GRAPH THEORY STUDY GUIDE

1. Definitions

Definition 1 (Partition of A)**.** A set $\mathcal{A} = A_1, ..., A_k$ of disjoint subsets of a set A is a partition of A if $\cup\mathcal{A}$ of all the sets $A_i \in A$ and $A_i \neq \emptyset$ for every i.

Definition 2 (Vertex set)**.** The set of vertices in a graph denoted by $V(G)$.

Definition 3 (Edge set)**.** The set of edges in a graph denoted by $E(G)$.

Definition 4 (Order)**.** the number of vertices of a graph G written $|G|$.

Definition 5 (Incident)**.** A vertex v is incident with an edge e if $v \in e$, then e is an edge at v.

Definition 6 (Adjacent)**.** Two vertices x, y of G are adjacent if xy is an edge of G.

Definition 7 (Complete)**.** If all vertices of G are pairwise adjacent, then G is complete.

Definition 8 (Independent)**.** A set of vertices or edges is independent if no two of its elements are adjacent.

Definition 9 (Isomorphic)**.** We call G and H isomorphic, and write $G \simeq H$, if there exists a bijection $\phi : V \to V'$ with $xy \in E \iff \phi(x)\phi(y) \in E'$ for all x, y in V.

Definition 10 (Property)**.** A class of graphs that is closed under isomorphism is called a graph property.

Definition 11 (Invariant)**.** A map taking graphs as arguments is called a graph invariant if it assigns equal values to isomorphic graphs.

Definition 12 (Induced)**.** If $G \subseteq G'$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an induced subgraph of G. We say that V' induces G' in G.

Definition 13 (Spanning)**.** $G' \subseteq G$ is a spanning subgraph of G if V' spans all of G or $V' = V$.

Definition 14 (Line Graph)**.** The line graph $L(G)$ of G is the graph of E in which $x, y \in E$ are adjacent as vertices if and only if they are adjacent as edges in G.

Definition 15 (N(G))**.** the set of neighbors of a vertex v.

Definition 16 (Degree)**.** The degree $(d(v))$ of a vertex v is the number $|E(v)|$ of edges at v or the number of neighbors of v.

Definition 17 (Isolated)**.** A vertex of degree 0 is isolated.

Definition 18 (Minimum degree)**.** $\delta(G)$ is the minimum degree of G.

Definition 19 (Maximum degree)**.** $\Delta(G)$ = maximum degree of G.

Definition 20 (Regular)**.** all the vertices of G have the same degree k, then G is k-regular, or regular.

Definition 21 (Average degree)**.** $d(G) = \frac{1}{|V|}\sum_{v\in V} d(v)$ and $\delta(G) \leq d(G) \leq \Delta(G)$.

Definition 22 (Number of Edges)**.** $|E| = \frac{1}{2}\sum_{v\in V} d(v) = \frac{1}{2}d(G) * |V|$.

Definition 23 (Path)**.** A path is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, \ldots, x_{k-1}x_k\}$ where the x_i are all distinct.

Definition 24 (H-path)**.** Given a graph H, we call P an H-path if P is non-trivial and meets H exactly in its end points.

Definition 25 (Cycle)**.** If $P = x_0, ...x_{k-1}$ is a path and $k \geq 3$, then the graph C $= P + x_{k-1}x_0$ is called a cycle.

Definition 26 (Girth)**.** The minimum length of a cycle in a graph G is the girth ($g(G)$).

Definition 27 (Circumference)**.** The maximum length of a cycle in G is its circumference.

Definition 28 (Chord)**.** An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of the cycle.

Definition 29 (Induced cycle)**.** an induced cycle in G, a cycle in G forming an induced subgraph , is one that has no chords.

Definition 30 (Distance)**.** The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x - y$ path in G.

Definition 31 (Diameter)**.** The greatest distance between any two vertices in G is the diameter of G, denoted by diam G.

Definition 32 (Central)**.** a vertex is central in G if its greatest distance from any other vertex is as small as possible.

Definition 33 (Radius)**.** the greatest distance between the central vertex and from any other vertex denoted rad G where $\text{rad}G = \min_{x\in V(G)} \max_{y\in V(G)} d_G(x, y)$ where $\text{rad}G \leq \text{diam}G \leq 2\text{rad}G$.

