Note on the Kleitman-Sha Bounds

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April 25, 2009

In the manuscript "The number of linear extensions of the subset ordering" ([1]), the authors use the facts that

$$\log \prod_r \binom{n}{r}^{\binom{n}{r}} = 2^n \left(-\frac{1}{2}\log(2\pi n) - \frac{1}{2} + (n+1)\log 2 + o(1) \right)$$

and

$$\log \prod_r \binom{n}{r}! = 2^n \left(-\frac{1}{2}\log(2\pi n) - \frac{3}{2} + (n+1)\log 2 + o(1) \right).$$

However, they do not offer a proof, simply commenting that "standard computation" leads one from Stirling's approximation inexorably to these bounds. Here we provide the missing argument, which the present author found rather difficult to reproduce.

We wish to estimate the quantities $\prod_r \binom{n}{r}!$ and $\prod_r \binom{n}{r}^{\binom{n}{r}}$. We use following version of Stirling's Approximation:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O(n^{-1})\right),$$

and its logarithm,

$$\log n! = \frac{1}{2}\log(2\pi n) + n\log n - n + O(n^{-1}).$$

Note that

$$\begin{aligned}
\log \prod_r \binom{n}{r}^{\binom{n}{r}} - \log \prod_r \binom{n}{r}! &= \sum_r \binom{n}{r}\log\binom{n}{r} - \log\binom{n}{r}! \\
&= \sum_r \binom{n}{r}\log\binom{n}{r} - \frac{1}{2}\log\left(2\pi\binom{n}{r}\right) \\
&\qquad - \binom{n}{r}\log\binom{n}{r} + \binom{n}{r} + O\left(\binom{n}{r}^{-1}\right) \\
&= \sum_r -\frac{1}{2}\log\left(2\pi\binom{n}{r}\right) + O\left(\binom{n}{r}^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2^n + O(1) - \sum_r \frac{1}{2}\log\left(2\pi\binom{n}{r}\right) \\
&= 2^n + O(n^2).
\end{aligned}
$$

Therefore,

$$
\frac{\log\prod_r \binom{n}{r}^{\binom{n}{r}} - \log\prod_r \binom{n}{r}!}{2^n} = 1 + o(1). \tag{1}
$$

Now,

$$
\begin{aligned}
\log\prod_r \binom{n}{r}^{\binom{n}{r}} &= \sum_r \binom{n}{r}\log\binom{n}{r} \\
&= \sum_r \binom{n}{r}\left(\log n! - \log r! - \log(n-r)!\right).
\end{aligned}
$$

Since $\binom{n}{r}\log(n-r)! = \binom{n}{n-r}\log(n-r)!$, this can be written

$$
\begin{aligned}
\log\prod_r \binom{n}{r}^{\binom{n}{r}} &= \sum_r \binom{n}{r}\left(\log n! - 2\log r!\right). \\
&= \sum_r \binom{n}{r}\bigg(\frac{1}{2}\log(2\pi n) + n\log n - n + O(n^{-1}) \\
&\qquad - \log(2\pi r) - 2r\log r + 2r + O(r^{-1})\bigg) \\
&= 2^n\left(\frac{1}{2}\log(2\pi n) + n\log n - n + O(n^{-1})\right) \\
&\qquad + \sum_r \binom{n}{r}\left(-\log(2\pi r) - 2r\log r + 2r + O(r^{-1})\right) \\
&= 2^n\left(\frac{1}{2}\log(2\pi n) + n\log n - n + o(1)\right) \\
&\quad + \sum_{r=n/2-n^{.6}}^{n/2+n^{.6}} \binom{n}{r}\left(-\log(2\pi r) - 2r\log r + 2r + O(r^{-1})\right),
\end{aligned}
$$

since the sum of all terms outside the range $r \in I = [n/2 - n^{.6}, n/2 + n^{.6}]$ amounts to $O(2^n n\log n \cdot e^{-n^{1.2}/2n}) = o(2^n)$ (by a standard Chernoff bound). When $r \in I$, we have

$$
\begin{aligned}
\log r &= \log\frac{n}{2} + \log\frac{2r}{n} \\
&= \log\frac{n}{2} + \log(1 + O(n^{-.1})) \\
&= \log\frac{n}{2} + O(n^{-.1}) \\
&= \log\frac{n}{2} + o(1).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\sum_{r\in I}\binom{n}{r}\big(-\log(2\pi r) & -2r\log r+2r+O(r^{-1})\big) \\
&= o(2^n)+\sum_{r\in I}\binom{n}{r}\left(-\log(\pi n)-2r\log r+2r\right) \\
&= 2^n(o(1)-\log(\pi n))-2\sum_{r=1}^{n}\binom{n}{r}\left(r\log r-r\right) \\
&= 2^n(o(1)-\log(\pi n)+n)-2\sum_{r=1}^{n}\binom{n}{r}r\log r.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{r=1}^{n}\binom{n}{r}r\log r &= o(2^n)+\sum_{r=n/2-n^{.6}}^{n/2+n^{.6}}\binom{n}{r}r\log r \\
&= o(2^n)+\sum_{s=-n^{.6}}^{n^{.6}}\binom{n}{n/2+s}\left(\frac{n}{2}+s\right)\log\left(\frac{n}{2}+s\right) \\
&= o(2^n)+\sum_{s=-n^{.6}}^{n^{.6}}\binom{n}{n/2+s}\left(\frac{n}{2}+s\right)\left(\log\left(\frac{n}{2}\right)+\log\left(1+\frac{2s}{n}\right)\right) \\
&= o(2^n)+\log\left(\frac{n}{2}\right)\sum_{r=1}^{n}\binom{n}{r}r \\
&\qquad+\sum_{s=-n^{.6}}^{n^{.6}}\binom{n}{n/2+s}\left(\frac{n}{2}+s\right)\log\left(1+\frac{2s}{n}\right) \\
&= o(2^n)+2^{n-1}n\log\left(\frac{n}{2}\right) \\
&\qquad+\sum_{s=-n^{.6}}^{n^{.6}}\binom{n}{n/2+s}\left(\frac{n}{2}+s\right)\left(\frac{2s}{n}-\frac{2s^2}{n^2}+O\left(\frac{s^3}{n^3}\right)\right).
\end{aligned}$$

