Graceful Labelings

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Dedication

To my parents, Rickey and Susan.

Acknowledgments

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Abstract

Let T be a tree on n vertices and let $f : V \to \{1, 2, \ldots, n\}$ be a valuation on $V(T)$. Then f is called a *graceful labeling* of T if $|f(v) - f(u)|$ is unique for each edge $vu \in E(T)$. We will establish the connection between gracefully labeled trees and the Ringel-Kotzig Conjecture, as well as survey several classes of trees that have been proven graceful as a result of efforts to show that all trees are graceful.

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Chapter 1

Introduction

The study of graph labelings has become a major subfield of graph theory. Very often, the problems from this area draw attention due to their application to real life situations or, in some cases, their history. Many of the most arduous problems of graph theory are the easiest to state. The four-color theorem, for example, states that a planar map can be colored using at most four colors in such a way that no two bordering regions have the same color. This problem, originally hypothesized in the early 1850's, remained unsolved until the 1980's, and its proof relies on the use of computer software to check a large number of cases. Subsequent, more elegant proofs have arisen, but the problem was unsolved for some one hundred eighty years.

Here, we investigate another somewhat longstanding problem in graph labelings. The *Graceful Tree Conjecture*, which states that every tree graph on n vertices has some vertex labeling using the numbers $1, 2, \ldots, n$ such that the edge labeling obtained from the vertex labeling by taking the absolute value of the difference of two adjacent vertex labels assigns distinct edge labels. Alexander Rosa was the first to consider such a labeling scheme as a method to prove the conjectures of Ringel and Kotzig [**15**]. Ringel's conjecture states that the complete graph K_{2m+1} can be decomposed into $2m+1$ subgraphs that are all isomorphic to a given tree on m edges. Kotzig later conjectured that such a decomposition could be achieved cyclically. Rosa established that these conjectures could be solved by showing that every tree can be labeled as described above, which was ultimately termed *graceful labeling.*

In this monograph, we begin by introducing the definitions needed to state the various conjectures and establish the early results of Rosa that facilitated an accessibility to this problem leading to the publication of hundreds of papers on the subject. While our main interest lies in the exposition of graceful labelings, we will see that Rosa defines four labeling in [**15**] that are intimately related to each other. At times we will give some of the results related to the other labelings. Chapter 2 contains Rosa's equivalence to the Ringel-Kotzig conjecture as well as other results pertaining to the labelings.

In Chapter 3, we first focus on graceful graphs in general and a problem posed by Golomb that utilizes graceful labelings in its solution. We then restrict to trees and present several of the results toward the verification of the graceful tree conjecture. Most of the results pertaining to this problem have come in the form of establishing the gracefulness of relatively small classes of trees. Our discussion is broken into two parts. First, we consider several families of trees shown to be graceful by more or less standard techniques such as counting arguments, exhaustive techniques, and label searches. The second phase of the discussion investigates the constructive methods developed to build larger and larger graceful trees. Throughout this chapter, we offer several examples of the constructions as it is often easier to understand them when they are presented visually.

Finally, we consider adjacency matrices and their properties with respect to gracefully labeled graphs. We establish several conditions that give us an easy way to check the gracefulness of a labeling given the graph's adjacency matrix. We also provide alternative ways to construct classes of graceful graphs by creating matrices from the adjacency matrices of smaller graphs that induce larger graceful graphs.

These conjectures, which were made in the mid 1960's, remain open to this day. Kotzig and Rosa have referred to the graceful tree conjecture as a "graphical disease" due to the inability to establish results reaching further than showing small classes of trees are graceful. In fact, Frank Harary was the first to label problems as diseases, and one of his originally diagnosed diseases was the aforementioned four-color theorem [**7**]. Nevertheless, work continues on the conjecture, more or less in the same direction. Our goal here is to give a brief overview of the problem, as well as a flavor for the techniques developed over the previous 4 decades. For a more comprehensive treatment on graceful labelings, the reader is encouraged to reference *A Dynamic Survey of Graph Labeling* maintained by Joseph A. Gallian [**5**].

Chapter 2

The Graph Labelings of Alexander Rosa

Here, we wish to survey the early types of graph labelings that were developed to be used in investigating the Ringel-Kotzig Conjecture. We begin with a few central definitions.

Definition 2.1. A *graph* $G = (V, E)$ consists of a set V of vertices and a set E of edges where each edge is of the form $e = v_i, v_j$ for some $v_i, v_j \in V$, $v_i \neq v_j$. The *order* of G is the number of vertices of G and is denoted by $|G|$. The *size* of G is the number of edges of G and is denoted by $||G||$.

In this thesis, a graph is a simple graph in that it has at most one edge between any two vertices in V, and no edge has the same vertex as its endpoints. When more than one graph is in question, we will sometimes denote the vertex and edge sets of a graph G by $V(G) = V_G$ and $E(G) = E_G$, respectively.

Definition 2.2. The *degree* of a vertex v in a graph G, denoted $\deg(v)$, is the number of edges incident with v.

Definition 2.3. For a vertex v in a graph G, the *neighborhood of* v is the set $N(v) = \{w \in V(G) | wv \in E(G)\}$.

Definition 2.4. A *labeling* (or *vertex labeling*) of a graph G, sometimes called a valuation, is an injective map $f : V \to \mathbb{N}$ that assigns to each vertex $v \in V$ a unique natural number.

Definition 2.5. A graph G is said to be *bipartite* if $V(G)$ can be partitioned into classes $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ such that the edge $x_i x_j \notin E(G)$ for any i, j and the edge $y_k y_\ell \notin E(G)$ for any k, ℓ.

Definition 2.6. A graph G on n vertices is called the *complete graph* on n vertices if the edge $v_i v_j \in E(G)$ for all $1 \leq i, j \leq n$, $i \neq j$. We denote the complete graph of order n by K_n.

Definition 2.7. A graph G is said to be the *complete bipartite* graph $K_{p,q}$ if $V(G)$ can be partitioned into classes $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ such that the edge $x_i x_j \notin E(G)$ for any i, j, the edge $y_p y_q \notin E(G)$ for any p, q, and $x_\ell y_m \in E(G)$ for all $1 \leq \ell \leq p$ and $1 \leq m \leq q$.

Definition 2.8. A *walk* between two vertices x and y in a graph G is a sequence of vertices $w_1 w_2 \ldots w_k$, where $x = w_1$, $y = w_k$, and for each $i \in \{1, 2, \ldots, k-1\}$ we have $w_i w_{i+1} \in E(G)$. The *length* of a walk is defined to be $k - 1$.

Definition 2.9. A *closed walk* in a graph is a walk that begins and ends with the same vertex.

Definition 2.10. A *path* between two vertices x and y in a graph G is a walk between x and y such that no vertex is repeated in the walk.

Definition 2.11. For vertices $x, y \in V(G)$, the distance, $d(x, y) = d(y, x)$, is the length of a shortest path between x and y.

Definition 2.12. The graph G is called *connected* if for any two vertices x and y in G, there is a path between x and y.

Definition 2.13. A *cycle* in a graph G is a closed walk of the form $w_1 w_2 \ldots w_k$, where $k \geq 3$, and if for some $i \neq j$ we have $w_i = w_j$, then $\{i, j\} = \{1, 2\}$.

Definition 2.14. A graph G is called *acyclic* if it contains no cycles.

Definition 2.15. A *forest* is an acyclic graph.

Definition 2.16. A *tree* is a connected acyclic graph.

Definition 2.17. A *leaf* of a tree T is a vertex $v \in V(T)$ such that $\deg(v) = 1$.

Definition 2.18. The *path graph* on n vertices, P_n, is the graph whose edges form a path of length $n-1$.

Definition 2.19. A *circuit* in a graph G is a closed walk that traverses an edge at most once.

Definition 2.20. A graph G is said to be *Eulerian* if there exists a circuit $C \subseteq G$ such that $E(C) \equiv E(G)$.

We now state several theorems noting that the proofs can be found in any introductory graph theory text.

Theorem 2.21. *A connected graph G is Eulerian if and only if the degree of each vertex in G is even.*

Theorem 2.22. *For any tree T, there is a unique path between any two vertices x and y in $V(T)$.*

Theorem 2.23. *Every tree is a bipartite graph.*

Theorem 2.24. *Every tree on $n \geq 2$ vertices has at least two leaves.*

Theorem 2.25. *A tree with n vertices has $n-1$ edges.*

In 1966, Alexander Rosa published *On Certain Valuations of the Vertices of a Graph* [**15**], which introduced several types of vertex labelings for simple graphs. These constructions were the result of his work on a problem Ringel had presented several years earlier in 1964, which was later strengthened by Kotzig. We now define the labelings as introduced by Rosa.

Let $G = (V, E)$ be a graph with n vertices and m edges, and let $f : V \to \mathbb{N}$ be a labeling of the vertices of V. Let $a_i = f(v_i)$, for $1 \leq i \leq n$. Define $V_{\Phi_G} = \{a_1, a_2, \ldots, a_n\}$ to be the set of labels of the vertices of G given by f. For any edge $e_k = v_i v_j$ in E, let $b_k = |f(v_i) - f(v_j)| = |a_i - a_j|$. We denote by E_{Φ_G} the set of values b_k of the edges of G induced by f.

Let f be a labeling of a graph G with m edges and consider the following conditions:

(1) $V_{\Phi_G} \subset \{0, 1, 2, \ldots, m\}$

(2) $V_{\Phi_G} \subset \{0, 1, 2, \ldots, 2m\}$

(3) $E_{\Phi_G} = \{1, 2, \ldots, m\}$

(4) $E_{\Phi_G} = \{x_1, x_2, \ldots, x_m\}$ where $x_i = i$ or $x_i = 2m + 1 - i$

(5) There exists x, with $x \in \{0, 1, 2, \ldots, m\}$, such that for any edge $v_i v_j$ of G, either $a_i \leq x$ and $a_j > x$, or $a_j \leq x$ and $a_i > x$ holds.

Then Rosa defines the following valuations:

- If f satisfies (1),(3), and (5), then f is called an α-valuation.
- If f satisfies (1) and (3), then f is called a β-valuation.
- If f satisfies (2) and (3), then f is called a σ-valuation.
- If f satisfies (2) and (4), then f is called a ρ-valuation.

From the definitions of these valuations, we see that there is a hierarchy structure relating them. That is, an α-valuation is also a β, σ, and ρ-valuation. A β-valuation is also a σ-valuation and a ρ-valuation. And finally, a σ-valuation is also a ρ-valuation. Moreover, if there is an α-valuation of G, then G is a bipartite graph. While the condition that G is bipartite is not sufficient for G to have an α-valuation, G being complete bipartite is. For this reason, α-valuations are also referred to as *bipartite labelings.*

THEOREM 2.26. *If a graph G has an α-valuation, then G is a bipartite graph.*

Proof. Let G have an α-valuation and partition $V_{\Phi_G} = \{a_0, a_1, \ldots, a_m\}$ as follows. Define two sets by

$$A := \{a_i \mid a_i > x\}$$
$$B := \{a_j \mid a_j \leq x\} \quad .$$

Clearly, $A \cap B = \emptyset$. Also, for any edge $e = a_\ell a_m$, the endvertices of e are in different partitions. Thus G is a bipartite graph. $\square$

Theorem 2.27. *The graph $K_{p,q}$ has an α-valuation.*

Proof. [**15**] Let X and Y partition E and let $V(X) = \{x_1, x_2, \ldots, x_p\}$ and $V(Y) = \{y_1, y_2, \ldots, y_q\}$. Then the valuation with $f(x_i) = i - 1$ and $f(y_j) = jp$ is an α-valuation. To see this, we note that $K_{p,q}$ has $m = pq$ edges. Without loss of generality, we may assume that $p \leq q$. Now, f maps $V(K_{p,q})$ into $\{0, 1, \ldots, pq\}$, and using $x = p - 1$ we have that for all edges $x_i y_j$, $f(x_i) \leq x$ and $f(y_j) > x$. Moreover, $|f(y_j) - f(x_i)| = f(y_j) - f(x_i) = jp - i + 1$. Since $i \in \{1, 2, \ldots, p\}$, we have that for a fixed j, $\{|f(y_j) - f(x_i)| | i \in \{1, 2, \ldots, p\}\} = \{(j-1)p + 1, (j-1)p + 2, \ldots, jp\}$, which gives that $E_{f(G)} = \{1, 2, \ldots, pq\}$. Thus f is an α-valuation. $\square$

These valuations provided the tools Rosa used to attack the conjectures of Ringel and Kotzig. These conjectures, along with the necessary definitions follow.

Definition 2.28. Let $G = (V, E)$ be a graph. Then the graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

Definition 2.29. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the *union* $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Definition 2.30. Let G be a graph. Then a *decomposition*, $\mathcal{D}$, of G is a collection of subgraphs $H_i \subseteq G$, $i = 1, 2, \ldots, k$, such that $E(H_i) \cap E(H_j) = \emptyset$ for all $1 \leq i < j \leq k$ and $G = \cup_{i=1}^{k} H_i$.

Definition 2.31. A *turning* of an edge $e = v_i v_j$ in K_n is an increase, by one, of the indices of the vertices of e so that the edge $e' = v_{i+1}v_{j+1}$ is obtained from e. The indices are taken modulo n. A turning of a subraph $H \subseteq G$ is the simultaneous turning of all edges of H.