Definition 34 (Walk)**.** A walk of length k is a non-empty alternating sequence of vertices and edges in G.

Definition 35 (Connected)**.** A non-empty graph G is called connected if any two of its vertices are linked by a path in G.

Definition 36 (Component)**.** A maximal connected subgraph of G is called a component.

Definition 37 (Separator)**.** If $A, B \subset V$ and $X \subseteq V \cup E$ are such that every $A - B$ path in G contains a vertex or an edge from X, we say that X separates the set A and B in G.

Definition 38 (Cutvertex)**.** A vertex which separates two other vertices of the same component is a cutvertex.

Definition 39 (Bridge)**.** An edge separating its ends is a bridge.

Definition 40 (k-connected)**.** G is called k-connected if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. (No vertices of G are separated by fewer than k other vertices.)

Definition 41 (Connectivity)**.** The greatest integer k such that G is k-connected is the connectivity, $\kappa(G)$.

Definition 42 (l-edge-connected)**.** If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than l edges, then G is called l-edge-connected.

Definition 43 (Edge-connectivity)**.** The greatest integer l such that G is l-edge-connected is $\lambda(G)$.

Definition 44 (Forest)**.** An acyclic, graph, one not containing any cycles is called a forest.

Definition 45 (Tree)**.** A connected forest is called a tree.

Definition 46 (Leaf)**.** The vertices of degree 1 of a tree.

Definition 47 (Tree-order)**.** Writing $x \geq y$ for $x \in rTy$ then define a partial ordering on $V(T)$, the tree -order associated with T and r.

Definition 48 (Normal Tree)**.** A rooted tree T contained in a graph G is called normal in G if the ends of every T-path in G are comparable in the tree order of T.

Definition 49 (R-partite)**.** A graph $G = (V, E)$ is called r-partitie if V admits a partition into r classes such that every edge has its ends in different classes.

Definition 50 (Complete Partite)**.** An r-partite graph in which every two vertices from different partitions classes are adjacent is called complete.

Definition 51 (Contraction)**.** Let $e = xy$ be an edge of the graph G. By G/e we denote the graph obtained from G by contracting the edge e into a new vertex v_e.

Definition 52 (Minor)**.** If $G = MX$ is a subgraph of another graph Y, we call X a minor of Y .

Definition 53 (Eulerian)**.** A closed walk in a graph is an Euler tour if it traverses every edge of the graph exactly once. Its called Eulerian if it admits an Euler tour.

Definition 54 (Vertex Space)**.** The vertex space of G is the vector space over the 2-element field of all functions $V \to F_2$, $\mathcal{V}(G)$.

Definition 55 (Edge Space)**.** The function $E \to F_2$ form the edge space of G, $\mathcal{E}(G)$.

Definition 56 (Cycle Space)**.** The cycle space is the subspace of the edge space spanned by all the cycles in G. $\mathcal{C} = \mathcal{C}(G)$.

Definition 57 (Cut Space)**.** The cut edges of G form a subspace of $\mathcal{E}(G)$ denoted by $\mathcal{C}^*(G)$.

Definition 58 (Hypergraph)**.** A hypergraph is a pair (V, E) of disjoint sets, where the elements of E are non-empty subset (of any cardinality) of V.

Definition 59 (Directed)**.** : A directed graph is a pair (V, E) of disjoint set (of vertices and edges) together with two maps init : $E \to V$ and ter : $E \to V$ assigning to every edge e an initial vertex init(e) and a terminal vertex ter(e).

Definition 60 (Orientation)**.** A directed graph D is an orientation of an (undirected) graph G if $V(D) = V(G)$ and$E(D) = E(F)$ and if $\{\text{init}(e), \text{ter}(e)\} = \{x, y\}$ for every edge $e = xy$.

Definition 61 (Multigraph)**.** A multigraph is a pair (V, E) of disjoint sets (of vertices and edges) together with a map $E \to V \cup [V]^2$ assigning to every edge either one or two vertices , its ends.

Definition 62 (Matching)**.** A set M of independent edges in a graph $G = (V, E)$is called a matching.

