GENERALIZED PATTERN FREQUENCY IN LARGE PERMUTATIONS

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Abstract. In the study of permutations, generalized patterns extend classical patterns by allowing the requirement that certain adjacent integers in a pattern must be adjacent in the permutation.

For any generalized pattern π_0^* of length k with $1 \leq b \leq k$ blocks, we prove that for all $\mu > 0$, there exists $0 < c = c(k, \mu) < 1$ so that whenever $n \geq n_0(k, \mu, c)$, all but $c^n n!$ many $\pi \in S_n$ admit $(1 \pm \mu)\frac{1}{k!}\binom{n}{b}$ occurrences of π_0^*. Up to the choice of c, this result is best possible.

We also give a lower bound on avoidance of the generalized pattern 12-34, which answers a question of S. Elizalde [8].

1. Introduction

Pattern and generalized pattern avoidance in permutations is a well-studied area (see, e.g.,[1–5, 7, 8, 10, 11]). Fix $1 \leq k \leq n$ and $\pi_0 \in S_k$ and let $\pi \in S_n$. An occurrence of a *pattern* π_0 in π is a sequence of integers $1 \leq \ell_1 < \cdots < \ell_k \leq n$ so that, for all $1 \leq i \neq j \leq k$,

$$\pi(\ell_i) < \pi(\ell_j) \quad \iff \quad \pi_0(i) < \pi_0(j). \tag{1}$$

In order to define generalized patterns, take a classical pattern $\pi_0 = (a_1, \ldots, a_k) = (\pi_0(1), \ldots, \pi_0(k))$, and fix $\pi_0^* = (a_1, \varepsilon_1, a_2, \varepsilon_2, \ldots, \varepsilon_{k-1}, a_k)$ where, for each $1 \leq i \leq k-1$, ε_i is either a dash '$-$' or the empty string. Then, $\pi \in S_n$ admits π_0^* as a *generalized pattern* if it contains an occurrence $1 \leq \ell_1 < \cdots < \ell_k \leq n$ of the classical pattern π_0 satisfying that,

$$\text{whenever } \varepsilon_i \neq -, \text{ then } \ell_{i+1} = \ell_i + 1. \tag{2}$$

More explicitly, suppose, for some positive integer sequence $\mathbf{q} = (q_1, \ldots, q_b)$, $q_1 + \cdots + q_b = k$,

$$\pi_0^* = \pi_0^{\mathbf{q}} = \big(a_1, \ldots, a_{q_1}, -, a_{q_1+1}, \ldots, a_{q_1+q_2}, -, \ldots, -, a_{k-q_b+1}, \ldots, a_k\big) = (A_1, -, A_2, -, \ldots, -, A_b). \tag{3}$$

Then, for some integers $1 \leq \hat{\ell}_1 < \cdots < \hat{\ell}_b \leq n$,

$$(\ell_1, \ldots, \ell_k) = \Big(\hat{\ell}_1, \ldots, \hat{\ell}_1 + q_1 - 1, \;\; \hat{\ell}_2, \ldots, \hat{\ell}_2 + q_2 - 1, \;\; \ldots, \;\; \hat{\ell}_b, \ldots, \hat{\ell}_b + q_b - 1\Big) = (L_1, \ldots, L_b). \tag{4}$$

We shall refer to the subsequences $A_1, \ldots, A_b$ and $L_1, \ldots, L_b$ as *blocks*.

Let $f_{\pi_0^*}(\pi)$ denote the frequency of the generalized pattern π_0^* in π, and set $F_{\pi_0}(\pi) = f_{\pi_0^*}(\pi)$ in the case that $\mathbf{q} = \mathbf{1}$ (i.e., classical patterns). In this notation, the celebrated result of Marcus and Tardos [13] (cf. Klazar [11]) asserts $F_{\pi_0}(\pi) \geq 1$ for all but C^n permutations $\pi \in S_n$, where $C = C(\pi_0) > 1$ and n is sufficiently large. The first author [7] proved that F_{π_0} is concentrated about its mean: $F_{\pi_0}(\pi) = (1 \pm o(1))\frac{1}{k!}\binom{n}{k}$ for all but $o(n!)$ permutations $\pi \in S_n$. Our main result shows, more generally, that $f_{\pi_0^*}$ is also concentrated about its mean, and we provide a sharp estimate for the error $o(n!)$ of concentration.

