m}\cdot\mathbf{z}}.$$

The transform $\mathfrak{F}_{\mathbf{m}}$ applied to this function yields $c_n e^{-u^2/2}H_n(u)$, which is both smooth in u and very rapidly decaying as $|u|\to\infty$. Very rapidly convergent lacunary series of this type give rise to smooth functions on X.

3. Examples of Discrete Convolutions

3.1. **Power-Law Decay.** A particularly convenient choice for $\sum_l a_l\omega^l$ is where a_l is defined so that $(1-\omega)^{-a}=\sum_{l=0}^{\infty}a_l\omega^l$ for all $|\omega|<1$, i.e. by the rule:

$$a_l=\begin{cases}\frac{a(a+1)\cdots(a+l-1)}{l!}=(-1)^l\binom{-a}{l}=\frac{\Gamma(a+l)}{\Gamma(a)\Gamma(l+1)} & l\geq 0,\\ 0 & l<0,\end{cases}\qquad a\in\mathbb{C}.$$

Using Sterling's asymptotic formula, or more precisely (6.1.47) of [2], and assuming a is not zero or a negative integer we find

$$a_l=\frac{l^{a-1}}{\Gamma(a)}\left[1+\frac{a(a-1)}{2l}+\mathrm{O}\left(l^{-2}\right)\right]\qquad \text{as } l\to\infty.$$

If a is zero or a negative integer then $a_l=0$ for all sufficiently large l. Now if we take $\sum_k b_k\omega^k$ as $(1-\omega)^{-b}$ then as we have seen

$$(1-\omega)^{-a-b}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_{n-k}b_k\right)\omega^n=\sum_{n=0}^{\infty}\frac{\Gamma(a+b+n)}{\Gamma(a+b)\Gamma(n+1)}\omega^n,$$

and we therefore have that (assuming $a+b$ is not zero or a negative integer)

$$\sum_{k=-\infty}^{\infty}a_{n-k}b_k=\sum_{k=0}^{n}a_{n-k}b_k=\frac{n^{a+b-1}}{\Gamma(a+b)}\left[1+\frac{(a+b)(a+b-1)}{2n}+\mathrm{O}\left(n^{-2}\right)\right]$$

as $n\to\infty$. Thus in this example the convolution of two sequences, both having power-like asymptotic behavior, also has power-like asymptotic behavior, where the powers obey the rule $a+b-1=(a-1)+(b-1)+1$.

That this rule continues to hold when $\Re a<0$ or $\Re b<0$ is a bit surprising since the rule is more complicated if we examine a different example:

$$\tilde{a}_l=\begin{cases}(\frac{1}{2}+l)^{a-1} & l\geq 0,\\ 0 & l<0,\end{cases}\qquad \tilde{b}_k=\begin{cases}(-\frac{1}{2}+k)^{b-1} & k\geq 1,\\ 0 & k\leq 0\end{cases}.$$

If $\Re a>0$ and $\Re b>0$ then we have

$$\begin{aligned}n^{1-a-b}\sum_{k=-\infty}^{\infty}\tilde{a}_{n-k}\tilde{b}_k&=\frac{1}{n}\sum_{k=1}^{n}n^{1-a}(\tfrac{1}{2}+n-k)^{a-1}n^{1-b}(-\tfrac{1}{2}+k)^{b-1}\\&=\frac{1}{n}\sum_{k=1}^{n}[1-\tfrac{1}{n}(k-\tfrac{1}{2})]^{a-1}[\tfrac{1}{n}(k-\tfrac{1}{2})]^{b-1}\\&\to\int_0^1(1-x)^{a-1}x^{b-1}\,dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\end{aligned}$$

as $n \to \infty$, a result which is in agreement with the above rule. However if $\Re a > 0$ and $\Re b < 0$ then we have

$$n^{1-a} \sum_{k=-\infty}^{\infty} \tilde{a}_{n-k}\tilde{b}_k = n^{1-a} \sum_{k=1}^{n} (\tfrac{1}{2} + n - k)^{a-1}(-\tfrac{1}{2} + k)^{b-1}$$
$$= \sum_{k=1}^{[\![\frac{n}{2}]\!]} [1 - \tfrac{1}{n}(k - \tfrac{1}{2})]^{a-1}(k - \tfrac{1}{2})^{b-1} + n^{1-a} \sum_{k=[\![\frac{n}{2}]\!]+1}^{n} (\tfrac{1}{2} + n - k)^{a-1}(k - \tfrac{1}{2})^{b-1}.$$

The first term can be asymptotically evaluated using the dominated convergence theorem, since the summand is absolutely bounded by $\max\{1, (\frac{1}{2})^{\Re a-1}\}(k-\frac{1}{2})^{\Re b-1}$, independently of n. Thus for the first term we have

$$\sum_{k=1}^{[\![\frac{n}{2}]\!]} [1 - \tfrac{1}{n}(k - \tfrac{1}{2})]^{a-1}(k - \tfrac{1}{2})^{b-1} \to \sum_{k=1}^{\infty} (k - \tfrac{1}{2})^{b-1} = \sum_{k=1}^{\infty} \tilde{b}_k$$

as $n \to \infty$. The second term may be estimated crudely by

$$n^{1-\Re a}([\![\tfrac{n}{2}]\!] + \tfrac{1}{2})^{\Re b-1} \sum_{k=0}^{[\![\frac{n+1}{2}]\!]-1} (k + \tfrac{1}{2})^{\Re a-1} = \mathrm{O}\,(n^{\Re b})$$

as $n \to \infty$. Hence $\sum_{k=-\infty}^{\infty} \tilde{a}_{n-k}\tilde{b}_k \sim n^{a-1} \sum_{k=1}^{\infty} \tilde{b}_k$ as $n \to \infty$, a marked departure from the rule for the case $\Re b > 0$. The case where both $\Re a$ and $\Re b$ are negative is covered by a theorem of Omey [19], and will not concern us here.

Our first example, namely $\sum_k a_k\omega^k = (1-\omega)^{-a}$ and $\sum_k b_k\omega^k = (1-\omega)^{-b}$, evades the above arguments because when $\Re b < 0$ we have that

$$\sum_{k=0}^{\infty} b_k = \lim_{x\to 1^-} \sum_{k=0}^{\infty} b_k x^k = \lim_{x \to 1^-} (1-x)^{-b} = 0.$$

In this sense our first example is not generic for $\Re b < 0$; on the other hand it leads to a simple universal rule for the decay rates of convolutions. This simplicity is desirable for our purposes, provided it holds for more general classes of sequences. For $\Re b < 0$ (b not a negative integer) the sequences of our first and second examples have the same leading-order asymptotic behavior (up to a constant multiple) and yet they lead to different leading-order asymptotic behaviors for the convolution. Hence we will not be able to generalize our first example by merely imposing a specific leading-order asymptotic behavior on the sequences involved.

3.2. **Examples with Exponential Decay.** The convolution $\sum_{k=-\infty}^{\infty} a_{n-k}b_k$ can easily be made to decay exponentially by a simple modification of our first example. Thus we may choose $\sum_l a_l\omega^l = (1-\omega/\omega_0)^{-a}$, where ω_0 is a complex number with $|\omega_0| > 1$, so that $a_l = (\frac{-1}{\omega_0})^l\binom{-a}{l}$ for $l \geq 0$ and $a_l = 0$ for $l < 0$. We need the following lemma to analyze some of the cases involving different exponential rates of decay.

Lemma. *Suppose* $\sum_l a_l\omega^l = (1-\omega/\alpha_0)^{-a}$ *and* $\sum_k b_k\omega^k = (1-\omega/\beta_0)^{-b}$, *where* a, b, α_0, β_0 *are complex numbers with* $\alpha_0 \neq 0$, $\beta_0 \neq 0$, *and* $\arg\alpha_0 \neq \arg\beta_0$. *Then* *for all sufficiently large n:*

$$\sum_{k=0}^{n} a_{n-k}b_k = \frac{(1-\frac{\alpha_0}{\beta_0})^{-b}}{\alpha_0^n\Gamma(a)}\frac{\Gamma(n+a+b)}{\Gamma(n+1+b)}F(b,1-a;n+1+b;\tfrac{\beta_0}{\beta_0-\alpha_0})$$
$$+\frac{(1-\frac{\beta_0}{\alpha_0})^{-a}}{\beta_0^n\Gamma(b)}\frac{\Gamma(n+a+b)}{\Gamma(n+a+1)}F(a,1-b;n+a+1;\tfrac{\alpha_0}{\alpha_0-\beta_0}),$$

where $F(a,b;c;z)$ is the hypergeometric function and all the fractional powers are computed using the principal branch.

Proof. Note that if

$$f(\omega) = \left(1-\frac{\omega}{\alpha_0}\right)^{-a}\left(1-\frac{\omega}{\beta_0}\right)^{-b} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\omega^n$$

then by the Cauchy integral formula we have

$$\sum_{k=0}^{n} a_{n-k}b_k = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{1}{z^{n+1}}\left(1-\frac{z}{\alpha_0}\right)^{-a}\left(1-\frac{z}{\beta_0}\right)^{-b}dz,$$

where C is a circle centered at 0 with radius r_0 smaller than $|\alpha_0|$ and $|\beta_0|$ and oriented in the counterclockwise direction. This quantity is analytic in a and b, so we first attempt to evaluate it under the restrictions $\Re a < 1$, $\Re b < 1$. Let the contour C of integration be enlarged within the set $\mathbb{C}\setminus(\alpha_0[1,\infty)\cup\beta_0[1,\infty))$ to consist of most of a large circle of radius R centered at $z=0$, together with two semicircles centered at $z=\alpha_0$ and $z=\beta_0$ and four line integrals. Since $\Re a<1$ and $\Re b<1$ the contributions of the semicircles vanishes as their radii tend to 0. For n sufficiently large the contribution of the large circle also vanishes as $R\to\infty$. Thus we have

$$\frac{f^{(n)}(0)}{n!} = \frac{-1}{2\pi i}\int_1^{\infty}\frac{1}{(s\alpha_0)^{n+1}}\left(1-\frac{(s-i0^+)\alpha_0}{\alpha_0}\right)^{-a}\left(1-\frac{s\alpha_0}{\beta_0}\right)^{-b}\alpha_0\,ds$$
$$+\frac{1}{2\pi i}\int_1^{\infty}\frac{1}{(s\alpha_0)^{n+1}}\left(1-\frac{(s+i0^+)\alpha_0}{\alpha_0}\right)^{-a}\left(1-\frac{s\alpha_0}{\beta_0}\right)^{-b}\alpha_0\,ds$$
$$+\frac{-1}{2\pi i}\int_1^{\infty}\frac{1}{(s\beta_0)^{n+1}}\left(1-\frac{s\beta_0}{\alpha_0}\right)^{-a}\left(1-\frac{(s-i0^+)\beta_0}{\beta_0}\right)^{-b}\beta_0\,ds$$
$$+\frac{1}{2\pi i}\int_1^{\infty}\frac{1}{(s\beta_0)^{n+1}}\left(1-\frac{s\beta_0}{\alpha_0}\right)^{-a}\left(1-\frac{(s+i0^+)\beta_0}{\beta_0}\right)^{-b}\beta_0\,ds$$
$$=\frac{\sin(\pi a)}{\alpha_0^n\pi}\int_1^{\infty}s^{-n-1}(s-1)^{-a}\left(1-\tfrac{s\alpha_0}{\beta_0}\right)^{-b}ds$$
$$+\frac{\sin(\pi b)}{\beta_0^n\pi}\int_1^{\infty}s^{-n-1}\left(1-\tfrac{s\beta_0}{\alpha_0}\right)^{-a}(s-1)^{-b}\,ds$$
$$=\frac{\sin(\pi a)}{\alpha_0^n\pi}\int_0^{\infty}(1+s)^{-n-1}s^{-a}\left(1-\tfrac{\alpha_0}{\beta_0}-\tfrac{s\alpha_0}{\beta_0}\right)^{-b}ds$$
$$+\frac{\sin(\pi b)}{\beta_0^n\pi}\int_0^{\infty}(1+s)^{-n-1}\left(1-\tfrac{\beta_0}{\alpha_0}-\tfrac{s\beta_0}{\alpha_0}\right)^{-a}s^{-b}\,ds$$

Since $\Re a<1$, $\Re b<1$, and $n>-\Re a-\Re b$ we can apply the integral representation of the hypergeometric function (see (12), page 60 of [9]) and the reflection formula for the Gamma function (see (6.1.17) of [2]) to obtain the claimed equality. Analytic continuation then removes the restrictions on $\Re a$ and $\Re b$ for all sufficiently large n. □

So when $\sum_l a_l \omega^l = (1-\omega/\alpha_0)^{-a}$ and $\sum_k b_k \omega^k = (1-\omega/\beta_0)^{-b}$ as in the above lemma, where we also assume $|\alpha_0| > 1$ and $|\beta_0| > 1$, we can understand precisely the asymptotics of the convolution.

(1) In the case where $\alpha_0 = \beta_0 = \omega_0$ we have (assuming $a+b$ is not zero or a negative integer)
$$\sum_{k=0}^{n} a_{n-k} b_k = \frac{n^{a+b-1}}{\omega_0^n \Gamma(a+b)} \left[1 + \frac{(a+b)(a+b-1)}{2n} + \mathrm{O}\left(n^{-2}\right)\right]$$
as $n \to \infty$. If $a+b$ is zero or a negative integer then the convolution vanishes for all sufficiently large n. If $a+b=1$ we have that $\sum_{k=-\infty}^{\infty} a_{n-k} b_k = \omega_0^{-n}$ for all $n \geq 0$, so that the error term is identically zero.