Definition 2.32. A decomposition, $\mathcal{D}$, of K_n is called *cyclic* if for any graph $H \in \mathcal{D}$, the graph H', obtained by turning H is also in $\mathcal{D}$. In this case, we write $G = \langle H \rangle$ to denote that the decomposition is generated by H.

Conjecture 2.33 (Ringel, 1964). *The complete graph K_{2m+1} can be decomposed into $2m+1$ subgraphs, each of which is isomorphic to a given tree with m edges.*

Conjecture 2.34 (Kotzig, 1964). *The complete graph K_{2m+1} can be cyclically decomposed into trees isomorphic to a given tree with m edges.*

Example 2.35. Let $m = 2$. Then Theorem 2.34 implies there is a cyclic decomposition of K_5 into trees isomorphic to the tree with 2 edges, the path P_3. The cyclic decomposition $\mathcal{D}_{K_5} = \langle P_3 \rangle$ is shown below with each step of the turning shown. This is in fact a decomposition as shown in Figure 2.2 with each edge of K_5 belonging to exactly one copy of P_3.

These conjectures remain open, though Rosa does confirm them for certain classes of given graphs.

Definition 2.36. For a given fixed labeling $V(G) = \{v_1, v_2, \ldots, v_n\}$, the *length* of an edge $v_i v_j$ in $E(G)$ is a number $d_{ij} = \min\{|i-j|, n-|i-j|\}$.

Theorem 2.37. *A cyclic decomposition of K_{2m+1} into subgraphs isomorphic to a given graph G with m edges exists if and only if there is a ρ-valuation of G.*

Proof. We may fix any labeling $\{v_0, v_1, \ldots, v_{2m}\}$ of K_{2m+1} since each pair of vertices has an edge connecting them. From the definition of the length of an edge, it follows that in K_{2m+1} there are only edges of length $1, 2, \ldots, m$ and for any fixed

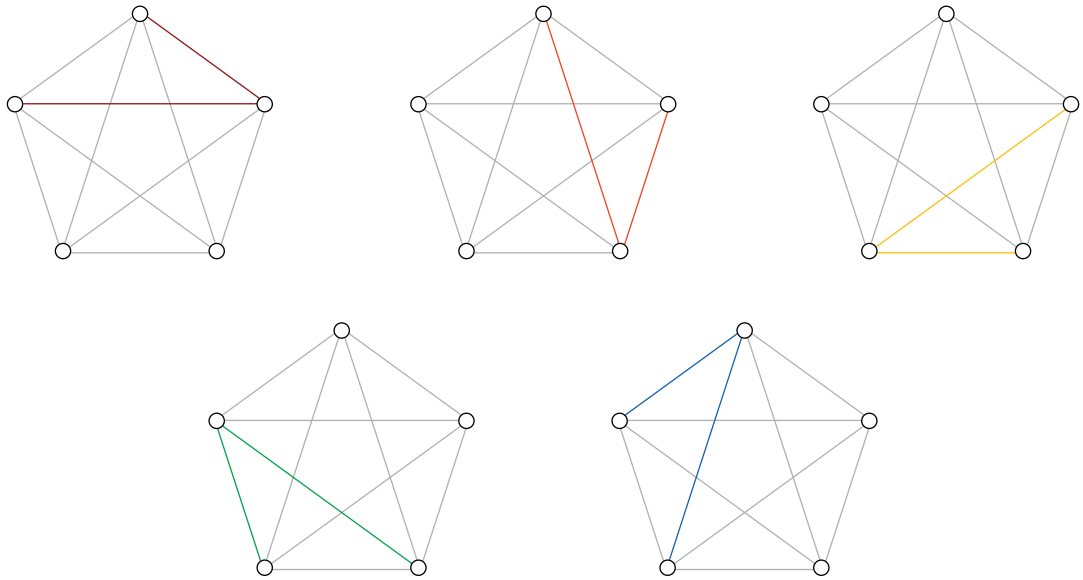

Figure 2.1: Cyclic Decomposition of K_5

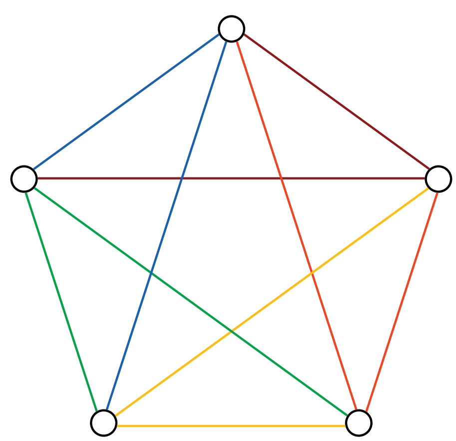

Figure 2.2: Edges of K_5

$i \in \{1, 2, \ldots, m\}$ exactly $2m + 1$ edges have length i. Moreover, those edges can be obtained from each other by a sufficient number of turnings. Also, the turn of an edge has the same length as the original edge.

To prove the sufficiency condition, assume G has a ρ-valuation, and let a_i be the value of w_i of G. We obtain an isomorphic copy G' of G in K_{2m+1} such that the vertex v_{a_i} of K_{2m+1} corresponds to the vertex x_i of G. Then

$$d_{ij} = \begin{cases} b_k & \text{if } b_k \leq n \\ 2n+1-b_k & \text{if } b_k > n \end{cases}$$

where b_k is the value of edge e_k of G and d_{ij} is the length of e_k in K_{2m+1}. This implies that the edges of G have mutually different lengths in K_{2m+1}, which in turn implies the existence of a cyclic decomposition of K_{2m+1} into subgraphs isomorphic to G, the last obtained by consecutively turning the graph G $2m$ times in K_{2m+1}.

Now, to prove the necessity, let a cyclic decomposition of K_{2m+1} into subgraphs isomorphic to G be given. Take an arbitrary subgraph $G_\star \cong G$ of the $2m{+}1$ subgraphs of this decomposition. We will prove that the edges of $G_\star$ have mutually different lengths in K_{2m+1}, and then the labels i of v_i in $G_\star$ will give the required ρ-valuation of G. Suppose to the contrary that $G_\star$ contains two edges of length i, $1 \leq i \leq n$. For example, consider $v_x v_{x+i}$ and $v_y v_{y+i}$ with $x \neq y$ and assume $y > x$. By the definition of a cyclic decomposition, this decomposition contains the graph $G_\star^{(y-x)}$ obtained from G by turning it $y - x$ times. But then this graph contains the edge $v_y v_{y+i}$, which contradict the definition of a graph decomposition. Thus all edges of $G_\star$ have mutually different lengths in K_{2m+1} and G has a ρ-valuation. $\square$

THEOREM 2.38. *If a graph G with m edges has an α-valuation, then there exists a cyclic decomposition of K_{2km+1} into subgraphs isomorphic to G, where k is an arbitrary natural number.*

PROOF. Let G be a graph with m edges, that is, $||G|| = m$. Assume that G has an α-valuation, and x is the number given in Condition (5) defining the α-valuation. Without loss of generality $V \subseteq \{v_0, v_1, \ldots, v_m\}$ such that the label of v_i in the α-valuation is i. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $b_k = k$. From the conditions for an α-valuation for each $\ell \in \{1, 2, \ldots, m\}$ we are given i_ℓ, j_ℓ such that $e_\ell = v_{i_\ell} v_{j_\ell}$ and $i_\ell \leq x$ and $j_\ell > x$.

Let $A = \{v_q \in V(G) | q \leq x\}$ and $B = V(G) - A$. For $p \in \{1, 2, \ldots, k\}$ we will define the graph $G^p = (V^p, E^p)$ as follows: $V^p = A \cup \{v_{j+m(p-1)} : v_j \in B\}$ and $E^p = \{e^p = v_{i_\ell} v_{j_\ell + m(p-1)} | \, \ell \in \{1, 2, \ldots, m\}\}$. Note that by definition, G^p is isomorphic to G, moreover, $G^1 = G$. Also, any edge in G^p has an endpoint in A, and another endpoint in the set $\{v_{(p-1)m}, v_{(p-1)m+1}, \ldots, v_{pm}\}$, so the G^p are edge-disjoint. Moreover, for the edge e^p we have that

$$|j_\ell + m(p-1) - i_\ell| \;=\; (j_\ell - i_\ell + m(p-1)) = \ell + m(p-1) \qquad (1)$$

Let $G' = \bigcup_{p=1}^{k} G^p$. Since the G^p are edge disjoint, $||G'|| = k||G|| = km$. By our construction, $V(G') \subseteq \{v_0, \ldots, v_{km}\}$. Define the valuation $\Phi : V(G') \to \{0, 1, \ldots, km\}$ by $\Phi(v_q) = q$. As for any edge of G', there is a $p \in \{1, \ldots, k\}$ and $\ell \in \{1, 2, \ldots, m\}$ such that the edge is of the form $v_{i_\ell}, v_{(p-1)m+j_\ell}$, where $i_\ell \leq x$ and $(p-1)m + j_\ell \geq j_\ell > x$. Also, by equation (1) we have that $E_{\Phi(G')} = \{1, 2, \ldots, km\}$. Thus, ϕ is an α- (and thus also a ρ-)valuation of G', which, by Theorem 2.37 means that there is a cyclic decomposition of K_{2km+1} into subgraphs isomorphic to G'. But G' is an edge-disjoint union of k copies of G, so this decomposition gives a cyclic decomposition of K_{2km+1} into subgraphs isomorphic to G. $\square$

If it were shown that every tree has an α-valuation, then both Ringel and Kotzig's conjectures would be confirmed. It has been proven, however, that not every tree has an α-valuation. Since then, attention has been turned to investigating which graphs have β-valuation. If it could be shown that every tree has a β-valuation, then Kotzig's Conjecture would be proven. This implies every tree has a ρ-valuation, which would in turn prove Ringel's Conjecture. In 1972, Solomon Golomb published an article containing his results on graph labelings [**6**]. Among some of the labelings he investigates were β-valuations, which he termed *graceful labelings*. It is now standard to use the terms graceful labeling and graceful graph when referring to a graph that admits a β-valuation. We follow suit and will use Golomb's terminology from now on. To repeat, therefore, the graceful labeling of a graph is defined as follows:

DEFINITION 2.39. Let G be a graph on n vertices with m edges. G is said to have a *graceful labeling* if there is a map $f : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ such that $|f(x) - f(y)|$ is unique for each edge $xy \in E(G)$. We then necessarily have that $E(G) = \{1, 2, \ldots, m\}$.

Finally, we wish to note that shifting the vertex labels by a positive integer k does not affect whether or not the labeling is graceful. That is, if a graph is gracefully labeled using labels from $\{0, 1, \ldots, m\}$, we get the same edge labels if we add k to each vertex label, with each still being distinct. In general, we will be using labels from $\{0, 1, \ldots, m\}$. In other cases, we will specify when we are using labels from $\{k, 1 + k, \ldots, m + k\}$ by saying that we will be labeling beginning with k.

We now turn to the advancements in labeling graphs gracefully.

Chapter 3

Graceful Graphs

We now wish to restrict our discussion of graph labelings to one in particular — graceful labelings. There have been many results in this direction. The majority of the work done with this labeling applies to the class of trees, due obviously to its connection with the decomposition theorems mentioned earlier. We will first look at graceful graphs in general, then give results obtained for trees.

3.1. Graceful Graphs and a Classic Problem

Golomb [**6**] gives an application of a graph labeling, and gives a condition for the labeling to be graceful. Suppose we are given solid bar of integer length ℓ. We want to carve $n-2$ notches in it at integer distances from either end with the condition that the distances between any two notches, as well as the distances from any notch to either of the endpoints, are all distinct. This problem can be viewed as labeling the vertices of the complete graph K_n with the elements of the set $\{0, 1, \ldots, \ell\}$ and labeling each edge by taking the absolute value of the difference of its endvertices. The problem is to find the smallest integer ℓ so that K_n can be labeled this way. In the event that the vertices can be labeled using only numbers from $N = \{0, 1, 2, \ldots, \binom{n}{2}\}$, with each edge label being distinct, then each number from $N \setminus \{0\}$ is used once. But this simply implies that for this n, K_n has a graceful labeling.

Example 3.1. Looking at the complete graphs on n vertices for $n = 2, 3$ and 4, we have that the vertex labelings shown in Figure 3.1 are graceful.

It turns out that $n = 4$ is the largest n for which K_n can be labeled gracefully, as proven by Golomb.

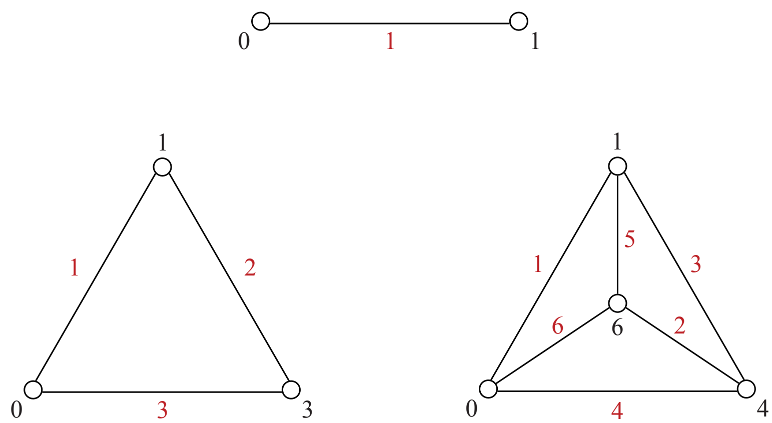

Figure 3.1: Gracefully Labeled K_n for $n = 2, 3, 4$

Theorem 3.2. *If $n > 4$, the complete graph K_n is not graceful.*

Proof. [**6**] We begin by noting that for $n > 4$, $m = |E(K_n)| = \binom{n}{2} \geq 10$. Suppose to the contrary that K_n is graceful. Then $V = V(K_n) \subseteq \{0, 1, 2, \ldots, m\}$ and the edges have distinct labels with $E = E(K_n) = \{1, 2, \ldots, m\}$.