Definition 63 (Factor)**.** A k-regular spanning subgraph is called k-factor.

Definition 64 (Alternating Paths)**.** A path in G which starts in A at an unmatched vertex and then contains, alternately,edges from $E - M$ and from M, is an alternating path with respect to M.

Definition 65 (Augmenting Path)**.** An alternating path P that ends in an unmatched vertex of B is called an augmenting path.

Definition 66 (Cover)**.** Let us call a set $U \subseteq V$ a (vertex) covering of E if every edge of G is incident with a vertex in U.

Definition 67 ($q(G)$)**.** Given a graph G, let us denote by C_G the set of its components and by $q(G)$ the number of its odd components, those of odd order.

Definition 68 (Block)**.** A maximal connected subgraph without a cutvertex is called a block.

Definition 69 (Plane Graph)**.** Graph drawing in the plane.

Definition 70 (Planar)**.** If G can be drawn in the plane without a crossing, G is plannar.

Definition 71 (Frontier)**.** The frontier of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighborhood of y meets both X and $\mathbb{R}^2 - X$.

Definition 72 (Geometric Graph)**.** A Geometric Graph is a planar embedding of G with straight lines.

Definition 73 (Topological Isomorphism)**.** We call σ a topological isomorphism between H and H' plane graphs if there exists a homeomorphism $\varphi : S^2 \to S^2$ such that $\psi := \pi \circ \varphi \circ \pi^{-1}$ induces σ on $V \cup E$.

Definition 74 (Combinatorial Isomorphism)**.** σ can be extended to an incidence preserving map: $\sigma : V \cup E \cup F \to V' \cup E' \cup F'$. (preserves incidence of not only vertices and edges but also of vertices and edges with faces.)

Definition 75 (Graph Theoretical Isomorphism)**.** σ is a graph theoretical isomorphism of plane H and H' if $\{\sigma(H[f]) \ : f \in F\} = \{H'[f'] \ : f \in F\}$.

Definition 76 (Simple)**.** We call a subset $\mathcal{F}$ of a graph's edge space $\mathcal{E}(G)$ simple if every edge of G lies in at most two sets of $\mathcal{F}$.

Definition 77 (Visible)**.** Given a polygon P, vertices w, v, w is visible from v, if line segment $\overline{vw} \cup \overline{P^c} = \emptyset$.

Definition 78 (Plane Multigraph)**.** A plane multigraph is a pair $G = (V, E)$ of finite sets (of vertices and edges, respectively) satisfying the following conditions:
1. $V \subseteq \mathbb{R}^2$
2. every edge is either an arc between two vertices or a polygon containing exactly one vertex (its endpoints).
3. apart from its own endpoints(s), and edge contains no vertex and no point of any other edge.

Definition 79 (Plane Dual)**.** A plane dual $(V^*, E^*) = G^*$ if there are bijections $F \to V^*$, $E \to E^*$, $V \to F^*$, where $f \mapsto v^*(f)$, $e \mapsto e^*$, $v \mapsto f^*(v)$ satisfying the following conditions
1. i. $v^*(v) \in f$ for all $f \in F$
2. $|e^* \cap G| = |\mathring{e}^* \cap \mathring{e}| = |e \cap G^*| = 1$ for all $e \in E$, and in each of e and e^* this point is an inner point of a straight line segment.
3. $v \in f^*(v)$ for all $v \in V$.

Definition 80 (Proper coloring)**.** If $c(v) \neq c(w)$ for all v adjacent w, then c is a proper coloring.

Definition 81 (k-colorable)**.** G is k-colorable if there exist proper coloring of the vertices with k colors.

Definition 82 (k-edge-colorable)**.** G is k-edge-colorable if there exists a proper edge coloring with k colors or proper coloring of line graph $L(G)$.

Definition 83 ($\chi(G)$)**.** minimum k so that G is k-colorable or chromatic number of G.

Definition 84 ($\chi'(G)$)**.** minimum k so that G is k-edge colorable or chromatic index of G.

Definition 85 ($c^{-1}(j)$)**.** color class of all the vertices colored j.