(2) In the case where $|\alpha_0| < |\beta_0|$ and $\arg \alpha_0 \neq \arg \beta_0$, we have (assuming a is not zero or a negative integer)
$$\sum_{k=0}^{n} a_{n-k} b_k = \frac{n^{a-1}}{\alpha_0^n \Gamma(a)} \left(1 - \frac{\alpha_0}{\beta_0}\right)^{-b} + \mathrm{O}\left(|\alpha_0|^{-n} n^{\Re a - 2}\right) \qquad \text{as } n \to \infty.$$
If a is zero or a negative integer but b is not zero or a negative integer then
$$\sum_{k=0}^{n} a_{n-k} b_k = \frac{n^{b-1}}{\beta_0^n \Gamma(b)} \left(1 - \frac{\beta_0}{\alpha_0}\right)^{-a} + \mathrm{O}\left(|\beta_0|^{-n} n^{\Re b - 2}\right) \qquad \text{as } n \to \infty.$$
If both a and b assume nonpositive integer values then then convolution vanishes for all sufficiently large n. This result follows immediately from the previous lemma, (6.1.47) of [2], and the asymptotic estimate (see (15.7.1) of [2])
$$F(a, b; c; z) = 1 + \frac{ab}{c} z + \mathrm{O}\left(|c|^{-2}\right), \qquad \text{as } |c| \to \infty.$$
When $|\alpha_0| < |\beta_0|$ but $\arg \alpha_0 = \arg \beta_0$ and and a is not zero or a negative integer the first asymptotic result continues to hold, but the proof will be delayed until the next section.

(3) In the case where $|\alpha_0| = |\beta_0|$, $\alpha_0 \neq \beta_0$, and $\Re a > \Re b$ we have
$$\sum_{k=0}^{n} a_{n-k} b_k = \frac{n^{a-1}}{\alpha_0^n \Gamma(a)} \left(1 - \frac{\alpha_0}{\beta_0}\right)^{-b} + \mathrm{O}\left(|\alpha_0|^{-n} n^{\max\{\Re a - 2, \Re b - 1\}}\right) \qquad \text{as } n \to \infty.$$
If instead $|\alpha_0| = |\beta_0|$, $\alpha_0 \neq \beta_0$, and $\Re a = \Re b$ then
$$\sum_{k=0}^{n} a_{n-k} b_k = \frac{n^{a-1}}{\alpha_0^n \Gamma(a)} \left(1 - \frac{\alpha_0}{\beta_0}\right)^{-b} + \frac{n^{b-1}}{\beta_0^n \Gamma(b)} \left(1 - \frac{\beta_0}{\alpha_0}\right)^{-a} + \mathrm{O}\left(|\alpha_0|^{-n} n^{\Re a - 2}\right)$$
as $n \to \infty$. The same sort of subtleties when a or b is a nonpositive integer as in cases (1) or (2) hold in this case as well.

Note that even though there is an explicit exponentially decaying leading-order term, the error term does not decay exponentially faster than it, but only faster by one power of n. Although a relative error term which is exponentially small can be arranged (such as in the first case when $a+b=1$) it is a highly nongeneric situation within this family of examples. It appears to be nongeneric in general, and so we will not pursue it further, and will focus on hypotheses which follow the pattern we observe in the above example.

3.3. **Examples with Super-Exponential Decay.** If a_k and b_k decay to zero faster than any exponential function of k, then the generating functions $\mathfrak{a}(\omega) = \sum_{k=0}^{\infty} a_k\omega^k$ and $\mathfrak{b}(\omega) = \sum_{k=0}^{\infty} b_k\omega^k$ will be entire functions, as will be their product. For example, if $\mathfrak{a}(\omega) = \sin\omega$ and $\mathfrak{b}(\omega) = \cos\omega$, so that

$$a_l = \begin{cases} \frac{(-1)^m}{l!} & l = 2m+1, m \geq 0, \\ 0 & l = 2m, m \geq 0, \\ 0 & l < 0, \end{cases} \qquad b_k = \begin{cases} \frac{(-1)^m}{k!} & k = 2m, m \geq 0, \\ 0 & k = 2m+1, m \geq 0, \\ 0 & k < 0, \end{cases}$$

then $\mathfrak{a}(\omega)\mathfrak{b}(\omega) = \frac{1}{2}\sin(2\omega)$ so that

$$c_n = \sum_{k=-\infty}^{\infty} a_{n-k}b_k = \sum_{k=0}^{n} a_{n-k}b_k = \begin{cases} \frac{(-1)^m 2^{n-1}}{n!} & n = 2m+1, m \geq 0, \\ 0 & n = 2m, m \geq 0, \\ 0 & n < 0. \end{cases}$$

Sterling's asymptotic formula shows that

$$\begin{aligned} a_l &\sim \frac{(-1)^{(l-1)/2}}{\sqrt{2\pi l}}\left(\frac{e}{l}\right)^l, \qquad l \text{ odd}, \ l \to \infty. \\ b_k &\sim \frac{(-1)^{k/2}}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k, \qquad k \text{ even}, \ k \to \infty. \\ c_n &\sim \frac{(-1)^{(n-1)/2}}{2\sqrt{2\pi n}}\left(\frac{2e}{n}\right)^n, \qquad n \text{ odd}, \ n \to \infty. \end{aligned}$$

In this case we have the sharp estimates involving limits superior:

$$e = \limsup_{n\to\infty} n|a_n|^{1/n} = \limsup_{n\to\infty} n|b_n|^{1/n} \qquad 2e = \limsup_{n\to\infty} n|c_n|^{1/n}.$$

For another example consider $\mathfrak{a}(\omega) = e^{\omega^2}$ and $\mathfrak{b}(\omega) = e^{-\omega^2}$. Thus

$$a_l = \begin{cases} \frac{1}{m!} & l = 2m, m \geq 0, \\ 0 & l = 2m+1, m \geq 0, \\ 0 & l < 0, \end{cases} \qquad b_k = \begin{cases} \frac{(-1)^m}{m!} & k = 2m, m \geq 0, \\ 0 & k = 2m+1, m \geq 0, \\ 0 & k < 0, \end{cases}$$

Again by Sterling's formula

$$|a_n| = |b_n| \sim \frac{1}{\sqrt{\pi n}}\left(\frac{2e}{n}\right)^{n/2}, \qquad n \text{ even}, \ n \to \infty.$$

However $\sum_{k=-\infty}^{\infty} a_{n-k}b_k = 0$ for all $n \in \mathbb{Z}$ except $n = 0$, where $a_0b_0 = 1$, since $\mathfrak{a}(\omega)\mathfrak{b}(\omega) = 1$ for all $\omega \in \mathbb{C}$.

These examples illustrate the principle that sharp super-exponentially decaying asymptotics are very complicated; they are too varied to give completely explicit leading-order asymptotics except in extremely limited circumstances. One could impose explicit leading-order asymptotics depending on many parameters, but for which values of n (as $n \to \infty$): even, odd, all? The author knows of no results along these lines. However in section 4.4 we will consider generalizations of the above example in which an assumption of a sharp estimate of the above limit superior type on a_n and on b_n, together with a nondegeneracy condition (excluding situations like our second example), results in a computable sharp estimate of the discrete convolution c_n of the same type.

4. Estimating Discrete Convolutions

4.1. **Review of Laurent Series.** In order to generalize the above examples we consider the following. If $0 \le r_1 < r_2 \le \infty$ then let $A(r_1, r_2) = \{\omega \in \mathbb{C} \mid r_1 < |\omega| < r_2\}$, and let $D(r_2) = \{\omega \in \mathbb{C} \mid |\omega| < r_2\}$. Let $\mathcal{H}(\Omega)$ denote the set of all holomorphic functions defined on the open set $\Omega \subset \mathbb{C}$. The series identity (*) from section 2.2 is true provided both of

$$\sum_{l=-\infty}^{\infty} a_l \omega^l, \qquad \sum_{k=-\infty}^{\infty} b_k \omega^k$$

are Laurent series for functions from $\mathcal{H}(A(r_1, r_2))$ for some (the same) nonempty annulus (see Berenstein and Gay, Prop. 2.4.8, page 142). If $\omega \in A(r_1, r_2)$ and $r_1 < \rho_1 < |\omega| < \rho_2 < r_2$, then by the Cauchy integral formula

$$h(\omega) = \frac{1}{2\pi i} \int_{|z|=\rho_2} \frac{h(z)}{z-\omega}\, dz - \frac{1}{2\pi i} \int_{|z|=\rho_1} \frac{h(z)}{z-\omega}\, dz$$

for all $h \in \mathcal{H}(A(r_1, r_2))$. The first contour integral defines a function $h_2(\omega)$ which is actually analytic in $D(\rho_2)$; the second contour integral defines a function $h_1(\omega)$ which is analytic in $A(\rho_1, \infty)$. These functions are independent of the choice of ρ_1, ρ_2, and hence they are analytic on $D(r_2)$ and $A(r_1, \infty)$ respectively. In fact since $h_1(1/\omega)$ tends to 0 as $\omega \to 0$ it has a removable singularity there, and determines an analytic function on $D(1/r_1)$. We have

$$h_2(\omega) = \sum_{l=0}^{\infty} a_l \omega^l, \qquad h_1(1/\omega) = \sum_{l=1}^{\infty} a_{-l} \omega^l,$$

where $a_l = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{h(z)}{z^{l+1}}\, dz$ for all $l \in \mathbb{Z}$, where $r_1 < \rho < r_2$ is arbitrary. Thus we have the estimates

$$(**) \qquad \limsup_{l\to\infty} |a_l|^{1/l} \le (r_2)^{-1}, \qquad \limsup_{l\to\infty} |a_{-l}|^{1/l} \le r_1.$$

Conversely if $\{a_l\}_{l\in\mathbb{Z}}$ satisfies the above two estimates then the functions $h_2(\omega)$ and $h_1(1/\omega)$ defined as above are analytic on $D(r_2)$ and $D(1/r_1)$ respectively, and hence $h(\omega) = h_2(\omega) - h_1(\omega)$ is analytic on $A(r_1, r_2)$.

If $\mathcal{A}$ is any class of functions which are holomorphic at least in some annulus then we denote by $\mathcal{L}\{\mathcal{A}\}$ the collection of sequences which are the Laurent coefficients of some function from $\mathcal{A}$. Hence $\mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$ is the set of all sequences $\{a_l\}_{l\in\mathbb{Z}}$ satisfying the above two estimates (**), i.e. the collection of sequences of Laurent coefficients of functions from $\mathcal{H}(A(r_1, r_2))$.

4.2. **Asymptotics of Convolutions with Laurent Series.** We begin with some notation. Suppose $0 \le r_1 < |\omega_0| < r_2$ and $\mathfrak{a}_0(\omega) = (1 - \frac{\omega}{\omega_0})^{-a}$. Let

$$\mathfrak{a}_0 \mathcal{H}(A(r_1, r_2))$$

denote the set of all functions (and Laurent series) $\mathfrak{a}(\omega) = \sum_n a_n \omega^n = \mathfrak{a}_0(\omega)\mathfrak{h}(\omega)$, where $\mathfrak{h} \in \mathcal{H}(A(r_1, r_2))$. Suppose $\{a'_l\}_{l\in\mathbb{Z}}$ is a sequence such as our main example, which (as the reader will recall) was $a'_l = \frac{\Gamma(a+l)}{\omega_0^l \Gamma(a)\Gamma(l+1)}$ for $l \geq 0$ and $a'_l = 0$ for $l < 0$. Let

$$\{a'_l\}_{l\in\mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$$

denote the set of all sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that $a_n = \sum_k a'_{n-k} h_k$, $n \in \mathbb{Z}$, where $\mathfrak{h}(\omega) = \sum_k h_k \omega^k$ satisfies $\mathfrak{h} \in \mathcal{H}(A(r_1, r_2))$, i.e. $\{h_k\}_{k\in\mathbb{Z}} \in \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$. It is not difficult to see that

$$\mathcal{L}\{\mathfrak{a}_0 \mathcal{H}(A(r_1, r_2))\} = \{a'_l\}_{l\in\mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$$

when $\mathfrak{a}_0(\omega) = \sum_l a'_l \omega^l$ for all $r_1 < |\omega| < |\omega_0|$.

If $\{a'_l\}_{l\in\mathbb{Z}}$ is such that $\tilde{\mathfrak{a}}_0(\omega) = \sum_l a'_l \omega^l$ is analytic in $A(r_1, |\omega_0|)$, and not identically zero, then for all $\{a_n\}_{n\in\mathbb{Z}} \in \{a'_l\}_{l\in\mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$ there is a unique $\{h_k\}_{k\in\mathbb{Z}} \in \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$ such that $a_n = \sum_k a'_{n-k} h_k$ for all $n \in \mathbb{Z}$. For if $\tilde{\mathfrak{a}}_0(\omega)\mathfrak{h}(\omega) = 0$ for all $\omega \in A(r_1, |\omega_0)$ then $\mathfrak{h}(\omega) = 0$ for all $\omega \in A(r_1, r_2)$.

We now intend to show that the leading-order term of the large n asymptotics of the sequence a_n is just a constant multiple of that of the sequence a'_n. This will provide a suitable generalization of our main example.

Lemma. *Suppose $\omega_0, a \in \mathbb{C}$, $0 \leq r_1 < |\omega_0| < r_2$. Suppose $\{a'_l\}_{l\in\mathbb{Z}}$ satisfies $a'_l = 0$ for $l < 0$, $\tilde{\mathfrak{a}}_0(\omega) = \sum_l a'_l \omega^l$ is analytic in $D(0, |\omega_0|)$, and*

$$a'_l = \frac{l^{a-1}}{\omega_0^l \Gamma(a)} + \mathrm{O}\left(\frac{l^{\Re a - 2}}{|\omega_0|^l |\Gamma(a)|}\right) \qquad \textit{as } l \to \infty.$$

Suppose $\{a_n\}_{n\in\mathbb{Z}} \in \{a'_l\}_{l\in\mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$. Assume a is not 0 or a negative integer. Let $\{h_k\}_{k\in\mathbb{Z}}$ in $\mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$ such that $a_n = \sum_k a'_{n-k} h_k$ for all $n \in \mathbb{Z}$, and define $\mathfrak{h}(\omega) = \sum_{l=-\infty}^{\infty} h_l \omega^l$ for all $r_1 < |\omega| < r_2$. Then

$$a_n = \frac{\mathfrak{h}(\omega_0) n^{a-1}}{\omega_0^n \Gamma(a)} + \mathrm{O}\left(|\omega_0|^{-n} n^{\Re a - 2}\right) \qquad \textit{as } n \to \infty.$$

However, if a is equal to 0 or a negative integer then

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq r_2^{-1};$$

in particular $\omega_0^n a_n$ will decay exponentially fast as $n \to \infty$ instead of having power-like behavior.