For K_n to have an edge labeled m, we have that 0 and m must be vertices since $|m - 0| = m$ is an edge label. In order to have an edge labeled $m - 1$, either 1 or $m - 1$ is also a vertex label. For any graceful graph G with m edges, the replacement of every vertex label a_i with $m - a_i$ does not change the edge labels. Thus we can choose the vertex label 1 for K_n instead of $m - 1$ with no loss of generality.

Now, to get an edge labeled $m - 2$, we have to adjoin the vertex label $m - 2$. This is because in order to get the edge label $m - 2$ from the difference of $m - 1$ and 1, we would have two edges labeled 1, one between vertices 0 and 1, and the other between $m - 1$ and m. If a vertex labeled 2 is added to get an edge labeled $m - 2$ as the difference of m and 2, again we get two edges labeled 1, one between vertices 0 and 1, and the other between vertices 1 and 2.

With vertices having labels 0, 1, $m - 2$, and m, we get edges labeled 1, 2, $m - 3$, $m - 2$, $m - 1$, and m. In order to have an edge labeled $m - 4$, we must use 4 as a vertex label. This is because if we use 2, then we get two edges labeled 2 as a difference of 2 and 0 and as a difference of $m-2$ and m.

If $n = 5$, then we should be done (we do have precisely 5 vertex labels). However, in this case $m = 10$, and we have that $m - 6 = 4$, so there actually is a pair of edges that have the same induced label, as shown in Figure 3.2. Thus, for $n = 5$ there is no good labeling. We need to only consider the case when $n > 5$. In this case $m > 10$ and so $4 < m - 6$ and there clearly is no repetition of edge labels at this point; moreover, we still need more labels. With vertices having labels 0, 1, 4, $m-2$,

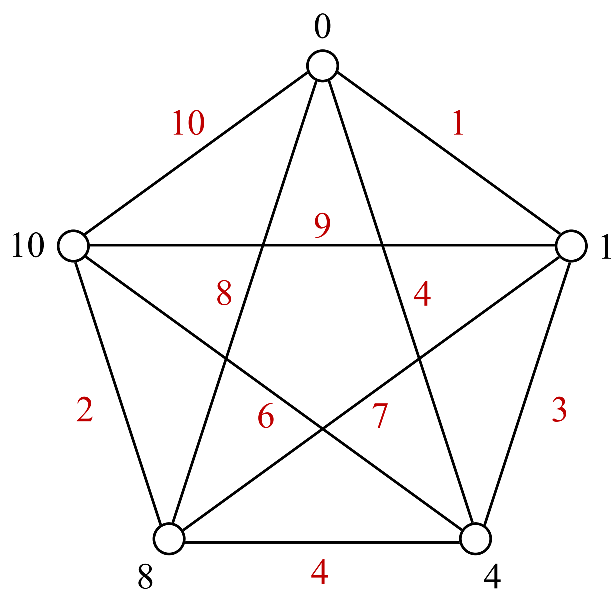

Figure 3.2: K_5 with Repeated Edge Labels

and m, we have edge labels 1, 2, 3, 4, $m-6$, $m-4$, $m-3$, $m-2$, $m-1$, and m. Note that for K_4, $m = 6$.

There is no way to obtain an edge labeled $m-5$ because all choices for ways to get such an edge label contains at least one vertex label that can't be used. Adding a vertex labeled $m-5$ gives a second edge labeled $m-6 = (m-5)-1$. Adding a vertex labeled $m-4$ gives two edges labeled $4 = m-(m-4)$. Adding a vertex labeled $m-1$ gives duplicate edges labeled $m-2=(m-1)-1$. We cannot add a vertex labeled 3 since we get two edges labeled $3-1=2$. Finally, our last choice is adding a vertex labeled 5. This is not possible either since we get two edges labeled $5-4=1$. This is a contradiction to our assumption that K_n is graceful for all cases when $m-5>4$, which corresponds to $n \geq 5$. $\square$

Rosa also gives a necessary condition for a graph to be graceful.

THEOREM 3.3. [**15**] *If G is an Eulerian graph with m edges such that $m \equiv 1$ or $2 \pmod 4$, then G cannot be labeled gracefully.*

PROOF. Suppose to the contrary that G is graceful graph of size $m \equiv 1$ or $2 \pmod 4$ with an Eulerian circuit, and suppose G is graceful. Taking the sum over all edge labels b_i , we have

$$\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} i = \frac{(m+1)m}{2}.$$

Now it is easy to see that if $m \equiv 1$ or $2 \pmod 2$, then

$$\sum_{i=1}^{m} b_i \equiv 1 \pmod 2$$

However, for an arbitrary closed path $C = a_1, a_2, \ldots, a_m, a_1$, the edge labels are given by

$$|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_m - a_1|,$$

and, as $|y| \equiv y \pmod 2$, the sum of the edge labels, with the indicies taken modulo $m+1$, is

$$\sum_{i=1}^{m} |a_i - a_{i+1}| \equiv \sum_{i=1}^{m} (a_i - a_{i+1}) \equiv 0 \pmod 2.$$

Thus, we have reached a contradiction and the claim is proven. $\square$

3.2. The Graceful Tree Conjecture

The conjectures of Ringel and Kotzig have led to one of the most easily stated, yet elusive conjectures in the realm of graph labelings.

Conjecture 3.4. *All trees are graceful.*

Many methods have been developed in hopes of resolving the nearly fifty year old problem. Initially, Rosa established the gracefulness of several classes of trees in [**15**]. Since then, other classes have been shown to admit graceful labelings. We will investigate several of these classes, dividing the proof methods into constructive and non-constructive classes.

3.2.1. Several Classes of Graceful Trees. Here we consider several classes of graceful trees. Unless otherwise stated, the graceful labelings given in this section are starting from 1 in keeping with the convention used by the majority of publications on graceful trees. The first class considered, caterpillars, was shown to be graceful by Rosa in [**15**].

Definition 3.5. A *caterpillar* is a tree such that either every vertex is a leaf, or by removing all leaves, we obtain a path. We call such a path a *central path* of the caterpillar.

Now, it follows that a caterpillar has at least 2 vertices, and if it has at least 3 vertices, then the central path exists and is unique. The next theorem establishes the gracefulness of caterpillars.

Theorem 3.6. *All caterpillars are graceful.*

Proof. [**16**] Let T be a caterpillar with n vertices. If $n \leq 2$, then T is a path of length one and we are done. So assume $n \geq 3$, and let $P = v_0 v_1 \ldots v_k$ be the path obtained from T by deleting the leaves of T. Since $n \geq 3$, P has at least one vertex.

We now partition the vertex set of T as follows:

$$\begin{aligned} X &= \{x | x \in V(T),\ d(v_0, x) \equiv 0 \pmod 2\} \\ Y &= \{y | y \in V(T),\ d(v_0, y) \equiv 1 \pmod 2\} = V(T) - X. \end{aligned}$$

We note that $v_i \in X$ if i is even and $v_i \in Y$ if i is odd. Label v_0 with n. Label the neighbors of v_0 with $1, 2, 3, \ldots$ where the neighbor getting the largest label is v_1. Assign labels $n-1, n-2, n-3, \ldots$ to the neighbors of v_1, other than v_0, with the largest label going to v_2. Continue as follows. After v_{2i} receives its label, assign increasing integer labels to its neighbors other than v_{2i-1} starting with the smallest unused label, assigning the largest label to v_{2i+1}. Then, assign labels to the neighbors of v_{2i+1}, other than v_{2i}, in decreasing order starting with the largest unused integer smaller than n ending by labeling v_{2i+2}.

The resulting labeling gives labels $n, n-1, \ldots, n-|X|+1$ to vertices in X while vertices of Y are labeled $1, 2, \ldots, |Y|$. Moreover, since $|X| + |Y| = n$, the labels of the vertices are all different.

Let ℓ_i be the label of the vertex v_i. Clearly, $\ell_1 < \ell_3 < \ldots$ and $\ell_0 > \ell_2 > \ldots$. Now for an even i, if $i \notin \{0, k\}$, the neighbors of v_i have labels $\ell_{i-1}, \ell_{i-1}+1, \ldots, \ell_{i+1}$ and the induced edge labels are $\ell_i - \ell_{i+1}, \ell_i - \ell_{i+1} - 1, \ldots, \ell_i - \ell_{i-1}$. For an odd i, if $i \neq k$, the neighbors of the vertex v_i have labels $\ell_{i-1}, \ell_{i-1}+1, \ldots, \ell_{i+1}$, and the induced edge labels are $\ell_{i-1} - \ell_i, \ell_{i-1} + 1 - \ell_i, \ldots, \ell_i + 1 - \ell_i$. The neighbors of v_0 have labels $1, 2 \ldots, \ell_1$, the neighbors of v_k have labels $\ell_{k-1}, \ell_{k-1}+1, \ldots, \ell_k - 1$ if k is even, and $\ell_k + 1, \ell_k + 2, \ldots, \ell_{k+1}$ if k is odd. From all this it is easy to see that the $n-1$ edge-labels are all different, the largest is $n-1$ and the smallest is 1. Thus the given labeling, as shown in Figure 3.3, is graceful. Note that the given labeling is also an α-labeling, since conditions (1) and (3) are already checked, and condition (5) clearly holds using $x = |Y|$. $\square$

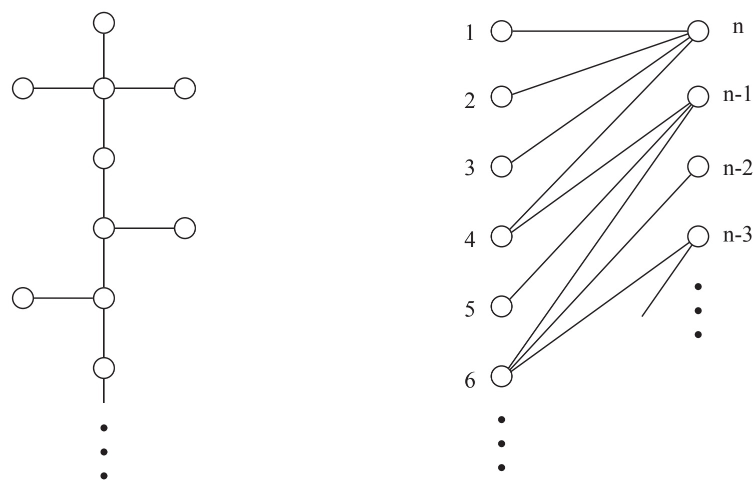

Figure 3.3: Gracefully Labeling a Caterpillar

Originally, it was shown that caterpillars admit α-valuations, and thus are also graceful [**15**]. Rosa demonstrated his method for labeling an arbitrary caterpillar gracefully without proof in [**15**]. We note that the labeling he gives is the inverse labeling of that from Theorem 3.6.

Definition 3.7. A *star* is any graph of the form $K_{1,k}$. Thus a star must be a tree on $k+1$ vertices with at least k leaves.

So stars are trees with one vertex that is adjacent to any number of leaves. It follows that paths and stars are simply caterpillars. That is, a path is a caterpillar with exactly two leaves, while stars with at least two leaves are caterpillars whose central paths have length 0.

Another early result of Golomb is that every tree on five vertices is graceful [**6**]. He proves this by simply exhibiting a graceful labeling for all trees on five vertices. This also follows easily from the above theorems as the smallest tree on at least two vertices that is not a caterpillar has 7 vertices, as seen in Figure 3.4.

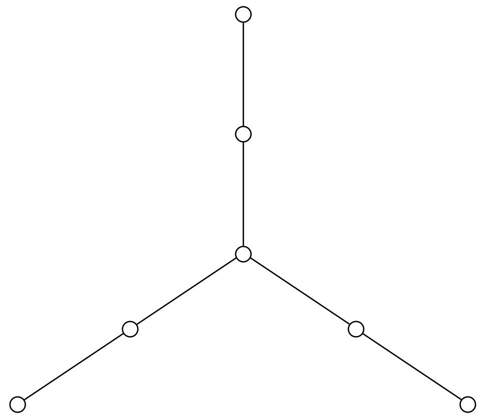

Figure 3.4: The Smallest Non-Caterpillar Tree

Definition 3.8. Let G be a graph. Then the *diameter of* G, denoted $\operatorname{diam}(G)$ is the length of a longest path in G.

We now show that any tree T with $1 \leq \operatorname{diam}(T) \leq 3$ is a caterpillar.

Theorem 3.9. *Let T be a tree with diameter at least 1 and at most 3. Then T is a caterpillar.*

Proof. Assume that the diameter of the tree T is i, where $i \in \{1, 2, 3\}$.

If $i = 1$, then clearly $T = K_2$, and we are done.