Definition 86 (k-constructible)**.** For each $k \in \mathbb{N}$, let us define the class of k-constructible graphs recursively as follows:
1. K_k is k-constructible.
2. If G is k-constructible and $x, y \in V(G)$ are non-adjacent, then also $(G + xy)/xy$ is k-constructible.
3. If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible.

2. Theorems

Proposition 87. *The number of vertices of odd degree in a graph is always even.*

Proposition 88. *Every graph G with at least one edge has a subgraph H with $\delta(H) > \epsilon(H) \geq \epsilon(G)$.*

Proposition 89. *Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$(provided that $\delta(G) \geq 2$).*

Proposition 90. *Every graph G containing a cycle satisfies $g(G) \leq 2diam(G) + 1$.*

Proposition 91. *A graph G of radius at most k and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.*

Theorem 92. *Let G be a graph. If $d(g) \geq d \geq 2$ and $g(G) \geq g \in \mathbb{N}$ then $|G| \geq n_0(d, g)$ where $n_0(d, g) = 1 + d\sum_{i=0}^{r-1}(d-1)^i$ if $g = 2r + 1$ is odd; $2\sum_{i=0}^{r-1}(d-1)^i$ if $g = 2r$ is even.*

Corollary 93. *If $\delta(G) \geq 3$ then $g(G) < 2log|G|$.*

Proposition 94. *The vertices of a connected graph G can always be enumerated say as $v_1, ..., v_n$ so that $G_i = G[v_1, ..., v_i]$ is connected for every i.*

Proposition 95. *If G is non-trivial then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Theorem 96. *Let $0 \neq k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has $(k+1)$-connected subgraph H such that $\epsilon(H) > \epsilon(G) - k$.*

Theorem 97. *The following assertions are equivalent fr a graph T:*

(1) *T is a tree.*
(2) *Any two vertices of T are linked by a unique path in T.*
(3) *T is minimally connected, i.e. T is connected by $T - e$ is disconnected for every edge e in T*
(4) *T is maximally acyclic, i.e. T contains no cycle but $T + xy$ does, for any tow non-adjacent vertices $x, y \in T$.*

Corollary 98. *The vertices of a tree can always be enumerated, say as $v_1, ..., v_n$ so that every v_i with $i \geq 2$ has a unique neighbor in $\{v_1, ..., v_{n-1}\}$.*

Corollary 99. *A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.*

Corollary 100. *If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$, then $T \subseteq G$, i.e. G has a subgraph isomorphic to T.*

Lemma 101. *Let T be a normal tree in G.*

(1) *Any two vertices x, y in T are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.*
(2) *If $S \subseteq V(T) = V(G)$ and S is down-closed, then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $T - S$.*

Proposition 102. *Every connected graph contains a normal spanning tree, with any specified vertex as its root.*

Proposition 103. *A graph is bipartite if and only if it contains no odd cycles.*

Proposition 104. (1) *Every TX is also an MX; thus, every topological minor of a graph is also its (ordinary) minor.*
(2) *If $\Delta(X) \leq 3$, then every MX contains a TX; thus every minor with maximum degree at most 3 of a graph is also its topological minor.*

Proposition 105. *The minor relation $\preccurlyeq$ and the topological minor relation are partial orderings on the class of finite graphs, ie. that are reflexive, antisymmetric and transitive.*

Theorem 106. *A connected graph is Eulerian if and only if Every vertex has even degree.*

Proposition 107. *The induced cycles in G generate its entire cycle space.*

Proposition 108. *The following assertions are equivalent for edge set $F \subseteq E$*

(1) *$F \in \mathcal{C}(G)$*
(2) *F is a disjoint union of (edge sets of) cycles in G.*
(3) *All vertex degrees of the graph (V,F) are even.*

Proposition 109. *Together with $\emptyset$, the cuts in G form a subspace $\mathcal{C}^*$ of $\mathcal{E}$. This space is generated by cuts of the form $E(v)$.*

Lemma 110. *Every cut is a disjoint union of bonds.*

Theorem 111. *The cycle space $\mathcal{C}$ and the cut space $\mathcal{C}^*$ of any graph satisfy $\mathcal{C} = \mathcal{C}^{*\perp}$ and $\mathcal{C}^* = \mathcal{C}^{\perp}$.*