Proof. To extract the large n asymptotics of $a_n = \sum_{l=-\infty}^{n} a'_{n-l} h_l$ consider positive numbers m, m' to depend on n such that $m \to \infty$, $m' \to \infty$, and $m/n \to 0$, $m'/n \to 0$ as $n \to \infty$. Then

$$\omega_0^n a_n = \sum_{l=-m'}^{m} \omega_0^{n-l} a'_{n-l} h_l \omega_0^l + \sum_{l=m+1}^{n} \omega_0^{n-l} a'_{n-l} h_l \omega_0^l + \sum_{l=m'+1}^{\infty} \omega_0^{n+l} a'_{n+l} h_{-l} \omega_0^{-l}.$$

Let the three terms on the right-hand-side of this equation be denoted I, II, and III respectively. Since a is not 0 or a negative integer we have the estimates

$$\left|\omega_0^k a'_k\right| \leq \mathrm{O}\left((1+k)^{\Re a - 1}\right), \qquad \omega_0^k a'_k = \frac{k^{a-1}}{\Gamma(a)} + \mathrm{O}\left(k^{\Re a - 2}\right).$$

The implied constants will depend on a but are independent of $k \geq 0$ in the first case and of $k > 0$ in the second case. Note that for $-m' \leq l \leq m$ we have that $n - l = n(1 - l/n) \to \infty$ as $n \to \infty$. Using the second estimate we see that

$$\omega_0^{n-l} a'_{n-l} = \frac{(n-l)^{a-1}}{\Gamma(a)} + \mathrm{O}\left((n-l)^{\Re a - 2}\right) = \frac{n^{a-1}}{\Gamma(a)} + \mathrm{O}\left((1+|l|)n^{\Re a - 2}\right).$$

Applying this in term I we get

$$\begin{aligned}\sum_{l=-m'}^{m} \omega_0^{n-l} a'_{n-l} h_l \omega_0^l &= \frac{n^{a-1}}{\Gamma(a)} \sum_{l=-m'}^{m} h_l \omega_0^l + \mathrm{O}\left(n^{\Re a - 2}\right) \sum_{l=-m'}^{m} (1+|l|)|h_l \omega_0^l| \\ &= \frac{n^{a-1}}{\Gamma(a)} \sum_{l=-\infty}^{\infty} h_l \omega_0^l + \mathrm{O}\left(n^{\Re a - 1}\right) \left[\sum_{l=-\infty}^{-m'-1} |h_l \omega_0^l| + \sum_{l=m+1}^{\infty} |h_l \omega_0^l| \right] \\ &\quad + \mathrm{O}\left(n^{\Re a - 2}\right) \sum_{l=-\infty}^{\infty} (1+|l|)|h_l \omega_0^l|.\end{aligned}$$

Our hypotheses imply that

$$\limsup_{l\to\infty} |h_l \omega_0^l|^{1/l} \leq |\omega_0| r_2^{-1} < 1, \qquad \limsup_{l\to\infty} |h_{-l} \omega_0^{-l}|^{1/l} \leq |\omega_0|^{-1} r_1 < 1.$$

Hence there exist numbers λ, λ' and an integer $L > 0$ such that $|\omega_0| r_2^{-1} < \lambda < 1$, $|\omega_0|^{-1} r_1 < \lambda' < 1$, and for all $l \geq L$ we have

$$|h_l \omega_0^l| \leq \lambda^l, \qquad |h_{-l} \omega_0^{-l}| \leq (\lambda')^l.$$

For n large enough so that $m + 1 \geq L$ and $m' + 1 \geq L$ we have

$$\sum_{l=m+1}^{\infty} |h_l \omega_0^l| \leq \frac{\lambda^{m+1}}{1-\lambda}, \qquad \sum_{l=m'+1}^{\infty} |h_{-l} \omega_0^{-l}| \leq \frac{(\lambda')^{m'+1}}{1-\lambda'}.$$

It is clearly possible to choose m, m' (consistent with our earlier assumptions) such that both λ^m and $(\lambda')^{m'}$ are $\mathrm{O}\left(n^{-1}\right)$ as $n \to \infty$. It should also be clear from this that $\sum_{l=-\infty}^{\infty} (1+|l|)|h_l \omega_0^l|$ converges. Hence term I satisfies:

$$\sum_{l=-m'}^{m} \omega_0^{n-l} a'_{n-l} h_l \omega_0^l = \frac{n^{a-1}}{\Gamma(a)} \mathfrak{h}(\omega_0) + \mathrm{O}\left(n^{\Re a - 2}\right),$$

as $n \to \infty$.

For sufficiently large n we find that term II can be estimated as follows:

$$\begin{aligned}\left| \sum_{l=m+1}^{n} \omega_0^{n-l} a'_{n-l} h_l \omega_0^l \right| &\leq C \sum_{l=m+1}^{n} (1+n-l)^{\Re a - 1} \lambda^l \\ &= \sum_{l=0}^{n-m-1} (1+l)^{\Re a - 1} \lambda^{n-l} \\ &= \lambda^m \sum_{l=0}^{n-m-1} (1+l)^{\Re a - 1} \lambda^{n-m-l}.\end{aligned}$$

Abbreviate $\tilde{n} = n - m$ and $\alpha = \Re a$. Note that $\tilde{n} = [\![\frac{\tilde{n}}{2}]\!] + [\![\frac{\tilde{n}+1}{2}]\!]$. Then

$$\begin{aligned}
\sum_{l=0}^{\tilde{n}-1} (1+l)^{\alpha-1}\lambda^{\tilde{n}-l} &= \sum_{l=0}^{[\![\frac{\tilde{n}-1}{2}]\!]} (1+l)^{\alpha-1}\lambda^{\tilde{n}-l} + \sum_{l=1}^{[\![\frac{\tilde{n}}{2}]\!]} (1+\tilde{n}-l)^{\alpha-1}\lambda^{l} \\
&\le \lambda^{1+[\![\frac{\tilde{n}}{2}]\!]} \sum_{l=0}^{[\![\frac{\tilde{n}-1}{2}]\!]} (1+l)^{\alpha-1} + \left(\sum_{l=1}^{[\![\frac{\tilde{n}}{2}]\!]} \lambda^{l}\right) \begin{cases} (1+[\![\frac{\tilde{n}+1}{2}]\!])^{\alpha-1} & \text{if } \alpha < 1 \\ \tilde{n}^{\alpha-1} & \text{if } \alpha \ge 1 \end{cases} \\
&\le \lambda^{1+[\![\frac{\tilde{n}}{2}]\!]} \sum_{l=1}^{[\![\frac{\tilde{n}+1}{2}]\!]} l^{\alpha-1} + \mathrm{O}\left(\tilde{n}^{\alpha-1}\right)\frac{\lambda}{1-\lambda}.
\end{aligned}$$

Since $\lambda < 1$ we clearly have

$$\lambda^{1+[\![\frac{\tilde{n}}{2}]\!]} \sum_{l=1}^{[\![\frac{\tilde{n}+1}{2}]\!]} l^{\alpha-1} = \lambda^{1+[\![\frac{\tilde{n}}{2}]\!]} \left\{\begin{matrix} \mathrm{O}(1) & \text{if } \alpha < 0 \\ \mathrm{O}(\ln \tilde{n}) & \text{if } \alpha = 0 \\ \mathrm{O}(\tilde{n}^{\alpha}) & \text{if } \alpha > 0 \end{matrix}\right\} = \mathrm{O}\left(\tilde{n}^{\alpha-1}\right) \qquad \text{as } n \to \infty.$$

In fact this part is exponentially small. Recalling that $\lambda^m = \mathrm{O}\left(n^{-1}\right)$ we see that term II is $\mathrm{O}\left(n^{\alpha-2}\right)$ as $n \to \infty$.

For sufficiently large n we can estimate term III as follows:

$$\begin{aligned}
\left|\sum_{l=m'+1}^{\infty} \omega_0^{n+l} a'_{n+l} h_{-l} \omega_0^{-l}\right| &\le C \sum_{l=m'+1}^{\infty} (n+l)^{\alpha-1}(\lambda')^{l} \\
&\le C(\lambda')^{-n} \sum_{l=n+m'+1}^{\infty} l^{\alpha-1}(\lambda')^{l} \\
&\le C(\lambda')^{-n} \int_{n+m'}^{\infty} t^{\alpha-1}(\lambda')^{t}\, dt
\end{aligned}$$

Suppose $\lambda' = e^{-\beta}$ for $\beta > 0$. Then we have

$$\int_{n+m'}^{\infty} t^{\alpha-1}(\lambda')^{t}\, dt = \beta^{-\alpha} \int_{\beta(n+m')}^{\infty} t^{\alpha-1} e^{-t}\, dt = \beta^{-\alpha}\Gamma(\alpha, \beta(n+m')).$$

The function $\Gamma(\alpha, z)$ is the incomplete Gamma function and it has the well-known asymptotic expansion $\Gamma(\alpha, z) \sim z^{\alpha-1}e^{-z}[1 + (\alpha-1)z^{-1} + \dots]$ as $z \to \infty$. This means that $|\Gamma(\alpha, z)| = \mathrm{O}\left(|z|^{\alpha-1}e^{-|z|}\right)$ as $z \to \infty$. Thus term III is bounded by

$$C' e^{\beta n}\beta^{-\alpha}[\beta(n+m')]^{\alpha-1} e^{-\beta(n+m')} = C'\beta^{-1}(n+m')^{\alpha-1}(\lambda')^{m'} = \mathrm{O}\left(n^{\alpha-2}\right)$$

as $n \to \infty$, as desired. The proof of the power-like result is now complete.

Now suppose a is either 0 or a negative integer. In this case we still have $\mathfrak{a}(\omega) = \sum_l a_l \omega^l = \tilde{\mathfrak{a}}_0(\omega)\mathfrak{h}(\omega)$, where $\mathfrak{h} \in \mathcal{H}(A(r_1, r_2))$, and $\tilde{\mathfrak{a}}_0(\omega)$ is a polynomial. Hence $\mathfrak{a} \in \mathcal{H}(A(r_1, r_2))$, and the stated estimate on $|a_n|$ is therefore clear. □

It is interesting to relate this lemma to the Tauberian theorems (c.f. [24], and Chapter 9 of [22]). The relation is especially clear when $\tilde{\mathfrak{a}}_0(\omega) = \mathfrak{a}_0(\omega) = (1 - \omega/\omega_0)^{-a}$. In that case $\mathfrak{a}(\omega) = \sum a_n \omega^n = \tilde{\mathfrak{a}}_0(\omega)\mathfrak{h}(\omega)$ is a precise statement of the behavior of $\mathfrak{a}(\omega)$ near the point $\omega = \omega_0$, which is analogous to a combination of an Abel convergence condition and a Tauberian condition. The conclusion of a typical

Tauberian theorem (Proposition 1, page 178, [24]) is that some Cesaro mean of the sequence a_n converges as $n \to \infty$.

4.3. **Exponential decay of convolutions.** We assume $|\alpha_0| > 1$, $|\beta_0| > 1$, $r_1 < |\alpha_0| < r_2$, and $r_1 < |\beta_0| < r_2$. Now suppose $\mathfrak{a}_0(\omega) = (1 - \frac{\omega}{\alpha_0})^{-a}$ and $\mathfrak{b}_0(\omega) = (1 - \frac{\omega}{\beta_0})^{-b}$, and

$$\{a_l\}_{l \in \mathbb{Z}} \in \mathcal{L}\{\mathfrak{a}_0 \mathcal{H}(A(r_1, r_2))\} \text{ and } \{b_k\}_{k \in \mathbb{Z}} \in \mathcal{L}\{\mathfrak{b}_0 \mathcal{H}(A(r_1, r_2))\}.$$

The lemma of the previous subsection allows us to understand the leading-order asymptotics of a_n and of b_n as $n \to \infty$. Now we can use the same lemma to understand the leading-order asymptotics of $\sum_{k=-\infty}^{\infty} a_{n-k} b_k$ as $n \to \infty$. Let $\mathfrak{a}(\omega) = \sum_l a_l \omega^l = \mathfrak{a}_0(\omega)\mathfrak{h}(\omega)$ and $\mathfrak{b}(\omega) = \sum_k b_k \omega^k = \mathfrak{b}_0(\omega)\mathfrak{l}(\omega)$, where $\mathfrak{h}, \mathfrak{l} \in \mathcal{H}(A(r_1, r_2))$. Then

$$\mathfrak{a}(\omega)\mathfrak{b}(\omega) = \sum_n (\sum_{k=-\infty}^{\infty} a_{n-k} b_k) \omega^n = \mathfrak{a}_0(\omega)\mathfrak{b}_0(\omega)\mathfrak{h}(\omega)\mathfrak{l}(\omega).$$

We have several cases.

(1) If $\alpha_0 = \beta_0 = \omega_0$ then $\{\sum_k a_{n-k} b_k\}_{n \in \mathbb{Z}} \in \mathcal{L}\{(1 - \frac{\cdot}{\omega_0})^{-a-b} \mathcal{H}(A(r_1, r_2))\}$ and the lemma may be applied to extract the large n asymptotics of this convolution.

(2) If $|\alpha_0| < |\beta_0|$ then $\{\sum_k a_{n-k} b_k\}_{n \in \mathbb{Z}} \in \mathcal{L}\{\mathfrak{a}_0 \mathcal{H}(A(r_1, |\beta_0|))\}$ and again the lemma may be applied.