If $i = 2$, then let $P = v_0v_1v_2$ be a longest path in T. Clearly v_0 and v_2 must be leaves, otherwise P could be extended. Thus, if T has any other vertex y, then there must be a path between y and v_1 that does not go through v_0 or v_2. We can extend this path by adding v_0 to its end. Since 2 is the length of the longest path, this means that y must be attached to v_1 by an edge. Thus T is a star, and we are done.

Finally, if $i = 3$, let $P = v_0v_1v_2v_3$ be a longest path in T. Clearly v_0 and v_3 must be leaves, otherwise P could be extended. If T has any other vertex y, then there must be a path from y to a vertex of P that does not contain any other vertex of P. This means we must have a path from y to v_1 or v_2. Assume the path is to v_1 (the other case can be handled similarly). If the path from y to v_1 is not a single edge, then we obtain a new path that is of length at least 4 by appending v_2 and v_3. Thus, any vertex y of T not on P is attached by a single edge to either v_1 or v_2. Therefore, T is a caterpillar. $\square$

Huang, Kotzig, and Rosa [**9**] believed that showing all trees of diameter four were graceful would be a major achievement in getting closer to proving the graceful tree conjecture.

Let $\mathcal{T}(4)$ be the set of all trees of diameter 4. Let $\mathcal{T}(0,4)$ be the singleton set containing the path of length four. Now let $\mathcal{T}(1,4)$ be the set of caterpillars of diameter 4 excluding P_5, and let $\mathcal{T}(2,4)$ be the set of remaining trees of diameter 4. Then $\mathcal{T}(4) = \mathcal{T}(0,4) \cup \mathcal{T}(1,4) \cup \mathcal{T}(2,4)$, and these sets partition $\mathcal{T}(4)$. The notation for these sets comes from $\mathcal{T}(i,n)$ containing the trees where there is a fixed longest path of length n and a vertex x whose distance to the path is i, and all other vertices have distance at most i from the path. Also, $\mathcal{T}(i,n)$ is non-empty provided $i \leq \lfloor \frac{n}{2} \rfloor$, and these sets partition $\mathcal{T}(n)$, the set of trees on n vertices. Clearly any tree from $\mathcal{T}(0,4) \cup \mathcal{T}(1,4)$ is graceful. Showing the set $\mathcal{T}(2,4)$ is graceful is slightly more challenging. Huang et al. prove the following:

Theorem 3.10. *If $T \in \mathcal{T}(2,4)$, then T does not have an α-labeling.*

The authors then proceed to show a special subclass of $\mathcal{T}(2,4)$ is graceful. Shi-Lin Zhao shows that the trees of $\mathcal{T}(2,4)$ are graceful in [**17**].

Theorem 3.11. *All trees of diameter 4 are graceful.*

Zhao notes that although he was able to confirm that all trees of diameter four are graceful, the result does not impact the general graceful tree conjecture as greatly as hypothesized by Huang, Kotzig, and Rosa.

In 2001, the team of Hrnčiar and Haviar established that trees of diameter 5 are graceful in [**8**].

Theorem 3.12. *All trees of diameter 5 are graceful.*

Definition 3.13. Let $G^1 = (V_1, E_1)$ and $G^2 = (V_2, E_2)$ be graphs with no common vertices, and let $u \in V_1$ and $v \in V_2$. By identifying the vertex u with the vertex v we mean the procedure that results in the graph $G = (V, E)$, where $V = (V_1 \cup V_2) - \{v\}$ and

$$E = E_1 \cup \Big(E_2 \setminus \{vx | vx \in E_2\} \Big) \cup \{ux | vx \in E_2\}$$

We will also call this the uv join of G^1 and G^2 and denote it by $G^1_u \oplus G^2_v$.

Now, if G^1 and G^2 are trees, then clearly their uv-join is also a tree.

Definition 3.14. Let $G^i = (V_i, E_i)$ be vertex-disjoint graphs with $v_i \in V_i$. By identifying the vertices $v_1, v_2, \ldots, v_k$ we mean the result of the following recursive procedure: For each $j \in \{2, 3, \ldots, k\}$ we define

$$G^1_{v_1} \oplus G^2_{v_2} \oplus \ldots \oplus G^j_{v_j} = \Big(G^1_{v_1} \oplus G^2_{v_2} \oplus \ldots \oplus G^{j-1}_{v_{j-1}} \Big)_{v_1} \oplus G^j_{v_j}$$

Definition 3.15. A *spider* is a tree consisting of m paths $P_{x_1}, P_{x_2}, \ldots, P_{x_m}$ where the vertices $v_1, v_2, \ldots, v_m$, with each v_i a leaf of P_{x_i}, are identified.

Huang et al. prove several results on spiders [**9**]. Recall that an α-valuation is called a bipartite labeling.

Lemma 3.16. *Let f_1 be a bipartite labeling with labels starting at 0 of a tree S with $f_1(u) = 0$ for some $u \in V(S)$. Let f_2 be a graceful labeling (bipartite labeling, resp.) with labels starting at 0 of a tree T with $f_2(v) = 0$ for some $v \in V(T)$. Then there is a graceful labeling (bipartite labeling, resp.) of $S_u \oplus T_v$.*

Here we only sketch the proof. Define the labeling g of $S_u \oplus T_v$ by

$$g(z) = \begin{cases} f_1'(z) & \text{if } z \in V(T_1) \setminus \{u\}, \text{ and } f_1'(z) \leq x \\ f_1'(z) + m & \text{if } z \in V(T_1) \setminus \{u\}, \text{ and } f_1'(z) > x \\ f_2(z) + x & \text{if } z \in V(T_2) \setminus \{v\} \\ x & \text{if } z = u \end{cases}$$

where x is the value from condition (5) pertaining to a bipartite labeling, $m_i = ||T_i||$, and the valuation f_1' is the inverse labeling of f_1. That is, $f_1'(z) = m_1 - f_1(z)$.

It is easy to see that an edge in T that has at least one endpoint on the vertices of T_2 has the same value induced by g as the value induced by f_2 on the corresponding edge of T_2. Thus, on these edges we have the induced edge-labels $1, 2, \ldots, m_2$. Therefore we only need to show that the edge labels induced by g on the edges of T_1 are $m_2 + 1, \ldots, m$. That easily follows from the fact that f_1' induces edge-labels $\{1, 2 \ldots, m_1\}$ and all edges of T_1 run between points $z_1 z_2$ with $f_1(z_1) \leq x$ and $f_1(z_2) > x$.

We pause to note that it has been shown in [**14**] that for any path P_n and any $v \in V(P_n)$, there exists a graceful labeling f of P_n such that $f(v) = 0$. Also, there is a bipartite labeling g of P_n such that $g(v) = 0$ for any $v \in V(P_n)$ except when $n = 5$ and v is the middle vertex of P_5.

THEOREM 3.17. *The spiders $S(x_1, x_2, x_3)$ with three legs are graceful.*

PROOF. This proof follows directly from the previous comment on labeling paths and from Lemma 3.16. Identify a leaf from the path P_{x_1} with a leaf of P_{x_2}. The resulting tree, T, is itself a path. Choose a graceful labeling of T such that the identified vertex has label 0. Then, choose a bipartite labeling of P_{x_3} such that one of its leaves has label 0. Finally, joining T and P_{x_3} and relabeling according to Lemma 3.16 results in the tree $S(x_1, x_2, x_3)$, which is labeled gracefully. $\square$

It follows that every spider with three legs, except for $S(2,2,2)$ has a bipartite labeling. $S(2,2,2)$ does not since there is no bipartite labeling of P_5 with the central vertex having label 0.

THEOREM 3.18. *The spiders $S(x_1, x_2, x_3, x_4)$ with 4 legs are graceful.*

PROOF. [**9**] If at least one of x_1, x_2, x_3, x_4 is not 2, then without loss of generality $x_1 + x_2 \neq 4$. Let u be the central vertex of the spider $S(x_1, x_2) = P_{x_1+x_2}$ and v be the central vertex of the spider $S(x_3, x_4) = P_{x_3+x_4}$. We know that there is a bipartite labeling of $S(x_1, x_2)$ that labels u with 0 and a graceful labeling of $S(x_3, x_4)$ that labels v with 0. The result follows from Theorem 3.16. The only other possibility is that $x_1 = x_2 = x_3 = x_4 = 2$. A graceful labeling of $S(2,2,2,2)$ is shown in Figure 3.5.

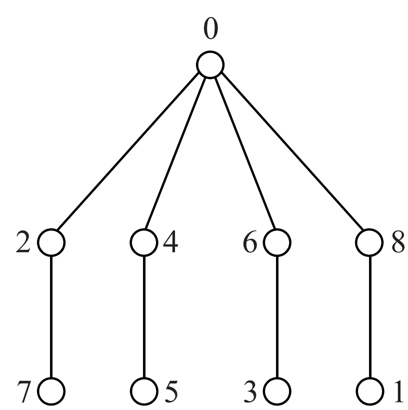

Figure 3.5: A Graceful Labeling of $S(2,2,2,2)$.

$\square$

We will revisit spiders in Chapter 4 and show other subclasses that are graceful.

In 1998, Aldred and McKay published a paper outlining a computer based algorithm they developed for testing trees for gracefulness [**1**]. Their result extended Rosa's result from [**15**] that all trees on at most 16 vertices are graceful.

THEOREM 3.19. *All trees on at most 27 vertices are graceful.*

We now present the algorithm noting that they used two different search methods in proving Theorem 3.19. For a tree T and an arbitrary labeling L of the vertices, let $z(T, L)$ denote the number of distinct edge labels induced by L. The goal of the algorithm is to find a labeling L such that $z(T, L) = |V(T)| - 1$ as this would imply the edge labels are distinct. For vertices $v, w \in V(T)$, define $S_w(L; v, w)$ to be the labeling achieved by swapping the labels on v and w given by L. Their first method, whch relies on the parameter M, follows:

(1) Start with any arbitrary vertex labeling of T with labels $\{0, 1, \ldots, n-1\}$.

(2) If $z(T, L) = n - 1$, stop.

(3) For each pair of vertices $\{v, w\}$, replace L by $L' = S_w(L; v, w)$ if $z(T, L') > z(T, L)$.

(4) If (3) finishes with L unchanged, replace L by $S_w(L; v, w)$, where the vertices $\{v, w\}$ are chosen at random from the set of all $\{v, w\}$ such that

   (a) $\{v, w\}$ has not been chosen during the most recent M times this step has been executed.

   (b) $S_w(L; v, w)$ is maximal subject to (a).

(5) Repeat from step (2).

The authors note that this method sometimes gets stuck, but the problem is resolved by restarting the algorithm using a different arbitrary initial labeling. The parameter M prevents the algorithm from repeatedly cycling through a small number of labelings. They add that $M = 10$ is usually suitable for smaller trees, but a useful value for M increases as the size of the tree does. Their second method was an exhaustive search over a restricted class of labelings. This second method was used when the first search method did not succeed fast enough. They note that a graceful labeling for a tree was typically used in determining the initial labeling for a larger tree, and that the first method usually terminated quickly with a graceful labeling. Their second method was invoked when this was not the case. Aldred and McKay also illustrate the volume of their computations by informing us that there are 279,793,450 trees of order 26 and 751,065,460 trees of order 27.

3.2.2. Constructive Methods. Constructive methods for finding families of graceful trees often originate with the goal of extending a known graceful tree or collection of trees to a larger tree with a graceful labeling. In 1976, Cahit [**3**] posed the question "Are all complete binary trees graceful?" This helped spark an interest in graceful labelings and led to early research in constructive methods for finding graceful trees. This section contains several constructions, each of which uses vertex labelings starting with label 1.

Definition 3.20. An *n-ary tree* is a rooted tree in which the root has degree 0 or degree n, and every other non-leaf vertex has degree $n+1$.

Definition 3.21. The *complete n-ary tree* is an n-ary tree in which every root-leaf path has length k. Denote by $T(n,k)$ the complete n-ary tree with root-leaf path length k.

It was Koh, Rogers, and Tan [**11**] who were able to verify an even stronger statement. Let T be a graceful tree on n vertices where the vertex v has label n. Let

$$T^p = \dot{\bigcup_{i=1}^{p}} T_i$$

for any $p \in \mathbb{N}$ and where each T_i is an isomorphic copy of T. That is, T^p is the disjoint union of p copies of T. Now, adjoin a new vertex $v^\star$ to T^p and the edge $\{v_i, v^\star\}$ for all $1 \leq i \leq p$, where v_i is the isomorphic image of v in T_i. Let T_v^p be the graph constructed in this way. The claim, then, is that T_v^p is graceful.

Theorem 3.22. *Let T be a gracefully labeled tree with n vertices, and let $v \in V$ be the vertex with label n. Then for any $p \geq 1$, the graph T_v^p is graceful and $v^\star$ has the label $np+1$.*

PROOF. [**11**] For each vertex w in T, let $d(w)$ (called the level of w) be the length of the shortest path joining v and w in T. Given any integer $p \geq 1$, we define a valuation $f^\star$ from the vertex set of T_v^p onto the set $\{1, 2, \ldots, np+1\}$ in terms of the valuation f and by the notion $d(w)$ as follows:

(1) $f^\star(v^\star) = pn + 1$

(2) for each w in $T_i \cong T$, $i = 1, 2, \ldots, p$,

$$f^\star(w) = \begin{cases} i \cdot n + 1 - f(w) & \text{if } d(w) \text{ is even in } T \\ (p+1-i)n + 1 - f(w) & \text{if } d(w) \text{ is odd in } T \end{cases}$$

Note that $f^\star(v_i) = (i-1)n + 1$ for each $i = 1, 2, \ldots, p$, and $f^\star(v^\star) = pn + 1 > f^\star(w)$ for each w in $T_v^p - \{v^\star\}$.