Theorem 1.1. *For every $k \geq 1$ and for all $\mu > 0$, there exists $0 < c < 1$ so that, for all sufficiently large integers n, the following holds. For every $\pi_0 \in S_k$ and for every sequence π_0^* with b blocks as in (3), all but $c^n n!$ many $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = (1 \pm \mu)\frac{1}{k!}\binom{n}{b}$.*

Remark 1. In Section 2, we offer two proofs of Theorem 1.1. The first, based on martingales, is fairly short. The second gives more detail, using a 'quasi-random' property (see Lemma 2.1) typical of random permutations. Lemma 2.1 extends some results from [7] and may be of independent interest.

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Up to the choice of $0 < c < 1$, Theorem 1.1 is best possible for 'nearly all' π_0^* (see Propositions 1.2 and 1.3 below).

Proposition 1.2. *Let $\pi_0 = (a_1, \dots, a_k) \in S_k$ be a non-identity[1] permutation, and let π_0^* be any sequence, as in (3) with $b \geq 2$ blocks, satisfying the following property:*

$$\text{for some pair } a_i > a_j \text{ with } i < j, \text{ the sequence } \pi_0^* \text{ places } a_i \text{ and } a_j \text{ in distinct blocks.} \tag{5}$$

Then, for all $0 < \gamma < 1/2$, there exist infinitely many integers n for which at least $\gamma^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) < \gamma n^b$.

Proposition 1.3. *For an integer k, let π_0^* be selected uniformly at random among all sequences of the form in (3). Then, with probability*

$$1 - O\left(\frac{1}{\sqrt{k}}\left(\frac{e}{4}\right)^k\right),$$

π_0^* satisfies the property in (5).

We prove Propositions 1.2 and 1.3 in Section 3.

Note, for example, that Proposition 1.2 does not apply to consecutive sequences, i.e., $b = 1$. In this and other cases, however, S. Elizalde proved a much stronger result (see Proposition 4.3 of [8]).

Theorem 1.4 (Elizalde [8])**.** *Let π_0^* be a sequence as in (3) having a block A_i of length at least 3. Then, there exists $0 < c < 1$ so that for all $n \geq k$, at least $c^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = 0$.*

An interesting issue, also considered by Elizalde [8], is to what extent Theorem 1.4 can be extended to sequences π_0^* whose every block has length at most two. He showed, in general, that Theorem 1.4 can't be extended to every π_0^*. To describe these results, let $A_n(\pi_0^*)$ denote the set of permutations $\pi \in S_n$ for which $f_{\pi_0^*}(\pi) = 0$, and let $\alpha_n(\pi_0^*) = |A_n(\pi_0^*)|$. For $(1-, 2, 3, -, 4) = 1\text{–}23\text{–}4$, Elizalde showed (see Corollary 6.2 in [8])

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{–}23\text{–}4)}{n!}\right)^{1/n} = 0 \tag{6}$$

and asked (see Section 7 of [8])

$$\textit{does } \lim_{n\to\infty}\left(\frac{\alpha_n(12\text{–}34)}{n!}\right)^{1/n} = 0\,? \tag{7}$$

We answer this question in the negative.

Theorem 1.5. *For odd integers n,*

$$\alpha_n(12\text{–}34) \geq \left(\frac{1}{2} - o(1)\right)^n n!.$$

We prove Theorem 1.5 in Section 4. In that section, we also provide a corollary of Theorem 1.1 to address a related problem.

2. Proofs of Theorem 1.1

For both of the following proofs, fix a positive integer k and fix $\mu > 0$.

[1] Note that $\pi_0 = (a_1, \dots, a_k)$ must contain a pair $a_i > a_j$ with $1 \leq i < j \leq k$.

2.1. **The martingale proof.** Let

$$c = \exp\left\{-\frac{\mu^2}{9k^4 k!^2}\right\} \tag{8}$$

and let n be a sufficiently large integer wherever needed. Fix π_0^* with b blocks as in (3). We show that all but $c^n n!$ many $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = (1 \pm \mu)\frac{1}{k!}\binom{n}{b}$.

To that end, let $\pi \in S_n$ be chosen uniformly at random. We use the 'exposure process' to define the following sequence of random variables. Set

$$X_0 = \mathbb{E}[f_{\pi_0^*}(\pi)], \text{ where from (4), we have } (1-o(1))\frac{1}{k!}\binom{n}{b} \le \mathbb{E}[f_{\pi_0^*}(\pi)] \le \frac{1}{k!}\binom{n}{b}. \tag{9}$$

For $r \in [n] = \{1, \dots, n\}$, let $\pi_{[r]}$ denote the restriction $\pi : [r] \to [n]$. Set

$$X_r = \mathbb{E}\left[f_{\pi_0^*}(\pi) \middle| \pi_{[r]} \right],$$

so that $X_n = f_{\pi_0^*}(\pi)$ is the variable we wish to estimate. Then, $X_0, X_1, \dots, X_n$ is the Doob martingale for the function $f_{\pi_0^*}$, to which we will apply Azuma's inequality.