(3) If $|\alpha_0| = |\beta_0|$ and $\alpha_0 \neq \beta_0$ then we may apply the lemma of section 3.2 to write $\{\sum_k a_{n-k} b_k\}_{n \in \mathbb{Z}}$ as the sum of two sequences, one from $\{a'_l\}_{l \in \mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$ and the other from $\{a''_l\}_{l \in \mathbb{Z}} * \mathcal{L}\{\mathcal{H}(A(r_1, r_2))\}$, where

$$a'_l = \frac{1}{\alpha_0^l \Gamma(a)} \frac{\Gamma(l + a + b)}{\Gamma(l + 1 + b)} F(b, 1 - a; l + 1 + b; \tfrac{\beta_0}{\beta_0 - \alpha_0})$$
$$a''_l = \frac{1}{\beta_0^l \Gamma(b)} \frac{\Gamma(l + a + b)}{\Gamma(l + a + 1)} F(a, 1 - b; l + a + 1; \tfrac{\alpha_0}{\alpha_0 - \beta_0})$$

whenever $l \geq 0$ and the above make sense, and $a'_l = a''_l = 0$ otherwise. Then we apply the lemma of section 4.2 to extract the large n asymtotics of each of these two convolutions, and add the results.

In cases (1) and (2) we merely obtain a constant multiple of the large n asymptotics for the example $\mathfrak{a}_0(\omega)\mathfrak{b}_0(\omega) = (1 - \omega/\alpha_0)^{-a}(1 - \omega/\beta_0)^{-b}$ discussed in section 3.2.

4.4. **Super-exponential decay of convolutions.** Here will will consider classes of sequences $\{a_l\}$ and $\{b_k\}$ such that their discrete convolution will satisfy a computable sharp super-exponential decay estimate.

Definition. Suppose ρ, τ are positive real numbers. The entire function $\mathfrak{a}(\omega)$ is said to be of *order* ρ if

$$\rho = \limsup_{|\omega| \to \infty} \frac{\ln \ln |\mathfrak{a}(\omega)|}{\ln |\omega|}.$$

The entire function $\mathfrak{a}(\omega)$ is said to be of *order* ρ and of *type* τ if

$$\tau = \limsup_{|\omega| \to \infty} \frac{\ln |\mathfrak{a}(\omega)|}{|\omega|^\rho}.$$

The *Lindelöf indicator function* $h_{\mathfrak{a},\rho}\colon S^1 \to [-\infty, \infty]$ of order ρ for $\mathfrak{a}(\omega)$ is defined to be

$$h_{\mathfrak{a},\rho}(e^{i\theta}) = \limsup_{r\to\infty} \frac{\ln|\mathfrak{a}(re^{i\theta})|}{r^\rho}, \qquad 0 \le \theta < 2\pi.$$

Lindelöf indicator functions of entire functions of order ρ and type τ are known to be continuous, and $\tau = \max_\theta h_{\mathfrak{a},\rho}(e^{i\theta})$ (see [5], Props. 5.2.2 and 5.2.5).

It is a well-known fact (Prop. 4.5.4, page 355 of [4]; Thm 2, page 4 of [12]) that if $0 < \rho, \tau < \infty$ then $\mathfrak{a}(\omega) = \sum_k a_k\omega^k$ is an entire function of order ρ and of type τ if and only if

$$\tau\rho e = \limsup_{n\to\infty} n|a_n|^{\rho/n}.$$

In particular for all $\epsilon > 0$

- there exists $m \ge 1$ such that for all $n \ge m$ we have $|a_n| \le \left[\frac{\tau\rho e+\epsilon}{n}\right]^{n/\rho}$;
- we have $|a_n| \ge \left[\frac{\tau\rho e-\epsilon}{n}\right]^{n/\rho}$ for infinitely many n.

This is the particular type of sharp super-exponential decay estimate that we wish to study.

Suppose that $\mathfrak{a}(\omega) = \sum_k a_k\omega^k$ is an entire function of order ρ_1 and of type τ_1 and $\mathfrak{b}(\omega) = \sum_k b_k\omega^k$ is an entire function of order ρ_2 and of type τ_2, where $\rho_1, \rho_2, \tau_1, \tau_2$ are positive real numbers. We need to be able to compute the order and type of $\mathfrak{a}(\omega)\mathfrak{b}(\omega)$. Suppose first that $\rho_1 > \rho_2$. Then by Theorem 12 of [12] we have that $\mathfrak{a}(\omega)\mathfrak{b}(\omega)$ has order ρ_1 and type τ_1. Hence the convolution $c_n = \sum_{k=0}^n a_{n-k}b_k$ satisfies the same sharp super-exponential decay estimate that a_k does.

Now suppose $\rho_1 = \rho_2 = \rho$. In general it can be shown that

$$h_{\mathfrak{ab},\rho}(e^{i\theta}) \le h_{\mathfrak{a},\rho}(e^{i\theta}) + h_{\mathfrak{b},\rho}(e^{i\theta})$$

for all θ. But there is a reasonable additional assumption on the entire functions $\mathfrak{a}$ and $\mathfrak{b}$ (i.e. completely regular growth, defined below) which allows us to conclude that equality (rather than inequality) holds in this relation, and which allows us to compute $h_{\mathfrak{a},\rho}(e^{i\theta})$ and $h_{\mathfrak{b},\rho}(e^{i\theta})$ in terms of the distribution of the zeros of the functions $\mathfrak{a}$ and $\mathfrak{b}$. Thus under this assumption the type of $\mathfrak{a}(\omega)\mathfrak{b}(\omega)$ is computable as the maximum value of $h_{\mathfrak{a},\rho} + h_{\mathfrak{b},\rho}$; again this allows sharp super-exponential decay estimates to be found for the convolution $c_n = \sum_{k=0}^n a_{n-k}b_k$.

Recall the *Hadamard Factorization Theorem* (Theorem 4.6.15 of [4]) has the following statement.

Theorem. *Suppose $\mathfrak{a}(\omega)$ is an entire function of order $\rho \in (0,\infty)$ and $\{\omega_n\}_{n\ge1}$ is a listing of the nonzero roots of $\mathfrak{a}(\omega)$, where each root is repeated in the list according to its multiplicity, and where the sequence $|\omega_n|$ is nondecreasing. Define $C_\mathfrak{a} = \{\lambda > 0 \mid \sum_{n\ge1} |\omega_n|^{-\lambda} < \infty\}$. Then there exists an integer $0 \le p \le [\![\rho]\!]$ such that $p+1 \in C_\mathfrak{a}$ and $p \notin C_\mathfrak{a}$. Furthermore*

$$\mathfrak{a}(\omega) = \omega^m \exp\left(\sum_{j=0}^{[\![\rho]\!]} c_j\omega^j\right) \prod_{n\ge1} G\left(\frac{\omega}{\omega_n}, p\right),$$

where $m \ge 0$ is the multiplicity of zero as a root of $\mathfrak{a}(\omega)$, $c_0, c_1, \ldots, c_{[\![\rho]\!]}$ are complex constants, and $G(z,p) = (1-z)\exp(\sum_{j=1}^p z^j/j)$.

This theorem shows a relation between the order of an entire function and an aspect of the distribution of its roots. Let q denote the degree of the polynomial

$\sum_{j=0}^{[\![\rho]\!]} c_j\omega^j$. When $q \geq 1$ the entire function $\mathfrak{e}(\omega) = \exp\left(\sum_{j=0}^{q} c_j\omega^j\right)$ has order q and type $|c_q|$; its Lindelöf indicator function is $h_{\mathfrak{e},q}(e^{i\theta}) = |c_q|\cos(q\theta + \arg c_q)$. If $\inf C_{\mathfrak{a}} > 0$ the entire function $\mathfrak{c}(\omega) = \prod_{n\geq 1} G\left(\frac{\omega}{\omega_n}, p\right)$ has order $\inf C_{\mathfrak{a}}$ (Theorem 7 of [12]), but its type and Lindelöf indicator function are more complicated to express (see below). If the entire function $\mathfrak{a}(\omega)$ has order $\rho \in (0,\infty)$ and type $\tau \in (0,\infty)$ then $\rho = \max\{q, \inf C_{\mathfrak{a}}\}$ and by Theorem 15 of [12] there are four cases to consider.

(1) $q > \inf C_{\mathfrak{a}}$. We have $\tau = |c_q|$ and $h_{\mathfrak{a},\rho} = h_{\mathfrak{e},q}$.
(2) $q = \inf C_{\mathfrak{a}}$ and $\rho = p+1$. We have $\tau = |c_q|$ and $h_{\mathfrak{a},\rho} = h_{\mathfrak{e},q}$.
(3) $q = \inf C_{\mathfrak{a}}$ and $\rho = p$. See below for a formula for $h_{\mathfrak{a},\rho}$.
(4) $q < \inf C_{\mathfrak{a}}$. We always have $p = [\![\rho]\!]$ and $h_{\mathfrak{a},\rho} = h_{\mathfrak{c},\rho}$. See below for a formula for $h_{\mathfrak{c},\rho}$.

Thus it remains to compute the Lindelöf indicator function of $\mathfrak{a}$ in the last two cases when $p = [\![\rho]\!]$.

We say the set $\{\omega_n\}_{n\geq 1}$ is *regularly distributed relative to* ρ if there exists a countable set $N \subset S^1$ such that for all $\alpha < \beta$ in $\mathbb{R}$, where $e^{i\alpha} \notin N$ and $e^{i\beta} \notin N$, we have that

$$\mu((\alpha,\beta]) \stackrel{\text{def}}{=} 2\pi\rho \lim_{r\to\infty} \frac{|\{n \geq 1 \mid \alpha < \arg\omega_n \leq \beta, |\omega_n| < r\}|}{r^\rho}$$

exists as a finite number; when ρ is an integer we also require the existence of

$$\lim_{r\to\infty} \sum_{|\omega_n|<r} \omega_n^{-\rho}$$

as a finite number. When the set $\{\omega_n\}_{n\geq 1}$ is regularly distributed relative to ρ then the above defines a measure μ on $\mathbb{R}$. It is clear from the Hadamard Factorization Theorem that there is a rich supply of entire functions of order ρ whose roots are regularly distributed relative to ρ. Theorems 1 and 2 (page 90–92) of Levin [12] allow us to explicitly compute the Lindelöf indicator functions in this case:

- If ρ is not an integer and $p = [\![\rho]\!]$ then

$$h_{\mathbf{a},\rho}(e^{i\theta}) = \frac{\pi}{\sin\pi\rho} \int_{(\theta-2\pi,\theta]} \cos\rho(\theta - \psi - \pi)\,\mu(d\psi).$$

- If ρ is an integer and $p = \rho$ then

$$h_{\mathbf{a},\rho}(e^{i\theta}) = \tau_{\mathfrak{a}}\cos(\rho\theta + \theta_{\mathfrak{a}}) - \int_{(\theta-2\pi,\theta]} (\theta-\psi)\sin[\rho(\theta-\psi)]\,\mu(d\psi),$$

  where $\tau_{\mathfrak{a}} \geq 0$ and $\theta_{\mathfrak{a}} \in \mathbb{R}$ are such that

$$\tau_{\mathfrak{a}} e^{i\theta_{\mathfrak{a}}} = c_\rho + \frac{1}{\rho} \lim_{r\to\infty} \sum_{|\omega_n|<r} \omega_n^{-\rho},$$

  and where c_ρ appears in the Hadamard factorization of $\mathfrak{a}$. (This corrects some misprints in [12].)

Entire functions $\mathfrak{a}$ of order ρ whose roots are regularly distributed relative to ρ are said to be of *completely regular growth* because, again by a theorem of Levin [12], there exists a set $E \subset (0,\infty)$ such that $\lim_{r\to\infty} |E \cap (0,r)|/r = 0$ and $h_{\mathbf{a},\rho}(e^{i\theta}) = \lim_{r\to\infty, r\notin E} r^{-\rho}\ln|\mathfrak{a}(re^{i\theta})|$ uniformly on S^1. In fact Levin proves that this growth condition is *equivalent* to the regular distribution of the roots relative to ρ.

If both $\mathfrak{a}$ and $\mathfrak{b}$ are entire of order ρ and one of $\mathfrak{a}$ or $\mathfrak{b}$ has completely regular growth then we have the equality $h_{\mathfrak{ab},\rho} = h_{\mathfrak{a},\rho} + h_{\mathfrak{b},\rho}$. If both are of completely regular growth this equality also clearly follows from the Hadamard factorization and from the above explicit formulae.

Thus if $\mathfrak{a}(\omega) = \sum_k a_k \omega^k$ is an entire function of order ρ and of type τ_1 and $\mathfrak{b}(\omega) = \sum_k b_k \omega^k$ is an entire function of order ρ and of type τ_2, where ρ, τ_1, τ_2 are positive real numbers, and if one or both of $\mathfrak{a}$ or $\mathfrak{b}$ are of completely regular growth and $\tau = \max_{0 \le \theta < 2\pi}[h_{\mathfrak{a},\rho}(e^{i\theta}) + h_{\mathfrak{b},\rho}(e^{i\theta})] > 0$ then $\mathfrak{ab}$ is an entire function of order ρ and type τ. Hence $c_n = \sum_{k=0}^n a_{n-k} b_k$ satisfies the sharp super-exponential decay estimate with order ρ and type τ. If $\tau \le 0$ we will not assert anything about the decay of the convolution.