To show that $f^\star$ is a valuation, it suffices to show that $f^\star$ is one-to-one. Thus, let u and w be two distinct vertices in $\cup T_i$, $i = 1, 2, \ldots, p$, and suppose to the contrary that $f^\star(u) = f^\star(w)$.

Case 1. Both u and w are in T_i for some $i = 1, 2, \ldots, p$.

Clearly, $u \neq w$ in T. Since $f(u) \neq f(w)$, u and w can neither both lie on an even level nor both on an odd level in T. Thus, by symmetry, we may suppose that $d(u)$ is even and $d(w)$ is odd in T. By the definition of $f^\star$, we get $i \cdot n + 1 - f(u) = (p+1-i)n + 1 - f(w)$, which implies $n|p+1-2i| = |f(w) - f(u)| < n$ and forces that $p + 1 - 2i = 0$. But then it follows that $f(u) = f(w)$, which is not possible.

Case 2. u is in T_i and w is in T_j where $i, j = 1, 2, \ldots, p$, and $i \neq j$.

Assume that $d(u)$ is odd and $d(w)$ is even in T. From the fact that $f^\star(u) = f^\star(w)$, it follows that $(p+1-i)n+1-f(u) = jn+1-f(w)$. Thus we have $n|j-(p+1-i)| = |f(w) - f(u)| < n$ which implies that $j - (p+1-i) = 0$ and hence $f(u) = f(w)$, a contradiction.

Suppose either $d(u)$ and $d(w)$ are odd or $d(u)$ and $d(w)$ are even (say the latter). Then $i \cdot n + 1 - f(u) = jn + 1 - f(w)$, that is, $n|j-1| = |f(w) - f(u)|n$, which again implies that $j = i$, a contradiction.

Hence $f^\star$ is one-to-one and therefore is a valuation from the vertex set of T_v^p onto the set $\{1, 2, \ldots, pn+1\}$

To prove that $f^\star$ is a graceful valuation on T_v^p, it remains to be shown that the edge labels of T_v^p induced by $f^\star$ are distinct. For each edge uw in T, let $u_i w_i$ be the corresponding copy of uw in T_i, and let $f^\star(u_i w_i)$ be the label induced by $f^\star$ on the edge $u_i w_i$.

Now, since an edge in T must run between different a vertex of even and a vertex of odd level, we may assume without loss of generality that u is at even level and w is at odd level. Thus, we must have that

$$f^*(u_i w_i) = |in + 1 - f(u) - (p+1-i)n - 1 + f(w)| = |(2i - p - 1)n + f(w) - f(u)|.$$

This implies that

$$\{f^*(u_i w_i), f^*(u_{p+1-i} w_{p+1-i}\} = \{|2i-p-1|n + f(uw), |2i-p-1|n - f(u,w)\}$$

For each integer $i \in \{1, 2 \ldots, \lfloor p/2 \rfloor\}$ let

$$\begin{aligned} S_i &= \{f^*(u_i w_i), f^*(u_{p+1-i} w_{p+1-i}) | uw \in E(T)\} \\ &= \{(p+1-2i)n + f(uw), (p+1-2i)n - f(u,w) | uv \in E(T)\}. \end{aligned}$$

Clearly, unless p is odd and $i = \frac{p+1}{2}$, all integers in S_i are pairwise distinct, and for each x in S_i, either $(p-2i)n < x < (p+1-2i)n$ or $(p+1-2i)n < x < (p+2-2i)n$. As T is graceful, the $2(n-1)$ edges in $T_i \cup T_{p+1-i}$ are therefore assigned the following

$2(n-2)$ distinct integers:

$$
\begin{gathered}
(p-2i)n+1, \\
(p-2i)n+2, \\
\vdots \\
(p-2i)n+(n-1) \quad \text{and} \\
(p+1-2i)n+1, \\
\vdots \\
(p+1-2i)n+(n-1)
\end{gathered}
$$

If p is odd and $i = \frac{p+1}{2}$, then $i = p+1-i$, thus, by definition, $S_{(p+1)/2}$ becomes

$$
\begin{aligned}
S_{(p+1)/2} &= \{f^*(u_{(p+1)/2}w_{(p+1)/2}) | uw \in E(T)\} \\
&= \{f(uw) | uw \in E(T)\}
\end{aligned}
$$

Note that if p is even, then the p isomorphic copies $T_1, \ldots, T_p$ are grouped into $p/2$ pairs $\{T_1, T_p\}, \{T_2, T_{p-1}\}, \ldots, \{T_{p/2}, T_{p/2+1}\}$. If p is odd, let

$$S_{(p+1)/2} = \{f^\star(u_{(p+1)/2}w_{(p+1)/2}) | uw \text{ is an edge in } T\}$$

By the definition of $f^\star$, $f^\star(u_{(p+1)/2}w_{(p+1)/2}) = f(uw)$. Since $f^\star(v^\star v_1) = p, \ldots, f^\star(v^\star v_i) = (p+1-i)n, \ldots, f^\star(v^\star v_p) = n$, it follows that each integer $x = 1, 2, \ldots, pn$ has been assigned to an edge in T_v^p. Thus the proof is complete. □

The preceding proof allows us to take p copies of a graceful tree and identify the largest labeled vertex in each copy with a leaf of the star $K_{1,p}$. It follows immediately that every complete binary tree is graceful. Furthermore, all complete n-ary trees are also graceful since they can be constructed inductively with the base case corresponding to the star $K_{1,n-1}$ on n vertices with central vertex labeled n and each step of the induction producing a graceful tree using the labeling defined in Theorem 3.22.

EXAMPLE 3.23. Here we would like to illustrate how Theorem 3.22 can be used to build a larger graceful tree T_v^3 from a smaller graceful tree T. Let T be the tree in Figure 3.6, where the vertex labels are given by the valuation f and $f(v) = 6$.

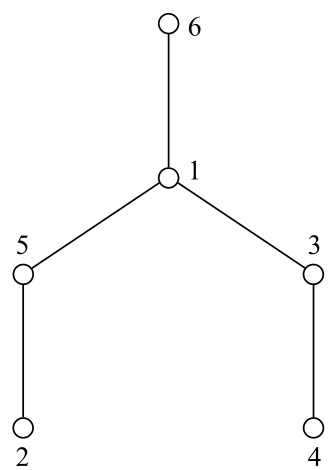

Figure 3.6: Graceful Tree T

Then, using the labeling provided in Theorem 3.22, we have that for each vertex $w \in T_1$, the corresponding labeling in $\uplus T_i$, for $i = 1, 2, 3$, is given by

$$f^{\star}(w) = \begin{cases} 7 - f(w) & \text{if } d(w) \text{ is odd} \\ 19 - f(w) & \text{if } d(w) \text{ is even.} \end{cases}$$

Similarly, for vertices $w \in T_2$ and T_3, we assign new labels using the rules

$$f^{\star}(w) = \begin{cases} 13 - f(w) & \text{if } d(w) \text{ is odd} \\ 13 - f(w) & \text{if } d(w) \text{ is even} \end{cases}$$

and

$$f^{\star}(w) = \begin{cases} 19 - f(w) & \text{if } d(w) \text{ is odd} \\ 7 - f(w) & \text{if } d(w) \text{ is even,} \end{cases}$$

respectively. Finally, adjoining the new vertex $v^\star$ and adding an edge between $v^\star$ and each vertex of T_1, T_2, and T_3 corresponding to $v \in T$, we obtain the gracefully labeled tree shown in Figure 3.7.

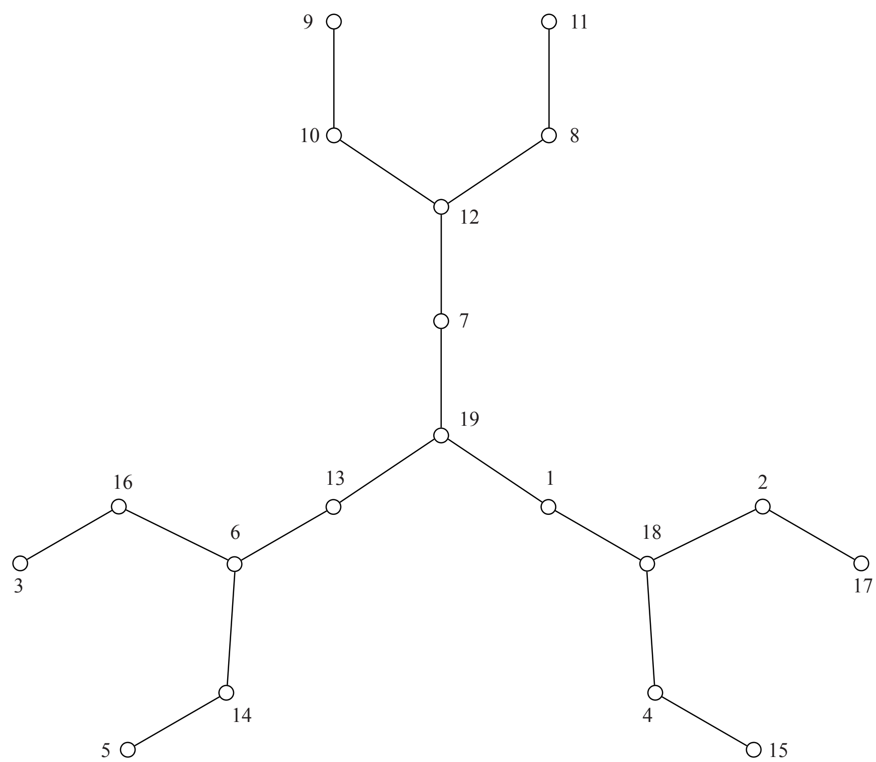

Figure 3.7: The Graceful Tree T_v^3

Koh, Rogers, and Tan extended the result of Theorem 3.22 a few years later in [**12**], giving a more general result. Let T be a graceful tree with valuation f such that for the vertex $w \in v(T)$, $f(w) = n$. Also, let $T_1, T_2, \ldots, T_p$ be p isomorphic copies of T and let w_i be the isomorphic image of w in each T_i. We define $(T_w^p)^\star$ to be the tree obtained by identifying the vertices $w_1, w_2, \ldots, w_p$. Koh, et. al. show the following.

Theorem 3.24. *Let T be a tree on n vertices with graceful valuation f. Let w be a vertex in T with $f(w) = n$, and let $N(w)$ be the neighborhood of w. If $\{f(v) - 1 | v \in N(w)\} \subseteq \{0\} \cup \{n - f(v) | v \in N(w)\}$, then there is a valuation $f^\star$ on $(T_w^p)^\star$ so that $(T_w^p)^\star$ is graceful.*

We omit the proof but note that it is similar to the proof of Theorem 3.22, and note that the valuation $f^\star$ of $(T_w^p)^\star$ given in the proof is as follows:

(1) $f^\star(w) = p(n-1)+1$ and

(2) for every $v \in T_i \setminus \{w\}$, with $i = 1, 2, \ldots, p$

$$f^\star(v) = \begin{cases} (i-1)(n-1) + f(v) & \text{if } d(w,v) \text{ is odd,} \\ (p-i)(n-1) + f(v) & \text{if } d(w,v) \text{ is even.} \end{cases}$$

Also appearing in [**12**] is a method for taking two arbitrary gracefully labeled trees and combining them to obtain a larger graceful tree. Let T and S be graceful trees under the valuations f_1 and f_2, respectively. Let $V(T) = \{w_1, w_2, \ldots, w_m\}$ and let v be an arbitrary fixed vertex of S. Adjoin, to each w_i, a copy S_i of S by identifying the isomorphic image of v in each S_i, and w_i. The tree obtained is denoted $T\Delta S$. The m copies of S are pairwise disjoint and clearly, no new edges are added by this operation.

THEOREM 3.25. *Let T and S be two graceful trees under the valuations f_1 and f_2, respectively, and let $|T| = m$ and $|S| = n$. Then there is a valuation f on $T\Delta S$ such that $T\Delta S$ is graceful.*

PROOF. [**12**] Define a mapping $f : T\Delta S \to \{1, 2, \ldots, mn\}$ as follows:
We will denote the copy of a vertex $z \in V(S)$ in the copy S_i by z_i. For each z in $V(S)$ and $i \in \{1, 2, \ldots, m\}$, we define

$$f(z_i) = \begin{cases} (f_1(w_i) - 1)n + f_2(z) & \text{if } d(v,z) \text{ is even,} \\ (m - f_1(w_i))n + f_2(z) & \text{if } d(v,z) \text{ is odd.} \end{cases}$$

Since $1 \le f_1(w_i) \le m$ and $1 \le f_2(z) \le n$, it follows from the above definition that $f(z_i) \in \{1, 2, \ldots, mn\}$. We now show that f is one-to-one. Let $u_i \in S_i$ and $z_j \in S_j$

with $u_i \neq z_j$ and $f(z_j) = f(u_i)$. If $d(v,u)$ is even and $d(v,z)$ is odd, then

$$\big(f_1(w_i) - 1\big)n + f_2(u) = \big(m - f_1(w_j)\big)n + f_2(z),$$

which gives that

$$n - 1 \geq |f_2(u) - f_2(z)| = n|[m + 1 - (f_1(w_i) + f_2(w_j))]| \geq 0.$$

Thus $f_2(u) = f_2(z)$, which is a contradiction. If both $d(v,u)$ and $d(v,z)$ are either even or odd, we again reach a contradiction by applying a similar argument, showing that f is one-to-one. Thus f is a valuation.