Theorem 112. *Let G be a connected graph and $T \subseteq G$ a spanning tree. Then the corresponding fundamental cycles and cuts form a basis of $\mathcal{C}(G)$ and of $\mathcal{C}^*(G)$, respectively. If G has n vertices and m edges, then $dim(\mathcal{C}(G)) = m - n + 1$ and $dim(\mathcal{C}^*(G)) = n - 1$.*

Theorem 113 (Konig)**.** *The maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover of its edges.*

Theorem 114 (Hall)**.** *G contains a matching of A if and only if $|N(S)| \geq |S|$ for $allS \subseteq A$.*

Corollary 115. *If G is $k - regular$, with $k \geq 1$, then G has a 1-factor.*

Corollary 116. *Every regular graph of positive even degree has a 2-factor.*

Theorem 117 (Tutte)**.** *A graph G has a 1-factor if and only if $q(G - S) \leq |S| for S \subseteq V(G)$.*

Corollary 118 (Petersen)**.** *Every bridgeless cubic graph has a 1-factor.*

Theorem 119. *Every graph $G = (V, E)$ contains a vertex set S with the following two properties:*

(1) *S is matchable to $\mathcal{C}_{G-S}$.*
(2) *Every component of $G - S$ is factor-critical.*

Given any such set S, the graph G contains a 1-factor if and only if $|S| = |\mathcal{C}_{G-S}|$

Theorem 120. *G is 2-connected iff either i. G is a cycle or ii. G can be obtained from a 2-connected H by adding an H-path.*

Lemma 121. *If G is 3-connected and $|G| > 4$, then G has one edge s.t. $G - e$ is 3-connected.*

Theorem 122. *G is 3-connected iff there exists a sequence of graphs $G_0, ..., G_n$ such that:*

(1) *$G_0 = K_4$; $G_n = G$*
(2) *G_{i+1} has an edge xy with $d(x)$, $d(y) \geq 3$ and $G_i = G_{i+1}/xy$, $\forall i < n$.*

Theorem 123 (Tutte)**.** *The cycle space of a 3-connected graph is generated by non-separating induced cycle.*

Theorem 124 (Menger)**.** *Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint $A - B$ paths in G.*

Corollary 125. *Let a, b be distinct vertices of G.*

(1) *If $a, b \nsubseteq E$, then minimum number of vertices $\neq ab$ separating a from b equal maximum number of independent $a - b$ path.*
(2) *the minimum number of edges separating a from b in G is equal to the max number of edges disjoint $a - b$ paths.*

Theorem 126 (Global Menger's Theorem)**.**

(1) *A graph is k-connected if and only if it contains k independent path between any two vertices.*
(2) *A graph is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.*

Theorem 127 (Jordan Curve Theorem for Polygons)**.** *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 - P$ has exactly two regions. Each of these has the entire polygon P as its frontier.*

Lemma 128. *Let P_1, P_2, P_3 be three arcs, between the same two endpoints but otherwise disjoint:*

(1) *$\mathbb{R}^2$ $(P_1 \cup P_2 \cup P_3)$ has exactly three regions, with frontiers $P_1 \cup P_2, P_2 \cup P_3, and P_1 \cup P_3$.*
(2) *If P is an arc between a point in $\mathring{P}_1$ and a point in$\mathring{P}_3$ whose interior lies in the region of $\mathbb{R}^2 - (P_1 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \emptyset$.*

Lemma 129. *Let $X_1, X_2 \subseteq \mathbb{R}^2$ be disjoint sets, each the union of finitely many points and arcs, and let P be an arc between a point in $X_1$and one in X_2 whose interior$\mathring{P}$ lies in a region O of $\mathbb{R}^2 - (X_1 \cup X_2)$. Then $O \backslash \mathring{P}$ is a region of $\mathbb{R}^2 - (X_1 \cup P \cup X_2)$.*

Theorem 130. *Let $\varphi : C_1 \to C_2$ be a homeomorphism between two circles on S^2, let O_1 be a region of C_1, and let O_2 be a region of C_2. Then φ can be extended to a homeomorphism $C_1 \cup O_1 \to C_2 \cup O_2$.*