Before we leave this topic let us return to the two examples discussed in section 3.3. If $\mathfrak{a}(\omega) = \sin\omega$ and $\mathfrak{b}(\omega) = \cos\omega$ then it is not difficult to see that both functions are of order $\rho = 1$ with the same Lindelöf indicator function $h_{\mathfrak{a},1}(e^{i\theta}) = h_{\mathfrak{b},1}(e^{i\theta}) = |\sin\theta|$. Hence both functions are of type 1. The Hadamard factorizations are as follows:

$$\sin\omega = \omega \prod_{k=1}^{\infty} \left[\left(1 - \frac{\omega}{k\pi}\right) e^{\frac{\omega}{k\pi}} \left(1 - \frac{\omega}{-k\pi}\right) e^{-\frac{\omega}{k\pi}} \right]$$

$$\cos\omega = \prod_{k=1}^{\infty} \left\{ \left[1 - \frac{\omega}{(k-\frac{1}{2})\pi}\right] e^{\frac{\omega}{(k-\frac{1}{2})\pi}} \left[1 - \frac{\omega}{-(k-\frac{1}{2})\pi}\right] e^{-\frac{\omega}{(k-\frac{1}{2})\pi}} \right\}$$

In both cases $C_{\mathfrak{a}} = C_{\mathfrak{b}} = (1,\infty)$ and $p = 1$; also in both cases the zeros are regularly distributed relative to $\rho = 1$, $c_0 = c_1 = 0$, and $\mu = 2\sum_{k=-\infty}^{\infty} \delta_{k\pi}$, where δ_θ is a Dirac delta measure on $\mathbb{R}$ located at θ. Also $\tau_{\mathfrak{a}} = \tau_{\mathfrak{b}} = 0$. Both $\mathfrak{a}$ and $\mathfrak{b}$ are of completely regular growth and the alternate formula for the indicator functions is easy to check in this case:

$$|\sin\theta| = -\int_{(\theta-2\pi,\theta]} (\theta-\psi)\sin[\rho(\theta-\psi)]\,\mu(d\psi).$$

Thus the indicator function of the product $\mathfrak{ab}$ is $2|\sin\theta|$, and the product is of order 1 and of type $\max_\theta 2|\sin\theta| = 2 > 0$.

The other example is $\mathfrak{a}(\omega) = e^{\omega^2}$ and $\mathfrak{b}(\omega) = e^{-\omega^2}$. Both are of order 2 and of type 1. $h_{\mathfrak{a},2}(e^{i\theta}) = \cos(2\theta)$ and $h_{\mathfrak{b},2}(e^{i\theta}) = -\cos(2\theta)$. Both functions have no roots so $C_{\mathfrak{a}} = C_{\mathfrak{b}} = (0,\infty)$ and $p = 0$; the infinite products in the Hadamard factorization are empty, so have the value 1; in both cases $\mu = 0$. For $\mathfrak{a}$ we have $(c_0, c_1, c_2) = (0,0,1)$, $\tau_{\mathfrak{a}} = 1$, $\theta_{\mathfrak{a}} = 0$; and for $\mathfrak{b}$ we have $(c_0, c_1, c_2) = (0,0,-1)$, $\tau_{\mathfrak{b}} = 1$, $\theta_{\mathfrak{b}} = \pi$. Thus the indicator functions have the alternate formulae: $\tau_{\mathfrak{a}}\cos(\rho\theta + \theta_{\mathfrak{a}}) = \cos(2\theta)$, and $\tau_{\mathfrak{b}}\cos(\rho\theta + \theta_{\mathfrak{b}}) = -\cos(2\theta)$. Since both $\mathfrak{a}$ and $\mathfrak{b}$ have completely regular growth the indicator function of the product $\mathfrak{ab}$ is $\cos(2\theta) - \cos(2\theta) = 0$. The maximum value of this is 0, which is not positive. Hence we cannot claim in this case even that the product function has order 2; in fact in this case the product function is the constant 1.

5. Asymptotics of the Discrete Time Correlation

5.1. **Asymptotics of Correlation for Hölder functions.** We return now to the setting and notation of section 2.2. First we will explore the meaning of the assumptions we have imposed in order to understand discrete convolutions. Suppose

$a_k = \hat{f}(\mathfrak{P}^k(\mathbf{m}_0))$ for all $k \in \mathbb{Z}$. Then the assumption

$$\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{L}\{(1 - \tfrac{\cdot}{\omega_0})^{-a}\mathcal{H}(A(r_1, r_2))\},$$

where $|\omega_0| > 1$ and where a is not zero or a negative integer, implies very precise information on a particular measure of the regularity of f. Note that $\mathbf{m}_0 = \mathbf{v}_+\alpha_+ + \mathbf{v}_-\alpha_-$ for some real constants α_+, α_-, where $\mathbf{v}_\pm$ are the eigenvectors of $\mathfrak{P}$ with eigenvalues $\lambda_\pm$ (see section 2.2). Hence $\mathbf{m} = \mathfrak{P}^k(\mathbf{m}_0) = \mathbf{v}_+\lambda_+^k\alpha_+ + \mathbf{v}_-\lambda_-^k\alpha_-$ and hence $\|\mathbf{m}\| \sim \|\mathbf{v}_+\alpha_+\|\lambda_+^k$ as $k \to \infty$. Therefore $\ln\|\mathbf{m}\| - \ln\|\mathbf{v}_+\alpha_+\| \sim k\ln\lambda_+$ as $k \to \infty$. It follows that

$$\begin{aligned} |\omega_0^{-k}| = |\omega_0|^{-k} &\sim |\omega_0|^{\frac{\ln\|\mathbf{v}_+\alpha_+\| - \ln\|\mathbf{m}\|}{\ln\lambda_+}} \\ &\sim |\omega_0|^{\frac{\ln\|\mathbf{v}_+\alpha_+\|}{\ln\lambda_+}} \exp\left(-\frac{\ln\|\mathbf{m}\|}{\ln\lambda_+}\ln|\omega_0|\right) \\ &\sim |\omega_0|^{\frac{\ln\|\mathbf{v}_+\alpha_+\|}{\ln\lambda_+}} \|\mathbf{m}\|^{-\frac{\ln|\omega_0|}{\ln\lambda_+}} \qquad \text{as } k \to \infty. \end{aligned}$$

Let $\gamma = \ln|\omega_0|/\ln\lambda_+$. The assumption

$$\sum_{k=-\infty}^{\infty} \hat{f}(\mathfrak{P}^k(\mathbf{m}_0))\omega^k = \left(1 - \frac{\omega}{\omega_0}\right)^{-a} \mathfrak{h}(\omega), \qquad r_1 < |\omega| < |\omega_0|,$$

for some $\mathfrak{h} \in \mathcal{H}(A(r_1, r_2))$ implies (see the lemma of section 4) that

$$\hat{f}(\mathbf{v}_+\lambda_+^k\alpha_+ + \mathbf{v}_-\lambda_-^k\alpha_-) = \frac{\mathfrak{h}(\omega_0)k^{a-1}}{\omega_0^k\Gamma(a)} + \mathrm{O}\left(|\omega_0|^{-k}k^{\Re a-2}\right), \qquad \text{as } k \to \infty.$$

Thus $\hat{f}(\mathbf{m})$ has a particular behavior as $\|\mathbf{m}\| \to \infty$ along the orbit $\{\mathfrak{P}^k(\mathbf{m}_0)\}_{k\in\mathbb{Z}}$. This behavior is consistent with the following asymptotic decay estimate:

$$|\hat{f}(\mathbf{m})| \sim \frac{|\mathfrak{h}(\omega_0)|}{|\Gamma(a)|}|\omega_0|^{\frac{\ln\|\mathbf{v}_+\alpha_+\|}{\ln\lambda_+}} \|\mathbf{m}\|^{-\gamma} \left[\frac{\ln\|\mathbf{m}\|}{\ln\lambda_+}\right]^{\Re a-1} = c\|\mathbf{m}\|^{-\gamma}[\ln\|\mathbf{m}\|]^{\Re a-1}$$

as $\|\mathbf{m}\| \to \infty$ along the given orbit, where the constant c depends on several quantities. Power-law decay of $|\hat{f}(\mathbf{m})|$ is associated with Hölder-type regularity of f of degree γ, although we only need to impose this sort of regularity in the direction of $\pm\mathbf{v}_+$. The number a introduces a logarithmic modification of this power-law behavior, which has a relatively subtle effect on the large-$\|\mathbf{m}\|$ asymptotics of $|\hat{f}(\mathbf{m})|$.

Now we can state our main result on the discrete time problem.

Theorem. *Suppose* $\mathbf{m}_0 \in S_1 \cup S_2$ *and* $f, g\colon \mathbb{T}^2 \to \mathbb{R}$ *satisfy* $\hat{f}(\mathbf{m}) = 0$ *for all* $\mathbf{m} \in \mathbb{Z}^2\backslash[\mathcal{O}(\mathbf{m}_0)\cup\mathcal{O}(-\mathbf{m}_0)\cup\{\mathbf{0}\}]$. *Suppose* $\alpha_0, \beta_0, a, b \in \mathbb{C}$, *where* $|\alpha_0| > 1, |\beta_0| > 1$, *and let* r_1, r_2 *be nonnegative real numbers such that* $r_1 < |\alpha_0| < r_2$ *and* $r_1 < |\beta_0| < r_2$. *Suppose*

$$\begin{aligned} &\{\hat{f}(\mathfrak{P}^k(\mathbf{m}_0))\}_{k\in\mathbb{Z}} \in \mathcal{L}\{(1 - \tfrac{\cdot}{\alpha_0})^{-a}\mathcal{H}(A(r_1, r_2))\} \text{ and} \\ &\{\overline{\hat{g}(\mathfrak{P}^{-k}(\mathbf{m}_0))}\}_{k\in\mathbb{Z}} \in \mathcal{L}\{(1 - \tfrac{\cdot}{\beta_0})^{-b}\mathcal{H}(A(r_1, r_2))\}. \end{aligned}$$

Define

$$\mathfrak{h}(\omega) = \left(1 - \frac{\omega}{\alpha_0}\right)^a \sum_{k=-\infty}^{\infty} \hat{f}(\mathfrak{P}^k(\mathbf{m}_0))\omega^k, \qquad r_1 < |\omega| < |\alpha_0|,$$

$$\mathfrak{l}(\omega) = \left(1 - \frac{\omega}{\beta_0}\right)^b \sum_{k=-\infty}^{\infty} \overline{\hat{g}(\mathfrak{P}^{-k}(\mathbf{m}_0))}\omega^k, \qquad r_1 < |\omega| < |\beta_0|.$$

These functions extend holomorphically to the annulus $A(r_1, r_2)$.

(1) *If* $\alpha_0 = \beta_0 = \omega_0$ *then*

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\, d^2\mathbf{z} = \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + 2\Re\left[\frac{\mathfrak{h}(\omega_0)\mathfrak{l}(\omega_0)n^{a+b-1}}{\omega_0^n\Gamma(a+b)}\right] + \mathrm{O}\left(\frac{n^{\Re a + \Re b - 2}}{|\omega_0|^n}\right)$$

as $n \to \infty$.

(2) *If* $|\alpha_0| < |\beta_0|$ *then*

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\, d^2\mathbf{z} = \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + 2\Re\left[\frac{\mathfrak{h}(\alpha_0)n^{a-1}}{\alpha_0^n\Gamma(a)} \sum_{k=-\infty}^{\infty} \overline{\hat{g}(\mathfrak{P}^{-k}(\mathbf{m}_0))}\alpha_0^k\right] + \mathrm{O}\left(\frac{n^{\Re a - 2}}{|\alpha_0|^n}\right) \qquad \text{as } n \to \infty.$$

(3) *If* $|\alpha_0| > |\beta_0|$ *then*

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\, d^2\mathbf{z} = \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + 2\Re\left[\frac{\mathfrak{l}(\beta_0)n^{b-1}}{\beta_0^n\Gamma(b)} \sum_{k=-\infty}^{\infty} \hat{f}(\mathfrak{P}^{k}(\mathbf{m}_0))\beta_0^k\right] + \mathrm{O}\left(\frac{n^{\Re b - 2}}{|\beta_0|^n}\right) \qquad \text{as } n \to \infty.$$

(4) *If* $|\alpha_0| = |\beta_0|$ *and* $\alpha_0 \neq \beta_0$ *then*

$$\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\, d^2\mathbf{z} = \hat{f}(\mathbf{0})\hat{g}(\mathbf{0}) + 2\Re\left[\frac{\mathfrak{h}(\alpha_0)\mathfrak{l}(\alpha_0)n^{a-1}}{\alpha_0^n\Gamma(a)}\left(1 - \frac{\alpha_0}{\beta_0}\right)^{-b} + \frac{\mathfrak{h}(\beta_0)\mathfrak{l}(\beta_0)n^{b-1}}{\beta_0^n\Gamma(b)}\left(1 - \frac{\beta_0}{\alpha_0}\right)^{-a}\right] + \mathrm{O}\left(\frac{n^{\Re a - 2}}{|\alpha_0|^n}\right) + \mathrm{O}\left(\frac{n^{\Re b - 2}}{|\beta_0|^n}\right) \qquad \text{as } n \to \infty.$$

This theorem has already been proved. Several remarks are in order. The error terms we have stated are not necessarily optimal in certain special circumstances, such as when a, b, or $a + b$ is zero or a negative integer. These more refined error estimates are already contained in the previous sections and have been omitted here to simplify the statement of the theorem. The reader may supply these refinements as an exercise.

The assumption that $\hat{f}(\mathbf{m}) = 0$ except for $\mathbf{m} \in \mathcal{O}(\mathbf{m}_0) \cup \mathcal{O}(-\mathbf{m}_0) \cup \{\mathbf{0}\}$ can be relaxed somewhat without changing the result. For example if there is a positive integer M such that for every $\mathbf{m}' \in (S_1 \cup S_2) \setminus \{\mathbf{m}_0\}$ we have $\hat{f}(\mathfrak{P}^k(\mathbf{m}')) = \hat{g}(\mathfrak{P}^k(\mathbf{m}')) = 0$ for all $|k| > M$ then the same result holds. This is because the contributions of the other orbits is eventually zero.

We can formulate results valid if f and g are linear combinations of functions satisfying the hypotheses of the theorem for different values of $\mathbf{m}_0, \alpha_0, \beta_0, a, b$. This can lead to explicit asymptotic results of great complexity. In fact it appears that from a detailed knowledge of the large-n asymptotics of $\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\,d^2\mathbf{z}$ one can recover a great deal of information about the functions f and g, although by no means everything about those functions. Thus the hyperbolic toral automorphism provides a sort of "perfectly regular" chaotic mixing, where the exact rate of decay of the covariance depends very sensitively on the functions involved.