We now show that the edge labels of $T\Delta S$ are distinct. First, note that $f(w_i w_j) = nf_1(w_i w_j)$ if $w_i w_j$ is an edge in T. Indeed,

$$\begin{aligned}
f(w_i w_j) &= |f(w_i) - f(w_j)| \\
&= |(f_1(w_i) - 1)n + f_2(v) - (f_1(w_j) - 1)n - f_2(v)| \\
&= n|f_1(w_i) - f_1(w_j)| \\
&= nf_1(w_i w_j).
\end{aligned}$$

Now consider an edge in $T\Delta S$ other than the ones of the form $w_i w_j$. This edge then must be an edge of S_i for some i, i.e. it is of the form $u_i z_i$ for some $uz \in E(S)$. Assume without loss of generality that $d(v,u)$ is even and $d(v,z)$ is odd. Observe that

$$\begin{aligned}
f(u_i z_i) &= |(f_1(w_i) - 1)n + f_2(u) - (m - f_1(w_i))n - f_2(z)| \\
&= |(2f_1(w_i) - m - 1)n + (f_2(u) - f_2(z))|,
\end{aligned}$$

which clearly is not a multiple of n. Thus it suffices to show that $f(u_i z_i) \neq f(u'_j z'_j)$ for any pair of distinct edges uv and $u'_j z'_j$ in $T\Delta S$ where $uz, u'z' \in E(S)$, and $i, j \in \{1, 2, \ldots, m\}$. To this end, assume also without loss of generality that $u'z'$ is an edge of S and say $d(u', v)$ is even and $d(z', v)$ is odd. Then we have

$$f(u'_j z'_j) = |(2f_1(w_j) - m - 1)n + (f_2(u) - f_2(z))|.$$

Assume $f(u_i z_i) = f(u'_j z'_i)$, and for simplicity, let $a = (2f_1(w_i) - m - 1)n$, $b = f_2(u) - f_2(z)$, $c = (2f_1(w_j) - m - 1)n$, and $d = f_2(u') - f_2(z')$. We want to show that the edges $u_i z_i$ and $u'_j z'_j$ must be the same. Note that from the definition, $|b| = f_2(uz) \leq n - 1$ and $|d| \leq f_2(u'z') \leq n - 1$. We may also assume without loss of generality that $a + b \geq c + d$. We have the following cases:

Case 1. $a + b \geq 0$ and $c + d \geq 0$.

In this case we have $a + b = c + d$. Thus, $a - c = d - b$. In other words,

$$(2n)|f_1(w_i) - f_1(w_j)| = |d - b| \leq 2(n - 1)$$

which forces that $f_1(w_i) = f_1(w_j)$. That is, $i = j$. From $i = j$ and $d = b$ we get $f_2(uz) = f_2(u'z')$, which means that $u = u'$ and $z = z'$. Thus $u'_j z'_j = u_i z_i$, and the two edges were not different to begin with.

Case 2. $a + b \geq 0$, $c + d \geq 0$. If this is the case we have that $a + b = -c - d$, which means that if $c \geq 0$ then

$$(2f_1(w_i) - m - 1)n + b = (m + 1 - 2f_1(w_j))n - d$$

otherwise

$$(2f_1(w_i) - m - 1)n + b = (2f_1(w_j) - m - 1))n - d.$$

Both of these lead to the conclusion that $2n$ divides $b+d$. But, since $|b+d| \leq |b|+|d| \leq 2(n - 1)$, this is only possible if $b = -d$, from which we get that $f_2(u) - f_2(z) = f_2(z') - f_2(u')$. But this means that the uv and $u'v'$ edges are the same in S, but $u = z'$ and $z = u'$. This contradicts the assumption that $d(u, v)$ is even but $d(z', v)$ is odd. Thus, this case can not occur and the proof is complete. $\square$

This Δ-construction essentially uses one graceful tree, say T, as a type of support, and adjoins to each vertex in T a copy of another graceful tree S by identifying each vertex of T with an arbitrary, but fixed vertex in each copy of S.

EXAMPLE 3.26. We now illustrate how the Δ-construction works. Let T and S be the gracefully labeled trees shown in Figure 3.8 with $|T| = 7$ and $|S| = 4$. Then

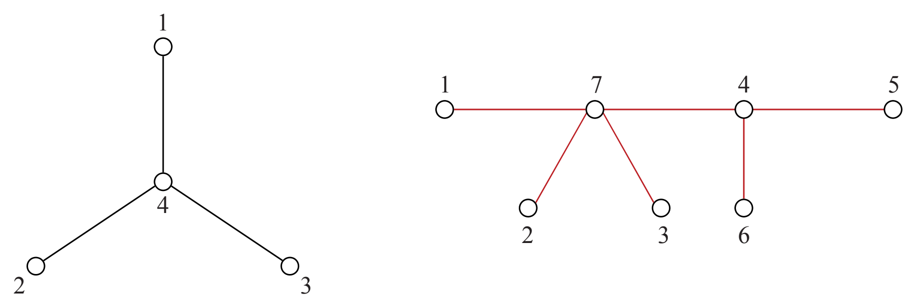

Figure 3.8: Graceful Trees S and T.

the construction will produce a new tree $T\Delta S$ of order 28 where each vertex of T is identified with the vertex $v^\star$, in this example the vertex labeled 1, of a different copy of S. Using the mapping

$$f(v) = \begin{cases} (f_1(w_i) - 1)n + f_2(v) & \text{if } d(v^\star, v) \text{ is even,} \\ (m - f_1(w_i))n + f_2(v) & \text{if } d(v^\star, v) \text{ is odd} \end{cases}$$

for each v in S_i, with $i = 1, 2, \ldots, 7$, we get the new graceful tree shown in Figure 3.9.

In 1998, Burzio and Ferrarese introduced the *generalized* Δ-*construction*, [**2**]. This generalization uses the fact that if $w_i w_j$ is an edge in T, then under f, as given in

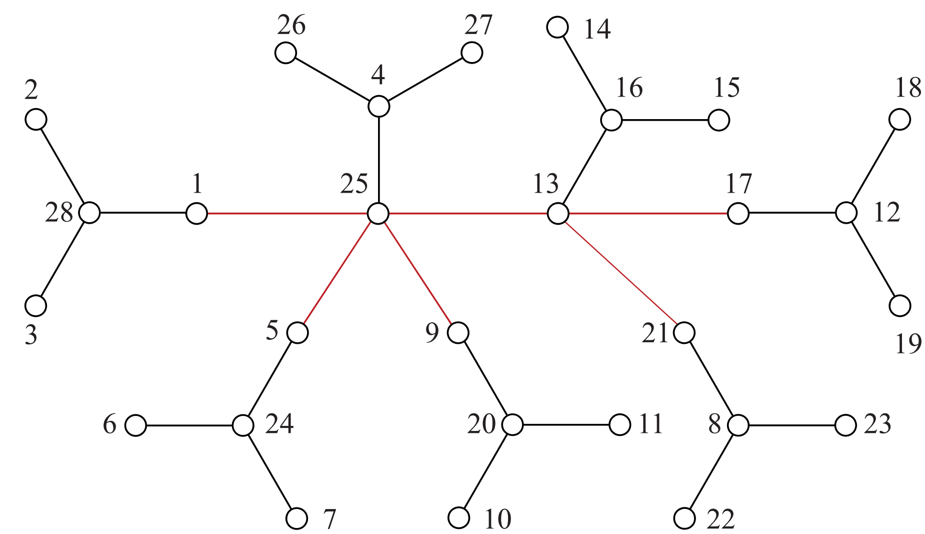

Figure 3.9: $T\Delta S$ Labeled Gracefully.

Theorem 3.25, the edge label is $f(w_iw_j) = nf_1(w_iw_j)$. Denote by z_i the vertex of S_i corresponding to the vertex z in S. If w_iw_j is an edge in T, then $|f(z_i) - f(z_j)| = f(w_iw_j)$ for each $z \in V(S)$. This gives that we can replace the w_iw_j edge by an edge z_iz_j for any $z \in V(S)$. This gives much more freedom in constructing new graceful trees. It should be noted that with the Δ-construction, we can construct up to n graceful trees of order mn, where $|T| = m$ and $|S| = n$. Using the generalized Δ-construction with the same trees and assuming $m \neq 1$, a possible n^{m-1} different graceful trees of order mn can be constructed, [**2**].

Burzio and Ferrarese [**2**] also introduced a method for constructing trees of order p, where p is a prime, based on the generalized Δ-construction defined in [**2**]. Using the same notation as above, let $v^\star$ be an arbitrary, fixed vertex of S and let w be the vertex of T such that $f_1(w) = m$. Now consider $T - \{w\}$, the graph obtained by removing w from T and all edges in T incident with w. The removal of this vertex leaves a forest. Let $\overleftarrow{v}$ (resp. $\overrightarrow{v}$) be the vertex of S with $f_2(\overleftarrow{v}) = 1$ (resp. $f_2(\overrightarrow{v}) = n$). Note that $\overleftarrow{v}\,\overrightarrow{v}$ is an edge of S since S is graceful under f_2 and this is the only way to get an edge of weight $n-1$. Thus $d(v^\star, \overleftarrow{v})$ and $d(v^\star, \overrightarrow{v})$ do not have the same parity. Define a new valuation $\tilde{f} = f_2$ if $d(v^\star, \overleftarrow{v})$ is even and $\tilde{f} = n+1-f_2$ if $d(v^\star, \overleftarrow{v})$ is odd. Construct a new graph $G = (T - \{w\})\Delta S$, which has $(m-1)n$ vertices and will generally be a proper forest. We define the valuation f on G as follows:

$$f(v_i) = \begin{cases} (f_1(w) - 1)n + \tilde{f}(v) & \text{if } d(v^\star, v) \text{ is even,} \\ (m - 1 - f_1(w_i))n + \tilde{f}(v) & \text{if } d(v^\star, v) \text{ is odd.} \end{cases}$$

for each vertex $v \in V(S_i)$, $i = 1, 2, \ldots, m-1$.

The values for each edge vv' of G, given by $|f(v) - f(v')|$, are distinct. Since $f_1(w) = n$, it follows that the edge label values not appearing are the multiples np_k, where each p_k is the weight of the edge ww_k, the edge incident with w in T. To recover the missing values, add to G a new vertex u and for each k such that ww_k is an edge in T, add the edges $u\overleftarrow{v}_{(k)}$ to G if $\tilde{f} = f_2$ (resp. $u\overrightarrow{v}$ if $\tilde{f} = f_2'$) where $\overleftarrow{v}_{(k)}$ (resp. $\overrightarrow{v}_{(k)}$) is the corresponding vertex of $\overleftarrow{v}$ (resp. $\overrightarrow{v}$) in S_k. We denote this graph by $T\Delta_+S$.

THEOREM 3.27. *The mapping $f_+ : T\Delta_+S \to \{1, 2, \ldots, (m-1)n+1\}$ defined by*

$$\begin{aligned} f_+(v) &= f(v) \qquad \textit{for each } v \in V(G) \\ f_+(u) &= (m-1)n + 1 \end{aligned}$$

is a graceful valuation on $T\Delta_+S$.

PROOF. [**2**] If $\tilde{f} = f_2$, we need only show that $|f_+(u) - f_+(\overleftarrow{v}_{(k)})| = np_k$. But

$$\begin{aligned} |f_+(u) - f_+(\overleftarrow{v}_{(k)})| &= |(m-1)n + 1 - [(f_1(w_k) - 1)n + 1]| \\ &= |(m-1)n + 1 - [(m - p_k - 1)n + 1]| \\ &= |p_k n| = p_k n \end{aligned}$$

The proof for when $\tilde{f} = f_2'$ is similar. $\square$

Using these results, Burzio and Ferrarese were able to show that subdivision graphs are graceful.

Definition 3.28. Let G be a graph. The graph obtained by replacing each edge uv of G by a new vertex w and the edges uw and vw is called the *subdivision graph of* G, denoted $\mathcal{S}(G)$.

Theorem 3.29. *The subdivision graph of a graceful tree is a graceful tree.*

Proof. [**2**] Let T be a graceful tree under valuation f_1 and a let $|T| = m$. Let $w \in V(T)$ be the vertex such that $f_1(w) = m$. Let $S = P_2$, the path on two vertices, be graceful under f_2 with vertices $\overleftarrow{v}, \overrightarrow{v}$ such that $f_2(\overleftarrow{v}) = 1$ and $f_2(\overrightarrow{v}) = 2$. Fix $v^\star = \overleftarrow{v}$ in S and obtain $T\Delta_+S$ with the Δ_+-construction. Using the generalized Δ-construction, connect the copies S_i and S_j in $T - \{w\}$ by the edge $\overleftarrow{v}_{(i)}\overleftarrow{v}_{(j)}$ (resp. $\overrightarrow{v}_{(i)}\overrightarrow{v}_{(j)}$) if $d(w_j, w) = d(w_i, w) + 1$ is odd (resp. even). Then $\mathcal{S}(T) = T\Delta_+S$ is the subdivision graph of T and $\mathcal{S}(T)$ is graceful by Theorem 3.27. $\square$

Koh et al., in [**10**], summarize their previous results from [**11**] and [**12**] as well as define and explore a class of trees they call interlaced trees. Here, they add conditions to those for graceful labelings and give methods for generating larger interlaced trees.

