MATH 574, Practice Problems
Set Theory Problems

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Determine which of the following statements are true and which are false, and prove your answer. (NB: The symbol '$\setminus$' has the same meaning as '$-$' in the context of set theory. Rosen uses the latter, but the former is actually more standard.)

1. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
2. There is a bijection between $\mathbb{R}$ and $(0, 1)$.
3. If $A \subset C$ and $A \subseteq B \subseteq C$, then either $A \subset B$ or $B \subset C$.
4. For any three sets A, B, and C, $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
5. For any three sets A, B, and C, $(A \oplus B) \cup C = (A \cup C) \oplus (B \cup C)$.
6. Suppose $f, g \in A^A$, and $f \circ g = g \circ f$. Then $f \circ g = \mathrm{id}_A$.
7. If f is a one-to-one function from the set X to the set Y and $A, B \subseteq X$, then $f(A \oplus B) = f(A) \oplus f(B)$.
8. If there is a bijection from the set A to the set B and from the set C to the set D, then there is a bijection between A^C and B^D.
9. For any two sets A and B, $B \setminus (B \setminus A) = A$.
10. There exists a one-to-one function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.
11. For any four sets A, B, C, and D, $A \cup B \cap C \cup D = A \cap B \cup C \cap D$.
12. Let $f \in A^A$. Define $A_0 = A$, $A_1 = f(A)$, $A_2 = f(A_1)$, ..., $A_n = f(A_{n-1})$ for $n \geq 1$. Let $A^* = \bigcap_{j=0}^{\infty} A_j$. Then $f(A^*) \subseteq A^*$.

Set Theory Problems: Solutions

1. *True.* Suppose $(a, c) \in A \times C$. Then $a \in A$ and, since $A \subseteq B$, we have that $a \in B$. Similarly, $c \in C$ and $C \subseteq D$ implies $c \in D$. Therefore, $a \in B$ and $c \in D$, so $(a, c) \in B \times D$. We may conclude that $A \times C \subseteq B \times D$. □

2. *True.* There are many such bijections; the following is just one example. Define the function $f : (0, 1) \to \mathbb{R}$ by $f(x) = \tan(\pi(x - 1/2))$. □

3. *True.* Suppose not. Then $A \subset C$, but $A \not\subset B$ and $B \not\subset C$. Then it must be that $A = B$ and $B = C$, so $A = C$, contradicting the fact that A is a proper subset of C. □

4. *True.* Suppose $(x, y) \in (A \cup B) \times C$. Then $x \in A \cup B$, so $x \in A$ or $x \in B$. WLOG, we may assume $x \in A$. Then, since $y \in C$, $(x, y) \in (A \times C)$, so $(x, y) \in (A \times C) \cup (B \times C)$. We may conclude that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$. In the other direction: Suppose $(x, y) \in (A \times C) \cup (B \times C)$. Then $(x, y) \in A \times C$ or $(x, y) \in (B \times C)$. WLOG, we may assume $(x, y) \in A \times C$. Then $x \in A$ and $y \in C$. Since $A \subseteq A \cup B$, we also have that $x \in A \cup B$. Therefore, $(x, y) \in (A \cup B) \times C$, and we may conclude that $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$. Therefore, $(A \cup B) \times C = (A \times C) \cup (B \times C)$. □

5. *False.* Let $A = \emptyset$, $B = \emptyset$, $C = \{\emptyset\}$. Then $(A \oplus B) \cup C = (\emptyset \oplus \emptyset) \cup \{\emptyset\} = \emptyset \oplus \{\emptyset\} = \{\emptyset\}$, but $(A \cup C) \oplus (B \cup C) = (\emptyset \cup \{\emptyset\}) \oplus (\emptyset \cup \{\emptyset\}) = \{\emptyset\} \oplus \{\emptyset\} = \emptyset$. □

6. *False.* Let $A = \mathbb{R}$, $f(x) = x^2$ and $g(x) = x^3$. Then $f \circ g = (x^2)^3 = x^6$, and $g \circ f = (x^3)^2 = x^6$, but $f \circ g \neq \mathrm{id}_{\mathbb{R}}$.

7. *True.* Suppose that $y \in f(A \oplus B)$. Then there exists $x \in A \oplus B$ so that $f(x) = y$. Then $x \in A \setminus B$ or $x \in B \setminus A$. WLOG, we may assume $x \in A \setminus B$. Then $x \in A$, so $f(x) \in f(A)$. Suppose $f(x) \in f(B)$ as well, so that there exists a $z \in B$ with $f(x) = f(z)$. Then, since f is one-to-one, it must be that $z = x$. But then $x \in B$, contradicting the fact that $x \in A \setminus B$. Therefore, $f(A \oplus B) \subseteq f(A) \oplus f(B)$. In the other direction: Suppose $y \in f(A) \oplus f(B)$. Then $y \in f(A) \setminus f(B)$ or $y \in f(B) \setminus f(A)$. WLOG, we may assume the former. Then there is an $x \in A$ so that $f(x) = y$. Suppose $x \in B$ as well. Then $y = f(x) \in f(B)$, contradicting the fact that $y \in f(A) \setminus f(B)$. Therefore, $y \in A \setminus B$, and we may conclude that $f(A) \oplus f(B) \subseteq f(A \oplus B)$. Since we have inclusion in both directions, $f(A \oplus B) = f(A) \oplus f(B)$. □