5.2. **Asymptotics of Correlation for Smooth functions.** When $f, g\colon \mathbb{T}^2 \to \mathbb{R}$ are smooth functions the theorem of the previous section does not yield much information. However the pattern of the results in the previous section suggests that the covariance should decay faster than any exponential function, since the exponential decay rate was tied directly to the Hölder regularity of the functions. Our results on super-exponential decay of convolutions in section 4 provide an approach to a result of this kind. In fact if $\mathfrak{a}(\omega) = \sum_{k=0}^{\infty} \hat{f}(\mathfrak{P}^k\mathbf{m}_0)\omega^k$ is an entire function of order $\rho \in (0,\infty)$ and type $\tau \in (0,\infty)$ then for all $\epsilon > 0$ there exists $m \geq 0$ such that for all $k > m$ we have the bound $|\hat{f}(\mathfrak{P}^k\mathbf{m}_0)| \leq \left[\frac{\tau\rho e+\epsilon}{n}\right]^{n/\rho}$. For $\mathbf{m} = \mathfrak{P}^k\mathbf{m}_0$ this translates into the following decay rate of $\hat{f}$ as $\|\mathbf{m}\| \to \infty$ along the orbit of $\mathbf{m}_0$:

$$|\hat{f}(\mathbf{m})| \sim c_1\|\mathbf{m}\|^{c_2 - \frac{1}{\rho \ln \lambda_+} \ln\ln \frac{\|\mathbf{m}\|}{\|\mathbf{v}_+\alpha_+\|}}$$

where real constants $c_1 > 0$ and c_2 depend on various quantities. This is consistent with f being a smooth function of a very specific degree.

For the sake of completeness we will now present a result of this type even though it will assert decay estimates which are not sharp in the strongest possible sense that we can produce an explicit leading-order term.

Theorem. *Suppose $\mathbf{m}_0 \in S_1 \cup S_2$ and $\rho_1, \rho_2, \tau_1, \tau_2$ are positive real numbers. Suppose $f, g\colon \mathbb{T}^2 \to \mathbb{R}$ are smooth functions satisfying the following.*

- *$\hat{f}(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}^2$ except $\mathbf{m} = \mathbf{0}$ and $\mathbf{m} = \pm\mathfrak{P}^k\mathbf{m}_0$ for $k \geq 0$. Assume $\tau_1\rho_1 e = \limsup_{k\to\infty} k|\hat{f}(\mathfrak{P}^k\mathbf{m}_0)|^{\rho_1/k}$ and that $\hat{f}(\mathfrak{P}^k\mathbf{m}_0) \in \mathbb{R}$ for all $k \geq 0$. Define the entire function $\mathfrak{a}(\omega) = \sum_{k=0}^{\infty} \hat{f}(\mathfrak{P}^k\mathbf{m}_0)\omega^k$ and let $h_{\mathfrak{a},\rho_1}(e^{i\theta})$ be its Lindelöf indicator function.*
- *$\hat{g}(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}^2$ except $\mathbf{m} = \mathbf{0}$ and $\mathbf{m} = \pm\mathfrak{P}^k\mathbf{m}_0$ for $k \leq 0$. Assume $\tau_2\rho_2 e = \limsup_{k\to\infty} k|\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0)|^{\rho_2/k}$ and that $\hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0) \in \mathbb{R}$ for all $k \geq 0$. Define the entire function $\mathfrak{b}(\omega) = \sum_{k=0}^{\infty} \hat{g}(\mathfrak{P}^{-k}\mathbf{m}_0)\omega^k$ and let $h_{\mathfrak{b},\rho_2}(e^{i\theta})$ be its Lindelöf indicator function.*

Define $\tilde{\rho} = \max\{\rho_1, \rho_2\}$ and

$$\tilde{\tau} = \begin{cases} \tau_j & \rho_1 \neq \rho_2, \tilde{\rho} = \rho_j, j \in \{1,2\}, \\ \max_{0\leq\theta<2\pi}\left[h_{\mathfrak{a},\rho_1}(e^{i\theta}) + h_{\mathfrak{b},\rho_2}(e^{i\theta})\right] & \rho_1 = \rho_2. \end{cases}$$

In the case where $\rho_1 = \rho_2$ we also assume at least one of the functions $\mathfrak{a}(\omega)$ or $\mathfrak{b}(\omega)$ has completely regular growth and that $\tilde{\tau} > 0$. Then for every $\epsilon > 0$ we have that

- *there exists $m_\epsilon \geq 1$ such that for all $n \geq m_\epsilon$ we have the estimate*

$$\left|\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\,d^2\mathbf{z} - \hat{f}(\mathbf{0})\hat{g}(\mathbf{0})\right| \leq \left[\frac{\tilde{\tau}\tilde{\rho}e+\epsilon}{n}\right]^{n/\tilde{\rho}};$$

- *for infinitely many $n \geq 1$ we have the estimate*

$$\left|\int_{\mathbb{T}^2} f(\mathcal{P}^{-n}(\mathbf{z}))g(\mathbf{z})\, d^2\mathbf{z} - \hat{f}(\mathbf{0})\hat{g}(\mathbf{0})\right| \geq \left[\frac{\tilde{\tau}\tilde{\rho}e - \epsilon}{n}\right]^{n/\tilde{\rho}}.$$

Remarks Note that $\tau_1 = \max_{0\leq\theta<2\pi} h_{\mathfrak{a},\rho_1}(e^{i\theta})$ and $\tau_2 = \max_{0\leq\theta<2\pi} h_{\mathfrak{b},\rho_2}(e^{i\theta})$, and so $\tilde{\tau} \leq \tau_1 + \tau_2$. But it is quite possible for it to be strictly less than $\tau_1 + \tau_2$. Larger values of ρ and of τ mean slower decay and less smoothness. In some sense the parameters (ρ, τ) play analogous roles for super-exponential decay as (ω_0, a) did for exponential decay. The first parameter controls the dominate features of the decay (or smoothness) and the second parameter controls minor modifications of the decay rate (or smoothness). The decay parameters of a convolution are such that the convolution decays no more rapidly than the rate associated with the least smooth of the two functions being convolved unless the two functions have exactly the same first parameter.

Although smooth functions are the most interesting case in physical applications, there is reason to believe that the pattern established for functions of Hölder regularity is the generic behavior in flows.

6. Asymptotics of Continuous Time Correlations

6.1. **Exponential Decay of Convolutions.** The Fourier transform gives us an effective tool for giving classes of functions $f, g\colon \mathbb{R} \to \mathbb{C}$ such that sharp large t asymptotics can be given for $f(t)$ and $g(t)$ as well as for the convolution $(f*g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\, ds$. As before, we are interested in general rules governing the parameters in these large t asymptotics.

As a first example consider the well-known distribution $t_+^{a-1}/\Gamma(a)$, where $a \in \mathbb{C}$ (see [10]). Here $t_+ = t$ when $t > 0$ and $t_+ = 0$ for $t \leq 0$. This distribution is defined by the locally integrable function when $\Re a > 0$ and by analytic continuation for all other values of a. The distributional Fourier transform, which is a function of the real variable w, of this distribution is in that reference shown to be $[i(w - 0i)]^{-a}$, where the fractional power is defined using the principal branch. Since $[i(w - 0i)]^{-a}[i(w-0i)]^{-b} = [i(w-0i)]^{-a-b}$ we have that

$$\frac{t_+^{a-1}}{\Gamma(a)} * \frac{t_+^{b-1}}{\Gamma(b)} = \frac{t_+^{a+b-1}}{\Gamma(a+b)}.$$

Other results on the asymptotic behavior of convolutions of functions with power-like decay at infinity can be found in [18]. The above family of examples can be easily modified to be exponentially decaying as $t \to \infty$:

$$e^{i\alpha_0 t}\frac{t_+^{a-1}}{\Gamma(a)},$$

where $\alpha_0 \in \mathbb{C}$, $\Im\alpha_0 > 0$. The Fourier transform of this distribution is $[i(w-\alpha_0)]^{-a}$. To understand the convolution of distributions of this type when $\beta_0 \in \mathbb{C}$, $\Im\beta_0 > 0$, and $\Re a > 0$, $\Re b > 0$, we note the identity (13.2.1, page 505, [2])

$$\int_0^t e^{i\alpha_0(t-s)}\frac{(t-s)^{a-1}}{\Gamma(a)}e^{i\beta_0 s}\frac{s^{b-1}}{\Gamma(b)}\, ds = e^{i\alpha_0 t}\frac{t^{a+b-1}}{\Gamma(a+b)}M(b, a+b, i(\beta_0 - \alpha_0)t),$$

which is valid for $t > 0$. In the above $M(b, a+b, z)$ is the confluent hypergeometric function (see 13.1.2, page 504, [2]), which is well-defined when $a+b$ is not a negative integer. Thus

$$\left(e^{i\alpha_0 t}\frac{t_+^{a-1}}{\Gamma(a)}\right) * \left(e^{i\beta_0 t}\frac{t_+^{b-1}}{\Gamma(b)}\right) = e^{i\alpha_0 t}\frac{t_+^{a+b-1}}{\Gamma(a+b)}M(b, a+b, i(\beta_0 - \alpha_0)t),$$

for all complex α_0 and β_0 such that $\Im\alpha_0 > 0$ and $\Im\beta_0 > 0$, and all complex a and b such that neither a nor b nor $a+b$ is zero or a negative integer. Its distributional Fourier transform is $[i(w-\alpha_0)]^{-a}[i(w-\beta_0)]^{-b}$. Of course if $\alpha_0 = \beta_0$ then the convolution simplifies to

$$e^{i\alpha_0 t}\frac{t^{a+b-1}}{\Gamma(a+b)},$$

since $M(b, a+b, 0) = 1$. If $\alpha_0 \neq \beta_0$ we apply a well-known asymptotic expansion (13.5.1, page 508, [2]) to get

$$\begin{aligned} e^{i\alpha_0 t}&\frac{t^{a+b-1}}{\Gamma(a+b)}M(b, a+b, i(\beta_0-\alpha_0)t) \\ &= \quad e^{i\alpha_0 t}\frac{t^{a-1}}{\Gamma(a)}[i(\alpha_0-\beta_0)]^{-b}\left[1 + \frac{b(1-a)}{i(\alpha_0-\beta_0)t} + \mathrm{O}(t^{-2})\right] \\ &\quad + e^{i\beta_0 t}\frac{t^{b-1}}{\Gamma(b)}[i(\beta_0-\alpha_0)]^{-a}\left[1 + \frac{a(1-b)}{i(\beta_0-\alpha_0)t} + \mathrm{O}(t^{-2})\right], \end{aligned}$$

as $t \to \infty$. This is by now a very familiar pattern: the convolution behaves asymptotically like the more slowly decaying of the two functions convolved, unless they both have exactly the same exponential decay rate. This corresponds to a large-time asymptotic expansion of the contour integral

$$\frac{1}{2\pi}\int_{\mathcal{C}} e^{iwt}[i(w-\alpha_0)]^{-a}[i(w-\beta_0)]^{-b}\,dw$$

with the dominant contribution arising from the singularity closest to the real w axis.

The distribution $e^{i\alpha_0 t}t_+^{a-1}/\Gamma(a)$ has a very clear large t asymptotic behavior, but it is not a particularly smooth function of t near $t=0$. We desire to generalize the above example by finding a class of smooth functions of t with large t asymptotic behavior like this example for which we can also understand the large t asymptotic behavior of convolutions of two functions from two such classes. There is no point in trying to make this class as large as possible, so the simplest thing to do is to consider the class of (distributional) convolutions $[e^{i\alpha_0 t}t_+^{a-1}/\Gamma(a)] * h(t)$, where h is a smooth function of compact support.

Lemma. *Suppose* $\Im\alpha_0 > 0$, $a \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$, *and* h *is a smooth function with compact support in* $\mathbb{R}$. *Then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}[i(w-\alpha_0)]^{-a}\hat{h}(w)\,dw = e^{i\alpha_0 t}\frac{t^{a-1}}{\Gamma(a)}\left[\hat{h}(\alpha_0) + O(t^{-1})\right]$$

as $t \to \infty$.

Thus the effect of the convolution is only to change the large t asymptotics by a complex constant factor.

Proof. Because $h(t)$ is a smooth function the above integral converges for any value of a; it can be expressed as the limit as $N \to \infty$ of the integral over the segment $\Re\alpha_0 - N/t \le w \le \Re\alpha_0 + N/t$. By Cauchy's Theorem we can change the contour of integration from $\Re\alpha_0 - N/t \le w \le \Re\alpha_0 + N/t$ to the contour parameterized by the rule $w = \alpha_0 + [w' + i(|w'| - 1)]/t$, with parameter $-N \le w' \le N$, where $0 < \epsilon < \pi/2$, plus the sum of the contour integrals (of the same integrand) over the two vertical line segments $w = \Re\alpha_0 \pm N/t + iy$, $0 \le y \le \Im\alpha_0 + (N-1)/t$. The absolute values of these two contour integrals over the vertical segments are bounded by

$$\int_0^{\Im\alpha_0+(N-1)/t} e^{-yt}|\Im\alpha_0 - y \pm iN/t|^{-\Re a}|\hat{h}(\Re\alpha_0 \pm N/t + iy)|\,dy.$$

Suppose $\operatorname{supp} h = [c_1, c_2]$ and $t > c_3 = \max\{|c_1|, |c_2|\}$. By the Paley-Wiener Theorem (the easy half) we have for each $M > 0$ there exists $c_M > 0$ such that

$$|\hat{h}(z)| \le c_M e^{c_3|\Im z|}(1 + |z|)^{-M}$$

for all $z \in \mathbb{C}$. Using this we have the further upper estimate

$$\begin{aligned} &c_M \int_0^{\Im\alpha_0+(N-1)/t} e^{-y(t-c_3)}|\Im\alpha_0 - y \pm iN/t|^{-\Re a}(1 + |\Re\alpha_0 \pm N/t + iy|)^{-M}\,dy \\ &\le c_M \int_0^{\infty} e^{-y(t-c_3)}|\Im\alpha_0 - y \pm iN/t|^{-\Re a}(1 + |\Re\alpha_0 \pm N/t + iy|)^{-M}\,dy \end{aligned}$$

For M sufficiently large the integrand is bounded independently of N by a constant times $e^{-y(t-c_3)}$, and it tends to zero as $N \to \infty$. Hence by the Dominated Convergence Theorem the two vertical segments make no contribution in the limit as $N \to \infty$. By this argument we have changed the contour in the original integral from $w \in \mathbb{R}$ to the (time dependent) contour $\mathcal{W}(t)$ parameterized by the rule $w = \alpha_0 + [w' + i(|w'| - 1)]/t$, $w' \in \mathbb{R}$.