Definition 3.30. Let T be a tree on n vertices. The vertex $b \in V(T)$ for which $f(b) = 1$ under the valuation f is the *base* of the valuation f.

For any vertex $v \in T$, let $\mathcal{E}(v)$ be the set of all vertices u in T for which $d(v, u)$ is even. Note that $v \in \mathcal{E}(v)$, and $d(v, v) = 0$. Let $s(v) = |\mathcal{E}(v)|$.

Definition 3.31. If T is graceful under f with base b, then the *size, s, of T under f* is $s = s(b)$.

Definition 3.32. Let T be a tree, and let b be the base of the valuation f of T. Then f is a *parity valuation* if it induces, by restriction, a bijection between $\mathcal{E}(b)$ and the set $\{1, 2, \ldots, s\}$.

DEFINITION 3.33. An *interlaced valuation* f of T is a graceful valuation that is also a parity valuation. Interlaced trees are trees admitting interlaced valuations.

In order to state and prove the following two results, let T_1 and T_2 be disjoint trees on n_1 and n_2 vertices having graceful valuations f_1 and f_2 with bases b_1 and b_2, respectively. Let s_1 and s_2 be the sizes of T_1 and T_2 when f_1 or f_2 is interlaced. These constructions give a way to combine interlaced trees to obtain graceful trees and give conditions for which the resulting tree is also interlaced.

THEOREM 3.34. *Let T_1 be an interlaced tree under f_1 and let $x \in V(T_1)$ be the vertex for which $f_1(x) = s_1$. Let T be the tree obtained from T_1 and T_2 by identifying $x \in V(T_1)$ with $b_2 \in V(T_2)$. Then T is graceful. Furthermore, if f_2 is also interlaced, then T is an interlaced tree.*

PROOF. [**10**] We define the valuation f of T as follows:

$$f(v) = \begin{cases} f_1(v) & v \in \mathcal{E}_{T_1}(b_1) \\ f_1(v) + n_2 - 1 & v \notin \mathcal{E}_{T_1}(b_1) \cup V(T_2) \\ f_2(v) + s_1 - 1 & v \in V(T_2) \setminus \{b_2\} \end{cases}$$

Now by definition, $f(\mathcal{E}_{T_1}) = f_1(\mathcal{E}_{T_1}) = \{1, 2, \ldots, s_1\}$, $f(V(T_1) \setminus \mathcal{E}_{T_1}) = \{s_1 + n_2, s_1 + n_2 + 1, \ldots, n_1 + n_2 - 1\}$ and $f(V(T_2) - \{b_2\}) = \{s_1 + 1, s_1 + 2, \ldots, s_1 + n_2 - 1\}$. Thus, f is a valuation assigning labels from $\{1, 2, \ldots, n_1 + n_2 - 1\}$.

Since the edges of T_1 are between vertices of $\mathcal{E}_{T_1}$ and $V(T_1) \setminus \mathcal{E}_{T_1}$, we get that $E_{f(T_1)} = \{n_2, n_2 + 1, \ldots, n_2 + n_1 - 2\}$. On the other hand, $E_{f(T_2)} = \{1, 2 \ldots, n_2 - 1\}$. Thus, f is graceful. Also, when f_2 is interlaced, f is an interlaced labeling of T with base b_1 and size $s_1 + s_2 - 1$, since b_1 and $b_2 = x$ are an even distance from each other. Consequently, $\mathcal{E}_{T(b_1)} = \mathcal{E}_{T_1}(b_1) \cup \mathcal{E}_{T_2}(b_2)$ and $f(\mathcal{E}_T) = \{1, 2, \ldots, s_1, s_1 + 1, \ldots, s_1 + s_2 - 1\}$. ∎

The next theorem gives a method for taking the disjoint union of two interlaced trees and adding an edge between a vertex from each to obtain a new graceful tree. Conditions are also given for when the resulting tree is interlaced.

THEOREM 3.35. *Suppose f_1 and f_2 are interlaced labelings of T_1 and T_2, respectively. Also, suppose there are vertices $u_1 \in V(T_1)$ and $u_2 \in V(T_2)$ such that either:*

(1) $f_1(u_1) - f_2(u_2) = s_1 < f_1(u_1)$, *or*

(2) $n_2 + f_1(u_1) - f_2(u_2) = s_1 \geq f_1(u_1)$

Let T be the tree obtained by joining u_1 and u_2 by a new edge. Then T is graceful. Furthermore, if in the two cases above we also have

(1) $f_2(u_2) \leq s_2$, *or*

(2) $f_2(u_2) > s_2$,

then T is also an interlaced tree.

PROOF. [**10**] Define f on $V(T)$ as follows:

$$f(v) = \begin{cases} f_1(v) & v \in \mathcal{E}_{T_1}(b_1) \\ f_1(v) + n_2 & v \notin \mathcal{E}_{T_1}(b_1) \cup V(T_2) \\ f_2(v) + s_1 & v \in V(T_2). \end{cases}$$

Then, $f(\mathcal{E}_{T_1}(b)) = \{1, 2 \ldots, s_1\}$, $f(V(T_1) \setminus \mathcal{E}_{T_1}(b_1)) = \{s_1 + n_2 + 1, s_1 + n_2 + 2, \ldots, n_1 + n_2\}$ and $f(V(T_2)) = \{s_1 + 1, s_1 + 2, \ldots, s_1 + n_2\}$, so f is a valuation that assigns labels in $\{1, 2, \ldots, n_1 + n_2\}$.

Now since edges of T_1 run between a vertex of $\mathcal{E}_{T_1}(b_1)$ and a vertex of $V(T) \backslash \mathcal{E}_{T_1}(b_1)$, $E_{f(T_1)} = \{n_2 + 1, n_2 + 1, \ldots, n_2 + n_1\}$, and $E_{f(T_2)} = \{1, 2, \ldots, n_2 - 1\}$. Moreover, in case *(1)*, since we have $u_1 \notin \mathcal{E}_{T_1}(b_1)$,

$$|f(u_1) - f(u_2)| = f(u_1) - f(u_2) = f_1(u_1) + n_2 - (f_2(u_2) + s_1) = n_2,$$

and in case *(2)*, when $u_1 \in \mathcal{E}_{T_1}$,

$$|f(u_1) - f(u_2)| = f(u_2) - f(u_1) = (f_2(u_2) + s_1) - f_1(u_1) = n_2.$$

Thus, in either case we have that the new edge has weight n_2. Hence the edges of T carry all possible weights in $\{1, 2, \ldots, n_1 + n_2 - 1\}$ and f is a graceful labeling of T. Additionally, if $f_2(u_2) \leq s_2$ when $f_1(u_1) > s_1$ or $f_2(u_2) > s_2$ when $f_1(u_1) \leq s_1$, then it follows that the bases b_1 and b_2 are an even distance apart. So T is an interlaced tree with size $s = s_1 + s_2$ and base $b = b_1$. $\square$

Chapter 4

Adjacency Matrices of Graceful Graphs

Definition 4.1. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

is called the *adjacency matrix* of G.

Definition 4.2. Let G be a graph with $m = ||G||$ and a valuation $f : V(G) \to \{1, 2, \ldots, m+1\}$. Then the $(m+1) \times (m+1)$ matrix $\mathbb{A}_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } xy \in E(G) \text{ for } f(x) = i \text{ and } f(y) = j \\ 0 & \text{otherwise.} \end{cases}$$

is called the *generalized adjacency matrix* of G induced by the valuation f. For simplicity, we will assume in this chapter that $V(G) \subseteq \{v_1, v_2, \ldots, v_{m+1}\}$ and f is given by $f(v_i) = i$.

Note that when G is a tree then the concepts in Definitions 4.1 and 4.2 agree. Also, the generalized adjacency matrix allows (all zero) rows/columns corresponding to missing labels; while in the adjacency matrix such rows correspond to vertices of degree 0. If two graphs have the same (or similar) adjacency matrix, then they are isomorphic, but if two graphs have the same generalized adjacency matrix, they may not be isomorphic. However, we get isomorphic graphs if we leave out the vertices of degree 0 from both.

Adjacency matrices are quite useful as they encode many of the corresponding graphs' characteristics. For example, vertex labeling, vertex adjacency, vertex degree, and other graph features are quickly ascertained by looking at a graph's adjacency matrix. In fact, the field of spectral graph theory makes use of adjacency matrices and their eigenvalues, eigenvectors, characteristic functions, etc., in investigating invariant graph properties. One particularly nice property of adjacency matrices is that one can find the number of walks of varying lengths between vertices by looking at multiplicative powers of the matrix. As stated earlier, we are only considering simple undirected graphs. This means that the corresponding adjacency matrices will be symmetric since there is no orientation on the edges and thus there is no distinction drawn between edges v_iv_j and v_jv_i. The property that all the graphs considered are simple guarantees that the main diagonal entries of the adjacency matrices will be zeros as a one on the main diagonal corresponds to a loop in the graph. Our interest in adjacency matrices is motivated by their properties when the corresponding graph is labeled gracefully.

Definition 4.3. Let A be an $n \times n$ matrix. Then the k^{th} *diagonal line of* A is the collection of entries $D_k = \{a_{ij} | \, j - i = k\}$, counting multiplicity.

So it is clear that $0 \leq |k| \leq n - 1$. When $A = \mathbb{A}_G$ is a (generalized) adjacency matrix, A is symmetric and $D_k = D_{-k}$. Thus we make the convention that the entry 1 corresponding to an edge between vertices v_i and v_j lies in the $|j-i|^{th}$ diagonal line and the edge label for that edge is $|j - i|$, assuming $f(v_i) = i$ for each $i = 1, 2, \ldots, n$ and f is a valuation on G. We now prove a theorem characterizing the adjacency matrices of graceful graphs.

Theorem 4.4. *Let G be a labeled graph and let $\mathbb{A}_G$ be the generalized adjacency matrix for G. Then $\mathbb{A}_G$ has exactly one entry 1 in each diagonal line, except the main diagonal of zeros, if and only if the valuation f on G that induces $\mathbb{A}_G$ is graceful.*

PROOF. We begin by noting that we need only consider the upper triangular part of $\mathbb{A}_G$ since it is symmetric. That is, for edges v_iv_j, we may assume $j > i$.

Suppose $\mathbb{A}_G$ has exactly one entry of 1 in each diagonal line, other than the main diagonal of zeros. Suppose to the contrary that the labeling of G that induces $\mathbb{A}_G$ is not a graceful labeling. Then there are distinct edges v_gv_h and v_qv_ℓ with edge labels $h - g = \ell - q = k > 0$. This implies

$$\sum_{\substack{i,j:j-i=k\\ j>i}} a_{ij} \geq 2$$

contradicting the assumption that $\mathbb{A}_G$ has exactly one entry in each diagonal (not including main diagonal).

Now suppose G is gracefully labeled by f and consider $\mathbb{A}_G$. Then for all $k = 1, 2, \ldots, |E(G)|$, there is exactly one non-zero entry $a_{ij} = 1$, such that $j > i$ and $j - i = k$, contributing to $|D_k|$ since each edge has a unique label. That is, $\mathbb{A}_G$ has exactly one entry of 1 in each non-main diagonal. $\square$

EXAMPLE 4.5. We now consider the caterpillar shown in Figure 4.1 below. This

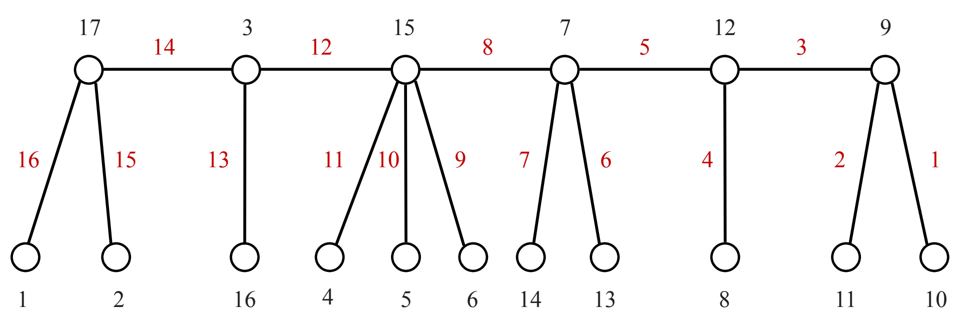

Figure 4.1: A Graceful Caterpillar

tree is gracefully labeled and has adjacency matrix

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

which satisfies Theorem 4.4. In fact, it is easy to see that the labeling described in Theorem 4.4 will always give a "staircase shaped" matrix.

The condition from Theorem 4.4 allows us to look at an adjacency matrix and immediately determine if the graph is labeled gracefully. We now consider several families of trees that can be shown to be graceful by building the adjacency matrix to be graceful.

Theorem 4.6. *Let T be a graceful bipartite graph on n vertices with generalized adjacency matrix $\mathbb{A}_T$. Then the matrix*

$$\mathbb{A} = \begin{pmatrix} O & \cdots & O & \mathbb{A}_T \\ \vdots & & \mathbb{A}_T & O \\ O & \iddots & \vdots & \vdots \\ \mathbb{A}_T & O & \cdots & O \end{pmatrix}_{pn \times pn}$$

with p submatrices $\mathbb{A}_T$ and where O is the $n \times n$ zero matrix is an adjacency matrix for

$$G = \dot{\bigcup_{i=1}^{p}} T_i,$$

the disjoint union of p copies of T.