8. *True.* Suppose $f : A \to B$ and $g : C \to D$ are bijections; then g^{-1} exists. Then, for a function $h \in A^C$, we may define a function $T : A^C \to B^D$ by $T(h) = f \circ h \circ g^{-1}$. That is, for $d \in D$, $T(h)(d) = f(h(g^{-1}(d)))$. Since $g^{-1} : D \to C$, the expression $g^{-1}(d)$ makes sense; because $h : C \to A$ and $g^{-1}(d) \in C$, the expression $h(g^{-1}(d))$ makes sense; and because $h(g^{-1}(d)) \in A$ and $f : A \to B$, the expression $f(h(g^{-1}(d)))$ makes sense. It remains only to prove that $R(h) = f \circ h \circ g^{-1}$ is a bijection. To do so, we simply provide an inverse. Claim: $R : h \mapsto f^{-1} \circ h \circ g$ exists and is an inverse to T. To see this, write

$$\begin{aligned} T \circ R(h) &= f \circ (f^{-1} \circ h \circ g) \circ g^{-1} \\ &= (f \circ f^{-1}) \circ h \circ (g \circ g^{-1}) \\ &= \mathrm{id}_B \circ h \circ \mathrm{id}_D \\ &= h. \end{aligned}$$

   □

9. *False.* By way of counterexample, let $B = \{1, 2\}$ and $A = \{2, 3\}$. Then $B \setminus (B \setminus A) = B \setminus \{1\} = \{2\} \neq A$. □

10. *True.* We could simply note that both sets are countable, and therefore equinumerous, so there exists such an injection (in fact, a bijection). However, it is more convincing to give an explicit example. Let f be defined as follows. When applied to the pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, we first write each of $|x|$ and $|y|$ in base 8; call the resulting strings S and T. Now, if x is negative, prepend the digit '8' to S to obtain a new string S'; do the same for y and T to obtain T'. Finally, concatenate S and T with a '9' between them, and interpret the result as an integer in base 10. (Example: $f(-101_{10}, 52_{10}) = 8145964$, because $101_{10} = 1 \cdot 64_{10} + 4 \cdot 8_{10} + 5 \cdot 1_{10} = 145_8$ and $52_{10} = 6 \cdot 8_{10} + 4 \cdot 1_{10} = 64_8$.) It is easy to see that this function is one-to-one. Indeed, if $f(x, y) = z$, then z contains exactly one digit '9' when written in base 10; splitting the base 10 representation of z into the part to the left of the '9' and the part to the right of the '9' yields two nonnegative integers x' and y'; if x' begins with an '8' in base 10, then interpret the rest of it in base 8 and take its negative to obtain x; similarly, one may obtain y'.

    Here is another example of an injection $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. This one is actually a bijection! First of all, define $g : \mathbb{Z} \to \mathbb{Z}^+$ by $g(x) = 2x$ if $x > 0$ and $g(x) = -2x + 1$ if $x \leq 0$. It is easy to check that this is a bijection. Defining g in this way lets us switch the problem to finding a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+$. We do so by defining the "walk the antidiagonals" function described in class (and the text) – although it is modified slightly here so as always to go left-to-right instead of back-and-forth. Let $h(n, m) = (n^2 + 2nm + m^2 - n - 3m + 2)/2$. (It's not hard to obtain this formula, although it does take some thinking.) Then we can define $f(x, y) = g^{-1}(h(g(x), g(y)))$. □

11. *False*. To obtain a counterexample, let $A = \{1\}$, $B = \emptyset$, $C = \{1, 2\}$, and $D = \emptyset$. Then $A \cup B \cap C \cup D = \{1\} \cap \{1, 2\} \cup \emptyset = \{1\} \cup \emptyset = \{1\}$, while $A \cap B \cup C \cap D = \emptyset \cup \{1, 2\} \cap \emptyset = \{1, 2\} \cap \emptyset = \emptyset$. $\square$

12. *True*. Suppose $x \in A^*$. Then $x \in A_j$ for all $j \in \mathbb{N}$, so $f(x) \in A_j$ for each $j \geq 1$. Since $A_1 = f(A) \subseteq A$, we also have $f(x) \in A = A_0$. Therefore, $f(x) \in A_j$ for all $j \in \mathbb{N}$, so $f(x) \in A^*$. We may conclude that $f(A^*) \subseteq A^*$. $\square$