If $p \ge 1$ is an integer then define the entire function $g_p(z) = [e^z - \sum_{j=0}^{p-1} z^j/j!]z^{-p}$, $z \in \mathbb{C}$. Clearly there exists $C > 0$ such that $|g_p(z)| \le C(1 + e^{\Re z})$ for all $z \in \mathbb{C}$. A short calculation shows that

$$\begin{aligned} \hat{h}(w) - \sum_{j=0}^{p-1} \frac{(w-\alpha_0)^j}{j!}\hat{h}^{(j)}(\alpha_0) &= \int_{c_1}^{c_2} e^{-i\alpha_0 s}[-i(w-\alpha_0)s]^p g_p(-i(w-\alpha_0)s)h(s)\,ds \\ &= [-i(w-\alpha_0)]^p \int_{c_1}^{c_2} e^{-i\alpha_0 s} g_p(-i(w-\alpha_0)s)s^p h(s)\,ds. \end{aligned}$$

Therefore we have that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}[i(w-\alpha_0)]^{-a}\hat{h}(w)\,dw &= \frac{1}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w-\alpha_0)]^{-a}\hat{h}(w)\,dw \\ = \sum_{j=0}^{p-1} \frac{(-i)^j \hat{h}^{(j)}(\alpha_0)}{j!} e^{i\alpha_0 t}\frac{t^{a-j-1}}{\Gamma(a-j)} \\ + \frac{(-1)^p}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w-\alpha_0)]^{p-a}\int_{c_1}^{c_2} e^{-i\alpha_0 s} g_p(-i(w-\alpha_0)s)s^p h(s)\,ds\,dw \end{aligned}$$

In the above we use the identity ($t > 0, \Im\alpha_0 > 0, a - j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$)

$$\frac{1}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w-\alpha_0)]^{-a+j}\,dw = e^{i\alpha_0 t}\frac{t^{a-j-1}}{\Gamma(a-j)}.$$

The double integral

$$\int_{\mathcal{W}(t)} e^{iwt}[i(w-\alpha_0)]^{p-a} \int_{c_1}^{c_2} e^{-i\alpha_0 s} g_p(-i(w-\alpha_0)s) s^p h(s)\, ds\, dw$$

is the remainder term $R(t)$. We must show that $e^{-i\alpha_0 t} t^{p+1-a} R(t)$ is bounded (independently of t) when t is sufficiently large. Now we transform the contour of integration in the outer integral of $R(t)$ by the rule $\omega = (w-\alpha_0)t = w' + i(|w'|-1)$, $w' \in \mathbb{R}$; let the new contour be denoted $\mathcal{V}$. Thus

$$e^{-i\alpha_0 t} t^{p+1-a} R(t) = \frac{(-1)^p}{2\pi} \int_{\mathcal{V}} e^{i\omega}[i\omega]^{p-a} \int_{c_1}^{c_2} e^{-i\alpha_0 s} g_p(-i\omega s/t) s^p h(s)\, ds\, d\omega.$$

Estimating we see that

$$\begin{aligned} \left| \int_{c_1}^{c_2} e^{-i\alpha_0 s} g_p(-i\omega s/t) s^p h(s)\, ds \right| &\le C \int_{c_1}^{c_2} e^{s\Im\alpha_0}[1 + e^{s\Im\omega/t}]|s|^p |h(s)|\, ds \\ &\le C[1 + e^{c_3|\Im\omega|/t}] \int_{c_1}^{c_2} e^{s\Im\alpha_0}|s|^p|h(s)|\, ds. \end{aligned}$$

Thus finally

$$|e^{-i\alpha_0 t} t^{p+1-a} R(t)| \le C' \int_{-\infty}^{\infty} e^{-(|w'|-1)}[1 + e^{c_3||w'|-1|/t}]|(iw' + 1 - |w'|)^{p-a}|\, dw',$$

which for $t > c_3 + 1$ is clearly finite and bounded independently of t. When $p = 1$ we have the desired result. □

Now suppose l is also a smooth function with compact support in $\mathbb{R}$. Now we consider the large t asymptotics of the convolution $[e^{i\alpha_0 t} t_+^{a-1}/\Gamma(a)] * h(t) * [e^{i\beta_0 t} t_+^{b-1}/\Gamma(b)] * l(t)$ of two functions of the type considered in the previous lemma.

Theorem. *Suppose h, l are smooth functions with compact support in $\mathbb{R}$. Suppose $\alpha_0, \beta_0, a, b \in \mathbb{C}$ satisfy $\Im\alpha_0 > 0, \Im\beta_0 > 0$. Consider the quantity*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt}[i(w-\alpha_0)]^{-a}[i(w-\beta_0)]^{-b} \hat{h}(w)\hat{l}(w)\, dw.$$

If $\alpha_0 = \beta_0$ and $a+b$ is not a nonpositive integer then this quantity is asymptotically

$$e^{i\alpha_0 t} \frac{t^{a+b-1}}{\Gamma(a+b)} \left[\hat{h}(\alpha_0)\hat{l}(\alpha_0) + O(t^{-1})\right]$$

as $t \to \infty$. If $\Im\alpha_0 < \Im\beta_0$ and a is not a nonpositive integer then the same quantity is asymptotically

$$e^{i\alpha_0 t} \frac{t^{a-1}}{\Gamma(a)} \left\{[i(\alpha_0 - \beta_0)]^{-b} \hat{h}(\alpha_0)\hat{l}(\alpha_0) + O(t^{-1})\right\}$$

as $t \to \infty$. If $\Re\alpha_0 \ne \Re\beta_0$, $\Im\alpha_0 = \Im\beta_0$, and neither a nor b is a nonpositive integer then the same quantity is asymptotically

$$\begin{aligned} &e^{i\alpha_0 t} \frac{t^{a-1}}{\Gamma(a)} \left\{[i(\alpha_0 - \beta_0)]^{-b} \hat{h}(\alpha_0)\hat{l}(\alpha_0) + O(t^{-1})\right\} \\ &+ e^{i\beta_0 t} \frac{t^{b-1}}{\Gamma(b)} \left\{[i(\beta_0 - \alpha_0)]^{-a} \hat{h}(\beta_0)\hat{l}(\beta_0) + O(t^{-1})\right\} \end{aligned}$$

as $t \to \infty$.

Proof. The case where $\alpha_0 = \beta_0$ is a consequence of the previous lemma. So suppose $\alpha_0 \neq \beta_0$. If $\Re\alpha_0 = \Re\beta_0$ and $\Im\alpha_0 < \Im\beta_0$ then change the contour to $w = \alpha_0 + [w' + i(|w'| - 1)]/t$, $w' \in \mathbb{R}$, denoted by $\mathcal{W}(t)$, as in the proof of the previous lemma. If $\Re\alpha_0 < \Re\beta_0$ then change the original contour to the union of $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$, where $\mathcal{W}_1(t)$ is parameterized by

$$w = \begin{cases} \alpha_0 + [w' + i(-w' - 1)]/t & w' \leq 0 \\ \alpha_0 + [\frac{1}{\pi}\Re(\beta_0 - \alpha_0)\tan^{-1} w' + i(w' - 1)]/t & w' \geq 0 \end{cases},$$

and where $\mathcal{W}_2(t)$ is parameterized by

$$w = \begin{cases} \beta_0 + [\frac{1}{\pi}\Re(\beta_0 - \alpha_0)\tan^{-1} w' + i(-w' - 1)]/t & w' \leq 0 \\ \beta_0 + [w' + i(w' - 1)]/t & w' \geq 0 \end{cases}.$$

The proof of the feasibility of these contour changes proceeds as in the lemma.

Now define

$$\tilde{g}_1(w) = \frac{\hat{h}(w)\hat{l}(w)[i(w - \beta_0)]^{-b} - \hat{h}(\alpha_0)\hat{l}(\alpha_0)[i(\alpha_0 - \beta_0)]^{-b}}{w - \alpha_0}$$

$$\tilde{g}_2(w) = \frac{\hat{h}(w)\hat{l}(w)[i(w - \alpha_0)]^{-a} - \hat{h}(\beta_0)\hat{l}(\beta_0)[i(\beta_0 - \alpha_0)]^{-a}}{w - \beta_0}.$$

Consider the case $\Re\alpha_0 = \Re\beta_0$ and $\Im\alpha_0 < \Im\beta_0$. Then

$$\frac{1}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w - \alpha_0)]^{-a}[i(w - \beta_0)]^{-b}\hat{h}(w)\hat{l}(w)\,dw$$
$$= \hat{h}(\alpha_0)\hat{l}(\alpha_0)[i(\alpha_0 - \beta_0)]^{-b}\frac{1}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w - \alpha_0)]^{-a}\,dw$$
$$+ \frac{-i}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w - \alpha_0)]^{1-a}\tilde{g}_1(w)\,dw$$

Since $\frac{1}{2\pi}\int_{\mathcal{W}(t)} e^{iwt}[i(w - \alpha_0)]^{-a}\,dw = e^{i\alpha_0 t}t^{a-1}/\Gamma(a)$ for all $t > 0$, let R denote the integral comprising the remainder term. We must show that $|e^{-i\alpha_0 t}t^{2-a}R|$ is bounded independently of t for t sufficiently large.

$$\begin{aligned} ie^{-i\alpha_0 t}t^{2-a}R &= \int_{\mathcal{W}(t)} e^{i(w-\alpha_0)t}[i(w - \alpha_0)t]^{1-a}\tilde{g}_1(w)t\,dw \\ &= \int_{\mathcal{W}(1)-\alpha_0} e^{i\omega}[i\omega]^{1-a}\tilde{g}_1(\alpha_0 + \omega/t)\,d\omega \\ &= \int_{-\infty}^{\infty} e^{i[w'+i(|w'|-1)]}[iw' - |w'| + 1]^{1-a}\tilde{g}_1\left(\alpha_0 + \frac{1}{t}[w' + i(|w'| - 1)]\right) \\ &\qquad [1 + i\ \mathrm{sgn}\ (w')]\,dw' \end{aligned}$$

Since $h * l$ has compact support we see that $|\hat{h}(w)\hat{l}(w)| \leq c_M e^{c_3|\Im w|}(1 + |w|)^{-M}$ for all $w \in \mathbb{C}$. Therefore $|\hat{g}_1(w)|$ satisfies estimates sufficient to see that the above integral converges for $t > c_3$ and the absolute value of this integral is bounded independently of $t \geq c_3 + 1$.

Now consider the case where $\Re\alpha_0 < \Re\beta_0$. There are three possibilities: $\Im\alpha_0 < \Im\beta_0$, $\Im\alpha_0 > \Im\beta_0$, and $\Im\alpha_0 = \Im\beta_0$. The first two are treated in the same way, so suppose $\Im\alpha_0 < \Im\beta_0$. The integral over $\mathcal{W}_1(t)$ gives rise to the leading-order term of the large-t asymptotics just like in the case discussed above, and the error term is also estimated to be $\mathrm{O}\,(t^{\Re a-2}e^{-\Im\alpha_0 t})$ in the same way as before. This leads to the necessary restriction that a is not a nonpositive integer. Now we must show that the integral over $\mathcal{W}_2(t)$ can be estimated to be smaller than that error term without any restriction on b.

$$\int_{\mathcal{W}_2(t)} e^{iwt}[i(w-\alpha_0)]^{-a}[i(w-\beta_0)]^{-b}\hat{h}(w)\hat{l}(w)\,dw$$
$$= t^{b-1}e^{i\beta_0 t}\int_{\mathcal{W}_2(1)-\beta_0} e^{i\omega}[i(\beta_0+\tfrac{\omega}{t}-\alpha_0)]^{-a}[i\omega]^{-b}(h*l)\hat{}\,(\beta_0+\tfrac{\omega}{t})\,d\omega.$$

Since this last integral is bounded independently of sufficiently large t, we see that the integral over $\mathcal{W}_2(t)$ is $\mathrm{O}\,(t^{\Re b-1}e^{-\Im\beta_0 t})$, which is much smaller than the error term in the asymptotic approximation of the integral over $\mathcal{W}_1(t)$.

The last remaining case: $\Re\alpha_0 < \Re\beta_0$ and $\Im\alpha_0 = \Im\beta_0$ is handled in the same manner. A leading-order term and an error term is generated from each of the contours $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ just as we have done above, under the stated assumptions on a and b. □

6.2. **Super-exponential Decay of Convolutions.** Fortunately for us if we wish to obtain sharp decay estimates of a convolution $(f*g)(t)$ as $t\to\infty$, where $f(t)$ and $g(t)$ decay super-exponentially as $|t|\to\infty$, we need only put together known results in the literature. We need the theory of entire functions discussed in section 4.4. Suppose $\alpha>1, a_+>0, a_->0, b_+>0, b_->0$ are real numbers. Define $Z^0_\alpha(a_+,a_-,b_+,b_-)$ to be the set of all entire functions $f(z)$ such that all of the following are true.