For that purpose, observe that for each $0 \le r \le n-1$,

$$X_r \le X_{r+1} \le X_r + k\binom{n}{b-1}. \tag{10}$$

To see this, the element $r+1$ belongs to between zero and $k\binom{n}{b-1}$ occurrences $(\ell_1, \dots, \ell_k)$ of π_0^* in π. Indeed, if $\ell_i = r+1$ belongs to block $L_{i'}$ (see (4)), then all of $L_{i'}$ is determined by $r+1 = \ell_i$ and $\mathbf{q} = \mathbf{q}(\pi_0^*)$. Thus, it remains to determine $L_1, \dots, L_{i'-1}, L_{i'+1}, \dots, L_b$, or equivalently, $\hat{\ell}_1, \dots, \hat{\ell}_{i'-1}, \hat{\ell}_{i'+1}, \dots, \hat{\ell}_b$, of which there are at most $\binom{n}{b-1}$.

Applying Azuma's inequality with $t = (\mu/2)X_0$ and using (9) and (10), we have

$$\begin{aligned}\mathbb{P}\big[|X_n - X_0| \ge t\big] &\le 2\exp\left\{-\frac{t^2}{2\sum_{r=0}^{n-1}(X_{r+1}-X_r)^2}\right\} \le \exp\left\{-\frac{\mu^2 \frac{1}{k!^2}\binom{n}{b}^2}{8nk^2\binom{n}{b-1}^2}(1-o(1))\right\} \\ &= \exp\left\{-\frac{\mu^2 n}{8k!^2k^2b^2}(1-o(1))\right\} \le \exp\left\{-\frac{\mu^2 n}{8k!^2k^4}(1-o(1))\right\} \le \exp\left\{-\frac{\mu^2 n}{9k!^2k^4}\right\} \stackrel{(8)}{=} c^n.\end{aligned}$$

Thus, with probability $1 - c^n$,

$$f_{\pi_0^*}(\pi) = \left(1 \pm \frac{\mu}{2}\right)\mathbb{E}[f_{\pi_0^*}(\pi)] \stackrel{(9)}{=} \left(1 \pm \frac{\mu}{2}\right)(1 \pm o(1))\frac{1}{k!}\binom{n}{b} = (1 \pm \mu)\frac{1}{k!}\binom{n}{b},$$

as desired.

2.2. **The quasi-random proof.** To present Lemma 2.1, we need a few concepts. For integers $n > t \ge j \ge 1$, define $I_j = [(j-1)\lfloor n/t \rfloor + 1, j\lfloor n/t \rfloor]$ and $R = [n] \setminus \bigcup_{j=1}^t I_j$. We call $[n] = I_1 \cup I_2 \cup \cdots \cup I_t \cup R$ the *t-partition* $\boldsymbol{P}_t$ of $[n]$. Now, fix $\pi \in S_n$, and consider partitions $\boldsymbol{P}_s = I_1 \cup \cdots \cup I_s \cup R_s$ and $\boldsymbol{P}_t = E_1 \cup \cdots \cup E_t \cup R_t$ of $[n]$, where $n > t \ge s \ge q \ge 1$. For [2] $\mathbf{i} = (i_1, \dots, i_q) \in ([s])_q$ and $j \in [t]$, let

$$E_{\mathbf{i}j}(\pi) = \left\{\hat{\ell} \in E_j : \hat{\ell} + q - 1 \in E_j \text{ and } \pi(\hat{\ell} + m - 1) \in I_{i_m} \text{ for all } m \in \{1, \dots, q\}\right\}. \tag{11}$$

For $\zeta > 0$, and $(\mathbf{i}, j) \in ([s])_q \times [t]$, we say $\pi \in S_n$ is *$(\mathbf{i}, j, \zeta, q)$-typical (w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$)* if

$$|E_{\mathbf{i}j}(\pi)| \ge (1-\zeta)\frac{1}{(s)_q}|E_j| = (1-\zeta)\frac{1}{(s)_q}\left\lfloor \frac{n}{t} \right\rfloor \tag{12}$$

and say $\pi \in S_n$ is *(ζ, q)-typical (w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$)* if it is $(\mathbf{i}, j, \zeta, q)$-typical for all $(\mathbf{i}, j) \in ([s])_q \times [t]$.