Lemma 131. *Let G be a plane graph, and e an edge of G.*

(1) *If X is the frontier of a face of G, then either $e \subseteq X or X \cap \mathring{e} = \emptyset$.*
(2) *If e lies on a cycle $C \subseteq G$, then e lies on the frontier of exactly two faces of G , and these are contained in distinct faces of C.*
(3) *If e lies on no cycle, then e lies on the frontier of exactly one face of G.*

Corollary 132. *The frontier of a face is always the point set of a subgraph.*

Proposition 133. *A plane forest has exactly one face.*

Lemma 134. *If a plane graph has different face with the same boundary, then the graph is a cycle.*

Proposition 135. *In a 2-connected plane graph, every face is bounded by a cycle.*

Proposition 136. *The face boundaries in a 3-connected plane graph are precisely its non-separating induced cycles.*

Proposition 137. *A plane graph of order at least 3 is maximally planar if and only if it is a plane triangulation.*

Theorem 138 (Euler's Formula)**.** *Let G be a connected plane graph with n vertices, m edges, and l faces, then $n - m + l = 2$.*

Corollary 139. *A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation with n vertices has $3n - 6$.*

Corollary 140. *A plane graph contains neither K_5 nor $K_{3,3}$ as a topological minor.*

Theorem 141.
(1) *Every graph-theoretical isomorphism between two plane graphs is combinatorial. Its extension to a face bijection is unique if and only if the graph is not a cycle.*
(2) *Every combinatorial isomorphism between two 2-connected plane graphs is topological.*

Proposition 142. *Every two embedding of a 3-connected Graph are equivalent.*

Theorem 143 (Kuratowski)**.** *The following are equivalent:*
(1) *G is planar.*
(2) *G has no K_5 and $K_{3,3}$ minor.*
(3) *G has no TK_5 or $TK_{3,3}$ minor.*

Lemma 144. *Every 3-connected G with no K_5 or $K_{3,3}$ minor is planar.*

Lemma 145. *Let χ be a set of 3-connected graphs. Let G be a graph with $\kappa(G) \leq 2$ and G_1, G_2 proper induced subgraph of G such that. $G = G_1 \cup G_2$ and $|G_1 \cap G_2| = \kappa(G)$. If G is edge maximal with respect to not having a topological minor in χ, then so are G_1 and G_2, and $G_1 \cup G_2 = K_2$.*

Lemma 146. *If $|G| \geq 4$ and G is edge-maximal with respect to having no TK_5 or $TK_{3,3}$ minor, then G is 3-connected.*

Theorem 147 (Maclane)**.** *G is planar if and only if its cycle space has a simple basis.*

Theorem 148. *A 3-connected graph is planar if and only if every edge lies on at most 2 non separating induced cycles.(equivalently: exactly).*

Proposition 149. *For any connected plane multigraph G, an edge set $E \subseteq E(G)$ is the edge set of a cycle in G if and only if $E^* := \{e^* | e \in E\}$ is a minimal cut in G^*.*

Proposition 150. *If G^* is an abstract dual of G, then the cut space of G^* is the cycle space of G, $\mathcal{C}^*(G^*) = \mathcal{C}(G)$.*

Theorem 151. *A graph is planar if and only if it has an abstract dual.*

Theorem 152. *All planar graphs are 4-colorable.*

Theorem 153. *Every planar graph is 5-colourable.*

Theorem 154. *Every planar graph not containing a triangle is 3-colourable.*

Proposition 155. *Every graph G with m edges satisfies $\chi(G) \leq \frac{1}{2} + \sqrt{(2m + \frac{1}{4})}$.*

Proposition 156. *Every graph G satisfies $\chi(G) \leq col(G) = \max \delta(H)|\ H \subseteq G + 1$.*

Corollary 157. *Every graph G has a subgraph of minimum degree at least $\chi(G) - 1$.*

Theorem 158 (Brooks)**.** *Let G be a connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 159 (Erdős)**.** *For every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.*

Theorem 160 (Hajós)**.** *Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.*

Proposition 161 (Konig)**.** *Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.*

Theorem 162 (Vizing)**.** *Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*