- For every $\epsilon>0$ and no $\epsilon<0$ as $z\to\infty$, $z\in\mathbb{C}$, $\Im z>0$,
$$f(z)=\mathrm{O}(\exp[(a_++\epsilon)|\Im z|^\alpha]).$$
- For every $\epsilon>0$ and no $\epsilon<0$ as $z\to\infty$, $z\in\mathbb{C}$, $\Im z<0$,
$$f(z)=\mathrm{O}(\exp[(a_-+\epsilon)|\Im z|^\alpha]).$$
- For every $\epsilon>0$ and no $\epsilon<0$ as $t\to\infty$, $t\in\mathbb{R}$,
$$f(t)=\mathrm{O}(\exp[(-b_++\epsilon)t^\alpha]).$$
- For every $\epsilon>0$ and no $\epsilon<0$ as $t\to-\infty$, $t\in\mathbb{R}$,
$$f(t)=\mathrm{O}(\exp[(-b_-+\epsilon)|t|^\alpha]).$$

Suppose $f\in Z^0_\alpha(a_+,a_-,b_+,b_-)$ and $h_{f,\alpha}(e^{i\theta})$ is the Lindelöf indicator function of f. It follows that

$$a_\pm=\sup_{\pm\sin\theta>0}\frac{h_{f,\alpha}(e^{i\theta})}{|\sin\theta|^\alpha},\qquad -b_+=h_{f,\alpha}(e^{i0}),\qquad -b_-=h_{f,\alpha}(e^{i\pi}).$$

Conversely, if $f(z)$ is an entire function of order α and of positive finite type and $a_\pm>0, b_\pm>0$ are determined by the above formulae then $f\in Z^0_\alpha(a_+,a_-,b_+,b_-)$. It was shown in [23] that a necessary condition on $a_+>0, a_->0, b_+>0, b_->0$ that $Z^0_\alpha(a_+,a_-,b_+,b_-)\neq\emptyset$ is

$$\frac{\max\{a_+,a_-\}}{\min\{b_+,b_-\}}\geq s_\alpha=\begin{cases}[\sin(\frac{\pi}{2}(\alpha-1))]^{-1} & 1<\alpha\leq 2,\\ [\sin(\frac{\pi}{2}(\alpha-1)^{-1})]^{\alpha-1} & \alpha>2.\end{cases}$$

This same paper [23] shows (in the proof of Theorem 1) that for all $b>0$ the set $Z^0_\alpha(s_\alpha b, s_\alpha b, b, b)$ is nonempty, and in fact contains a function of completely regular growth (see Theorem 3, page 94 of [12]). A fact (Theorem 1 (see proof) in [25] and Theorem 2 (see proof) of [23]) states that if $f \in Z^0_\alpha(a_+, a_-, b_+, b_-)$ then its Fourier transform F satisfies $F \in Z^0_\beta(K(\beta, b_+), K(\beta, b_-), K(\beta, a_-), K(\beta, a_+))$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and $K(\beta, c) = (\alpha c)^{1-\beta}/\beta$.

Suppose $f_1(z)$ and $f_2(z)$ are entire functions of orders $\alpha_1, \alpha_2 > 1$ respectively and both of positive finite type. Let $h_{f_1,\alpha_1}, h_{f_2,\alpha_2}$ be their respective Lindelöf indicator functions. Let the parameters $a_{i\pm}, b_{i\pm}$, be determined from f_i, $i = 1, 2$, as above; assume $a_{i\pm} > 0, b_{i\pm} > 0$, $i = 1, 2$. Then $f_i \in Z^0_{\alpha_i}(a_{i+}, a_{i-}, b_{i+}, b_{i-})$, $i = 1, 2$. We wish to obtain precise information about the asymptotics of $(f_1 * f_2)(t)$ as $t \to \infty$.

Let $F_i = \hat{f}_i$ be the Fourier transform of f_i, $i = 1, 2$. Then as noted above $F_i \in Z^0_{\beta_i}(K(\beta_i, b_{i+}), K(\beta_i, b_{i-}), K(\beta_i, a_{i-}), K(\beta_i, a_{i+}))$, where $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$, $i = 1, 2$. Thus we must inquire after the growth and decay properties of $F_1(w)F_2(w)$. We have two cases. If $\alpha_1 < \alpha_2$ then F_1F_2 has order $\beta_1 > \beta_2$ and has the exact same Lindelöf indicator function as F_1. Thus

$$F_1F_2 \in Z^0_{\beta_1}(K(\beta_1, b_{1+}), K(\beta_1, b_{1-}), K(\beta_1, a_{1-}), K(\beta_1, a_{1+})).$$

Since $K(\alpha_1, K(\beta_1, c)) = c$ and $2\pi(f_1 * f_2)(-t)$ is the Fourier transform of F_1F_2, this immediately implies that $f_1 * f_2 \in Z^0_{\alpha_1}(a_{1+}, a_{1-}, b_{1+}, b_{1-})$, and thus for every $\epsilon > 0$ and for no $\epsilon < 0$ we have

$$(f_1 * f_2)(t) = \mathrm{O}(\exp[(-b_{1+} + \epsilon)t^{\alpha_1}]), \qquad t \to \infty.$$

This result follows the same pattern that we have seen before, i.e. the asymptotics of the convolution agrees with that of the more slowly decaying function being convolved, provided the other one decays much faster.

For the second case suppose $\alpha_1 = \alpha_2 = \alpha$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and that one of F_1, F_2 has completely regular growth. Then the β-Lindelöf indicator function of F_1F_2 is the sum of the β-Lindelöf indicator functions of F_1 and F_2. Define

$$c_\pm = \sup_{\pm \sin\theta > 0} \frac{h_{F_1,\beta}(e^{i\theta}) + h_{F_2,\beta}(e^{i\theta})}{|\sin\theta|^\beta}.$$

Then

$$\begin{aligned} F_1F_2 &\in Z^0_\beta(c_+, c_-, K(\beta, a_{1-}) + K(\beta, a_{2-}), K(\beta, a_{1+}) + K(\beta, a_{2+})), \\ (f_1 * f_2)^\checkmark &\in Z^0_\alpha(K(\alpha, K(\beta, a_{1-}) + K(\beta, a_{2-})), K(\alpha, K(\beta, a_{1+}) + K(\beta, a_{2+})), \\ &\qquad K(\alpha, c_-), K(\alpha, c_+)), \\ f_1 * f_2 &\in Z^0_\alpha(K(\alpha, K(\beta, a_{1+}) + K(\beta, a_{2+})), K(\alpha, K(\beta, a_{1-}) + K(\beta, a_{2-})), \\ &\qquad K(\alpha, c_+), K(\alpha, c_-)). \end{aligned}$$

Thus for all $\epsilon > 0$ and for no $\epsilon < 0$ we have

$$(f_1 * f_2)(t) = \mathrm{O}(\exp[(-K(\alpha, c_+) + \epsilon)t^\alpha]), \qquad t \to \infty.$$

Thus to get the precise asymptotics in this case we must be able to compute the Lindelöf indicator functions of F_1 and F_2 and compute c_+, which may have no simple relationship to $K(\beta, b_{i\pm})$, $i = 1, 2$.

6.3. **Decay of Continuous Time Correlations.** Finally we are ready to state our two main results of the paper. The first result concerns functions f and g on X which have only Hölder regularity when restricted to constant time translates

$\phi_t(\mathcal{S})$ of the Poincaré section $\mathcal{S} \subset X$, but are smooth when restricted to individual trajectories $\{\phi_t(\mathbf{x})\}_{t\in\mathbb{R}} \subset X$, $\mathbf{x} \in X$.

Theorem. *Suppose $\alpha_0, \beta_0, a, b \in \mathbb{C}$ satisfy $\Im\alpha_0 > 0$, $\Im\beta_0 > 0$, and neither a nor b nor $a+b$ is a nonpositive integer. Suppose $h, l\colon \mathbb{R} \to \mathbb{R}$ are smooth functions with compact support. Suppose $\mathbf{m}_0 \in S_1 \cup S_2$. Suppose $f, g\colon X \to \mathbb{R}$ are continuous functions such that $\mathfrak{F}_{\mathbf{m}} f \equiv 0$ for all $\mathbf{m} \in S_1 \cup S_2 \setminus \{\mathbf{m}_0\}$, and*

$$(\mathfrak{F}_{\mathbf{m}_0} f)(-u) = \langle e^{i\alpha_0 t}\frac{t_+^{a-1}}{\Gamma(a)}, h(u-t)\rangle, \qquad \overline{(\mathfrak{F}_{\mathbf{m}_0} g)(u)} = \langle e^{i\beta_0 t}\frac{t_+^{b-1}}{\Gamma(b)}, l(u-t)\rangle$$

for all $u \in \mathbb{R}$ (distributions applied to test functions). Then the quantity

$$\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x} - \int_0^1 \Lambda_f(\tau - t)\Lambda_g(\tau)\,d\tau$$

has the following asymptotic behavior as $t \to \infty$ according to the following three cases.

- *If $\alpha_0 = \beta_0$ then it is*
$$2\Re\left[e^{i\alpha_0 t}\frac{t^{a+b-1}}{\Gamma(a+b)}\hat{h}(\alpha_0)\hat{l}(\alpha_0)\right] + O(e^{-\Im\alpha_0 t}t^{\Re a+\Re b-2}).$$
- *If $\Im\alpha_0 < \Im\beta_0$ then it is*
$$2\Re\left\{e^{i\alpha_0 t}\frac{t^{a-1}}{\Gamma(a)}[i(\alpha_0-\beta_0)]^{-b}\hat{h}(\alpha_0)\hat{l}(\alpha_0)\right\} + O(e^{-\Im\alpha_0 t}t^{\Re a-2}).$$
- *If $\Re\alpha_0 \neq \Re\beta_0$ and $\Im\alpha_0 = \Im\beta_0$ then it is*
$$2\Re\left\{e^{i\alpha_0 t}\frac{t^{a-1}}{\Gamma(a)}[i(\alpha_0-\beta_0)]^{-b}\hat{h}(\alpha_0)\hat{l}(\alpha_0)\right\} + O(e^{-\Im\alpha_0 t}t^{\Re a-2})$$
$$+ 2\Re\left\{e^{i\beta_0 t}\frac{t^{b-1}}{\Gamma(b)}[i(\beta_0-\alpha_0)]^{-a}\hat{h}(\beta_0)\hat{l}(\beta_0)\right\} + O(e^{-\Im\alpha_0 t}t^{\Re b-2}).$$

This theorem is an immediate consequence of the results of sections 2.3 and 6.1. A simple calculation like that in section 5.1 shows that f has Hölder-type regularity of order $\gamma = \Im\alpha_0/\ln\lambda_+$ with a logarithmic modification arising from a. Thus the decay rate of the correlation integral to its time-periodic leading order term depends directly on the order of regularity of the functions involved, with the least smooth of the functions determining the decay rate.

Our second main result concerns smooth functions $f, g\colon X \to \mathbb{R}$.

Theorem. *Suppose $\mathbf{m}_0 \in S_1 \cup S_2$. Suppose $f, g\colon X \to \mathbb{R}$ are smooth functions such that $\mathfrak{F}_{\mathbf{m}} f \equiv 0$ for all $\mathbf{m} \in S_1 \cup S_2 \setminus \{\mathbf{m}_0\}$, and $(\mathfrak{F}_{\mathbf{m}_0} f)(u)$ and $(\mathfrak{F}_{\mathbf{m}_0} g)(u)$ are entire functions of $u \in \mathbb{C}$ of positive finite orders $\alpha > 1, \beta > 1$ respectively and of positive and finite type. Suppose also that $(\mathfrak{F}_{\mathbf{m}_0} f)(u)$ and $(\mathfrak{F}_{\mathbf{m}_0} g)(u)$ are real-valued when $u \in \mathbb{R}$. Let $h(e^{i\theta})$ be the α-Lindelöf indicator function of $\mathfrak{F}_{\mathbf{m}_0} f$ and $l(e^{i\theta})$ be the β-Lindelöf indicator function of $\mathfrak{F}_{\mathbf{m}_0} g$, $-\pi < \theta \leq \pi$. Assume $h(e^{i0}) < 0, h(e^{i\pi}) < 0, l(e^{i0}) < 0, l(e^{i\pi}) < 0$. Then the quantity*

$$\int_X f(\phi_{-t}(\mathbf{x}))g(\mathbf{x})\,d\mathbf{x} - \int_0^1 \Lambda_f(\tau - t)\Lambda_g(\tau)\,d\tau$$

has the following asymptotic behavior as $t \to \infty$ according to the following three cases.

- *If $\alpha < \beta$ then for all $\epsilon > 0$ and for no $\epsilon < 0$ it is of the form*
$$O(\exp[(h(e^{i\pi}) + \epsilon)t^{\alpha}]).$$
- *If $\alpha > \beta$ then for all $\epsilon > 0$ and for no $\epsilon < 0$ it is of the form*
$$O(\exp[(l(e^{i0}) + \epsilon)t^{\beta}]).$$
- *If $\alpha = \beta$ then let F, G denote the Fourier transforms of $\mathfrak{F}_{\mathbf{m}_0} f, \mathfrak{F}_{\mathbf{m}_0} g$ respectively. Both F and G are entire functions of order α' where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Let $h_{F,\alpha'}(e^{i\theta})$ and $h_{G,\alpha'}(e^{i\theta})$ denote the α'-Lindelöf indicator functions of F, G respectively. Assume at least one of F and G has completely regular growth. Then for all $\epsilon > 0$ and for no $\epsilon < 0$ the desired asymptotic behavior is of the form $O(\exp[(-K(\alpha, c) + \epsilon)t^{\alpha}])$, where $K(\alpha, c) = (\alpha' c)^{1-\alpha}/\alpha$ and*
$$c = \sup_{0<\theta<\pi} \frac{h_{F,\alpha'}(e^{i(\theta-\pi)}) + h_{G,\alpha'}(e^{i\theta})}{(\sin\theta)^{\alpha'}}.$$
.

This theorem is a consequence of the results of sections 2.3 and 6.2. The restriction that the entire functions $(\mathfrak{F}_{\mathbf{m}_0} f)(u)$ and $(\mathfrak{F}_{\mathbf{m}_0} g)(u)$ be real-valued when $u \in \mathbb{R}$ is easy to satisfy, as can be seen from the Hadamard Factorization theorem (see section 4.4). Thus again we get a two parameter family of possible asymptotic behaviors, with subtleties appearing when the dominant parameters coincide.

Taken together these results show that each one of a large variety of asymptotic behaviors is possible, depending on the detailed regularity properties of the functions f and g.

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