Here we pause to note that all trees are bipartite. Thus this theorem holds for graceful trees.

Proof. We begin by noting that $\mathbb{A}$ is a (0,1)-matrix, and since $\mathbb{A}_T$ is symmetric, so is $\mathbb{A}$. Also, the main diagonal of $\mathbb{A}$ has all zero entries. This can be seen by the fact that the main diagonal does not contain entries from any submatrix $\mathbb{A}_T$ when p is even, and goes through exactly the main diagonal of one of the $\mathbb{A}_T$ when p is odd. So $\mathbb{A}$ is a generalized adjacency matrix. We now want to show that it is the generalized adjacency matrix for $G = \uplus T_i$, $i = 1, 2, \ldots, p$.

Consider $V(T)$. Note that if $\mathbb{A}_T$ is graceful, then there is an edge with induced label m. Thus, there are adjacent vertices with labels 1 and $m+1$. Let $x \in V(T)$ be the vertex such that $f(x) = 1$ where f is the graceful valuation of T. Since T is bipartite, we can fix a bipartition V_1, V_2 of the vertices such that $x \in V_1$. Moreover, $\{v \in V(T) | d(x,v) \text{ is even }\} \subseteq V_1$ and $\{v \in V(T) | d(x,v) \text{is odd }\} \subseteq V_2$, with equality holding if T is connected. Also, in the generalized adjacency matrix $\mathbb{A}_T$ of the graph T, we have that $a_{ij} = 1$ implies that $v_i \in V_1$ and $v_j \in V_2$ or vice versa. We will denote by $\mathcal{C}$ and $\mathcal{D}$ the set of labels on vertices of V_1 and V_2. That is,

$$\begin{aligned} \mathcal{C} &= \{\ell : v_\ell \in V_1\} \text{ and} \\ \mathcal{D} &= \{\ell : v_\ell \in V_2\} \end{aligned}$$

Now consider the p copies of $\mathbb{A}_T$ in the generalized adjacency matrix $\mathbb{A}$. We will call a copy the j-th copy if it is the j-th counting from the bottom left corner of $\mathbb{A}$.

For a set of integers S, let $a + S = \{a + s | s \in S\}$ for any $a \in \mathbb{Z}$. It is easy to see that for $j \in \{1, 2 \ldots, p\}$ the j-th copy of $\mathbb{A}_T$ in $\mathbb{A}$ defines edges between vertices with labels

$$\big((j-1)n + \mathcal{C}\big) \cup \big((p-j)n + \mathcal{D}\big),$$

and there is an edge $v_{(j-1)n+i}v_{(p-j)n+\ell}$ for some $i \in \mathcal{C}$ and $\ell \in \mathcal{D}$ precisely when $v_i \in V_1$, $v_\ell \in V_2$ and $v_iv_j \in E(T)$. Thus, the j-th copy of $\mathbb{A}_T$ induces an isomorphic copy T_j of T on the corresponding vertex set. It is easy to see that the T_j are vertex disjoint. $\square$

There are a few things we wish to note about the matrix constructed in Theorem 4.6. First off, since $\mathbb{A}_T$ is the generalized adjacency matrix of a graceful graph, each diagonal line of $\mathbb{A}$ has at most one 1. In fact, the only diagonal lines with all zero entries are the diagonal lines D_k, with $k = 0, \pm n, \pm 2n, \ldots, \pm(p-2)n, \pm(p-1)n$. This means that the edge labels for the edges of G are distinct though not all labels from $\{1, 2, \ldots, pn-1\}$ are used.

Definition 4.7. Let G_1 and G_2 be two bipartite graphs with $m = ||G_1|| = ||G_2||$. Let f_1 and f_2 be labelings of G_1 and G_2 from the set $\{1, 2, \ldots, m+1\}$, respectively. We call f_1 and f_2 *compatible labelings*, if there are disjoint sets $\mathcal{C}, \mathcal{D}$, where $\mathcal{C} \cup \mathcal{D} = \{1, 2 \ldots, m+1\}$ and bipartitions (C_1, D_1) of G_1, (C_2, D_2) of G_2 such that $f_1(C_1), f_2(C_2) \subseteq \mathcal{C}$ and $f_1(D_2), f_2(D_2) \subseteq \mathcal{D}$. It follows that isomorphic graphs are easily seen to be compatible.

Example 4.8. Let G_1 and G_2 be the labeled bipartite graphs in Figure 4.2.

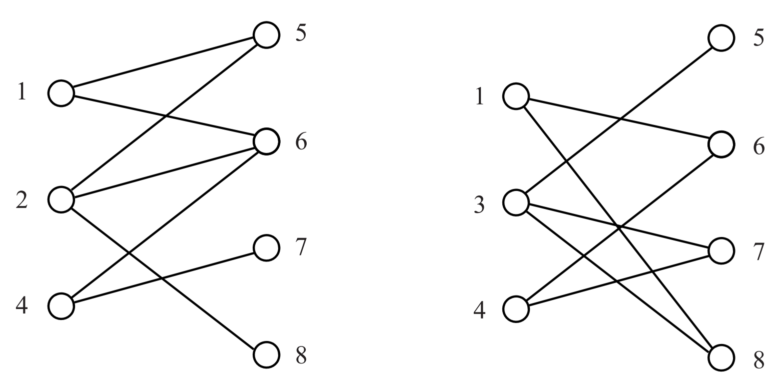

Figure 4.2: Bipartite Graphs G_1 and G_2 with Compatible Labelings.

Then clearly the labelings of G_1 and G_2 are compatible labelings. Note, however, that these labelings are not graceful.

THEOREM 4.9. *Let $T_1, T_2, \ldots, T_p$ be graceful bipartite graphs such that T_i and T_{p+1-i} have compatible graceful labelings, and denote the generalized adjacency matrices of these compatible labelings by $\mathbb{A}_{T_i}$. Then the matrix*

$$\mathbb{A} = \begin{pmatrix} O & \cdots & O & \mathbb{A}_{T_p} \\ \vdots & & \mathbb{A}_{T_{p-1}} & O \\ O & \iddots & \vdots & \vdots \\ \mathbb{A}_{T_1} & O & \cdots & O \end{pmatrix}_{pn \times pn}$$

with p submatrices of the form $\mathbb{A}_{T_i}$ and where O is the $n \times n$ zero matrix is a generalized adjacency matrix for

$$G = \dot{\bigcup_{i=1}^{p}} T_i,$$

the disjoint union of vertex-disjoint copies of T. Note that if T is a tree, then $\mathbb{A}$ is the usual adjacency matrix of G.

The proof of this theorem is essentially the same as that of the previous theorem, thus we omit it.

Let T be a graceful tree on n vertices with valuation f. We want to construct a new tree by identifying a leaf of $K_{1,p}$ with the vertex x such that $f(x) = 1$ in each of p copies of T. We denote this tree by $T \circ K_{1,p}$. It has been shown that the tree constructed this way, previously called T_v^p, is graceful in Theorem 3.22. The next definition gives us a way to describe the graph resulting from matrix manipulations.

DEFINITION 4.10. The matrix A is said to *gracefully induce* G if A is an adjacency matrix for G, and A satisfies the conditions of Theorem 4.4.

THEOREM 4.11. *Let T be a graceful tree on n vertices. Then the graph $T \circ K_{1,p}$ is graceful.*

PROOF. We begin by constructing the matrix $\mathbb{A}$ from Theorem 4.6 for the given T to get

$$\mathbb{A} = \begin{pmatrix} O & \cdots & O & \mathbb{A}_T \\ \vdots & & \mathbb{A}_T & O \\ O & \iddots & \vdots & \vdots \\ \mathbb{A}_T & O & \cdots & O \end{pmatrix}_{pn \times pn} .$$

We now extend $\mathbb{A}$ to the new matrix

$$\mathcal{A} = \begin{pmatrix} O & \cdots & O & \mathbb{A}_T & v \\ \vdots & & \mathbb{A}_T & O & v \\ O & \iddots & \vdots & \vdots & \vdots \\ \mathbb{A}_T & O & \cdots & O & v \\ v^t & v^t & \cdots & v^t & 0 \end{pmatrix}_{pn+1 \times pn+1} .$$

where

$$v = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

and v^t is the transpose of v. This new matrix induces a new graph that has an added vertex labeled $np+1$ that is adjacent to the vertices labeled $1, n+1, 2n+1, \ldots, (p-1)n+1$ from the graph induced by $\mathbb{A}$. These entries of 1 from the added vertex are in the diagonals D_k with $k = (pn+1) - (jn+1) = (p-j)n$ for $j = 0, ..., p-1$. Thus $\mathcal{A}$ is graceful since these are the exact diagonal lines, other than the main diagonal, of $\mathbb{A}$ that had all zero entries, and $\mathcal{A}$ gracefully induces $T \circ K_{1,p}$. $\square$

EXAMPLE 4.12. We now give an example of this construction. Let T be the gracefully labeled tree shown in Figure 4.3.

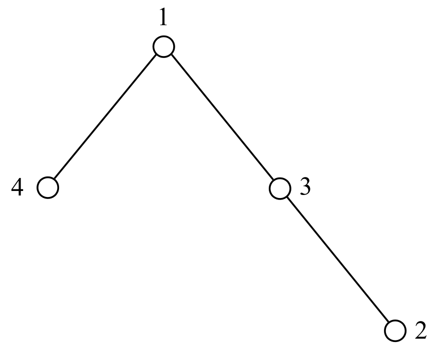

Figure 4.3: The Graceful Tree $T = P_4$.

Then this labeling induces the adjacency matrix

$$A_T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, we construct the matrix $\mathcal{A}$ as in Theorem 4.11 to get

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

This matrix, in turn, gracefully induces the graph $T \circ K_{1,3}$ shown in Figure 4.4.

The gracefulness of several classes of trees follows directly from Theorem 4.11.

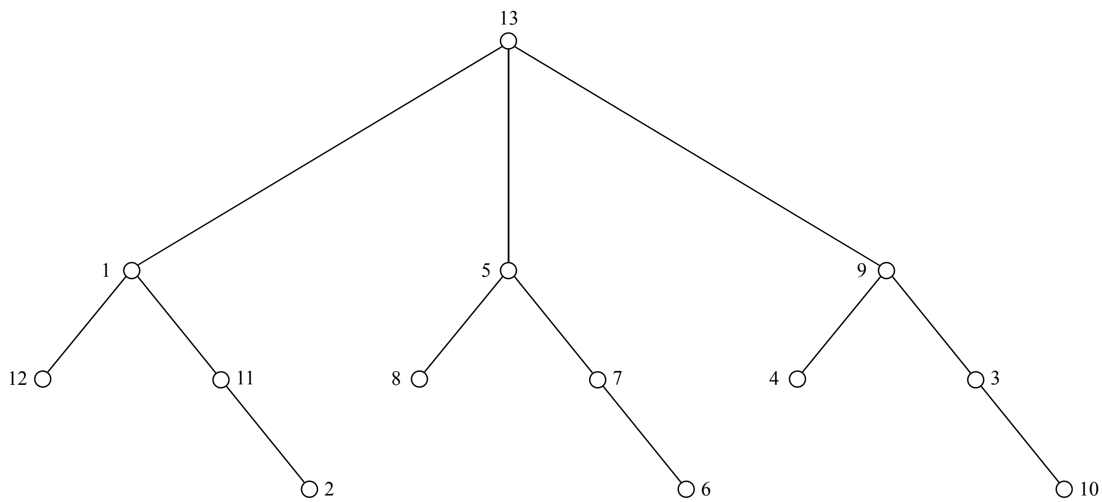

Figure 4.4: The Induced Graph $T \circ K_{1,3}$.

THEOREM 4.13. *The spider $S(k, k, \ldots, k)$ with p legs, each of which having length k, is graceful.*

PROOF. Let $\mathbb{A}_T$ be the adjacency matrix for $T = P_{k-1}$ where P_{k-1} is gracefully labeled and one of its leaves has label 1. Then the matrix $\mathcal{A}$ constructed as in Theorem 4.11 with p copies of $\mathbb{A}_T$ gacefully induces the spider $S(k, k, \ldots, k)$ with p legs of length k. $\square$

DEFINITION 4.14. A *banana tree* is a collection of p stars, each of which having one leaf connected to an added vertex.

Alternatively, we can think of a banana tree as a collection of p stars, each having exactly one leaf identified with a unique leaf of a $K_{1,p}$. Let $B(x_1, x_2, \ldots, x_p)$ be the banana tree with p stars, each having x_i leaves.

THEOREM 4.15. *The banana tree $B(k, k, \ldots, k)$ with p stars, each having k leaves, is graceful.*

PROOF. Let $\mathbb{A}_T$ be the adjacency matrix for $T = K_{1,k}$ where $K_{1,k}$ is gracefully labeled and one of its leaves is labeled 1. Then the matrix $\mathcal{A}$ constructed as in Theorem 4.11 with p copies of $\mathbb{A}_T$ gracefully induces $B(k, k, \ldots, k)$. $\square$

It is beyond the scope of this thesis to elaborate much more on graceful trees, though certainly there is plenty more that can be said. Therefore, we end here and refer the interested reader to Gallian's *A Dynamic Survey of Graph Labelings* [**5**] for a comprehensive treatment of the Graceful Tree Conjecture.

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