When $|s|\le n^{.6}$, the quantity $n\cdot s^3/n^3$ is $O(n^{-.2})$, so the final term in this sum can be absorbed into the $o(2^n)$, leaving

$$\begin{aligned}
\sum_{r=1}^{n}\binom{n}{r}r\log r &= o(2^n)+2^{n-1}n\log\left(\frac{n}{2}\right) \\
&\qquad+\sum_{s=-n^{.6}}^{n^{.6}}\binom{n}{n/2+s}\left(\frac{n}{2}+s\right)\left(\frac{2s}{n}-\frac{2s^2}{n^2}\right). \\
&= o(2^n)+2^{n-1}n\log\left(\frac{n}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{r=1}^{n} \binom{n}{r} r \left(\frac{2r}{n} - 1 - \frac{1}{2} \left(\frac{2r}{n} - 1 \right)^2 \right) \\
&= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{2n^2} \sum_{r=1}^{n} \binom{n}{r} r \left(8rn - 4r^2 - 3n^2\right) \\
&= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) - \frac{3}{2} \sum_{r=1}^{n} \binom{n}{r} r + \frac{2}{n^2} \sum_{r=1}^{n} \binom{n}{r} r^3 \\
&= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) - 3 \cdot 2^{n-2} n + \frac{2}{n^2} \sum_{r=1}^{n} \binom{n}{r} (2r^2 n - r^3).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} r^2 &= \sum_{r=1}^{n} \binom{n}{r} (r(r-1) + r) \\
&= 2^{n-2} n(n-1) + 2^{n-1} n,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} r^3 &= \sum_{r=1}^{n} \binom{n}{r} (r(r-1)(r-2) + 3r(r-1) + r) \\
&= 2^{n-3} n(n-1)(n-2) + 3 \cdot 2^{n-2} n(n-1) + 2^{n-1} n,
\end{aligned}
$$

by the first, second, and third derivatives of the Binomial Theorem, so that

$$
\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} (2r^2 n - r^3) &= 2^{n-1} n^2 (n-1) + 2^n n^2 \\
&\qquad - 2^{n-3} n(n-1)(n-2) - 3 \cdot 2^{n-2} n(n-1) - 2^{n-1} n \\
&= 2^{n-3}(3n^3 + n^2).
\end{aligned}
$$

Putting the pieces back together,

$$
\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} r \log r &= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) - 3 \cdot 2^{n-2} n \\
&\qquad + \frac{2}{n^2} \left(2^{n-3} n(n-1)(n-2) + 3 \cdot 2^{n-2} n(n-1) + 2^{n-1} n\right) \\
&= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) - 3 \cdot 2^{n-2} n + 2^{n-2}\,(3n+1) \\
&= o(2^n) + 2^{n-1} n \log\left(\frac{n}{2}\right) + 2^{n-2},
\end{aligned}
$$

so that

$$
\sum_{r \in I} \binom{n}{r} \big(-\log(2\pi r) - 2r \log r + 2r + \frac{1}{6r} + O(r^{-2}) \big)
$$

$$
\begin{aligned}
&= 2^n(o(1) - \log(\pi n) + n) - 2^n n \log\left(\frac{n}{2}\right) - 2^{n-1} \\
&= 2^n \left(o(1) - \log(\pi n) + n - n\log\left(\frac{n}{2}\right) - \frac{1}{2} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\log \prod_r \binom{n}{r}^{\binom{n}{r}} &= 2^n \left(\frac{1}{2}\log(2\pi n) + n\log n - n + o(1) \right) \\
&\quad + 2^n \left(o(1) - \log(\pi n) + n - n\log\left(\frac{n}{2}\right) - \frac{1}{2} \right), \\
&= 2^n \left(-\frac{1}{2}\log(2\pi n) - \frac{1}{2} + (n+1)\log 2 + o(1) \right).
\end{aligned}
$$

By (1), we also have

$$
\log \prod_r \binom{n}{r}! = 2^n \left(-\frac{1}{2}\log(2\pi n) - \frac{3}{2} + (n+1)\log 2 + o(1) \right).
$$

References

[1] J. C. Sha, D. J. Kleitman, The number of linear extensions of subset ordering. Special issue: ordered sets (Oberwolfach, 1985). *Discrete Math.* **63** (1987), no. 2-3, 271–278.