[2] For a set X, we write $(X)_m$ for the family of m-permutations of X, and we write $(|X|)_m$ for $|(X)_m|$, when $|X|$ is finite.

Lemma 2.1. *For all $\zeta > 0$ and integers $q \geq 1$, there exists an integer s_0 so that for all integers $s \geq s_0$, there exists an integer t_0 so that for all integers $t \geq t_0$, there exists $c_0 > 0$ so that for all sufficiently large integers n, all but $\exp\{-c_0 n\} n!$ permutations $\pi \in S_n$ are (ζ, q)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$.*

Lemma 2.1 follows by a standard (albeit tedious) probablistic analysis, which we give in Section 5.

To show that Lemma 2.1 implies Theorem 1.1, define auxiliary constants $\delta, \zeta > 0$ so that

$$\delta = \frac{\mu}{k!} \qquad \text{and} \qquad (1-2\zeta)^{k+2} > 1 - \delta. \tag{13}$$

For $q \in [k]$, let $s_0(q)$ be the constant guaranteed by Lemma 2.1. Fix an integer s so that

$$s \geq \max\{s_0(1), \ldots, s_0(k)\} \qquad \text{and} \qquad \binom{s}{k} \geq \frac{s^k}{k!}(1-2\zeta). \tag{14}$$

For $q \in [k]$, let $t_0(q)$ be the constant guaranteed by Lemma 2.1. Fix an integer t with

$$t \geq \max\left\{t_0(1), \ldots, t_0(k)\right\} \qquad \text{and so that for all } b \in [k], \qquad \binom{t}{b} \geq \frac{t^b}{b!}(1-2\zeta). \tag{15}$$

For $q \in \{1, \ldots, k\}$, let $c_0(q) > 0$ be the constant guaranteed by Lemma 2.1. Define

$$c_0 = \min\left\{c_0(1), \ldots, c_0(k)\right\} \qquad \text{and} \qquad c = \exp\{-c_0/4\}. \tag{16}$$

In all that follows, let n be a sufficiently large integer.

Fix a permutation $\pi_0 \in S_k$, and let $\pi_0^* = \pi_0^{\mathbf{q}} = (A_1, -, \ldots, -, A_b)$ be given as in (3) where $\mathbf{q} = (q_1, \ldots, q_b)$. Apply Lemma 2.1 (cf. (14)–(16)) to conclude that all but

$$\Big(\exp\big\{-c_0(q_1)n\big\} + \cdots + \exp\big\{-c_0(q_b)n\big\}\Big) n! \leq k \exp\{-c_0 n\} n! \leq \exp\left\{-\frac{c_0}{2} n\right\} n!$$

permutations $\pi \in S_n$ are (ζ, q_x)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$ for all $x \in [b]$. For such a $\pi \in S_n$, we show

$$f_{\pi_0^{\mathbf{q}}}(\pi) \geq (1-\delta)\frac{1}{k!}\binom{n}{b} \overset{(13)}{>} (1-\mu)\frac{1}{k!}\binom{n}{b}. \tag{17}$$

Indeed, fix indices $1 \leq i_1 < \cdots < i_k \leq s$ and $1 \leq j_1 < \cdots < j_b \leq t$. For $x \in [b]$, recall the block

$$A_x = \big(a_{q_1+\cdots+q_{x-1}+1}, \ldots, a_{q_1+\cdots+q_x}\big) = \big(\pi_0(q_1+\cdots+q_{x-1}+1), \ldots, \pi_0(q_1+\cdots+q_x)\big)$$

of $\pi_0^{\mathbf{q}}$ (cf. (3)). Consider the injection defined by, for each $x \in [b]$,

$$j_x \longmapsto \mathbf{i}_x \overset{\text{def}}{=} (i_a)_{a \in A_x} = \Big(i_{a_{q_1+\cdots+q_{x-1}+1}}, \ldots, i_{a_{q_1+\cdots+q_x}}\Big) = \big(i_{\pi_0(q_1+\cdots+q_{x-1}+1)}, \ldots, i_{\pi_0(q_1+\cdots+q_x)}\big). \tag{18}$$

For each $x \in [b]$, arbitrarily select $\hat{\ell}_x \in E_{\mathbf{i}_x j_x}(\pi)$ (cf. (11)). We claim that the sequence

$$(L_1, L_2, \ldots, L_b), \qquad \text{where for each } x \in [b], \qquad L_x = \big(\hat{\ell}_x, \hat{\ell}_x + 1, \ldots, \hat{\ell}_x + q_x - 1\big), \tag{19}$$

is exactly an occurrence in π of the generalized pattern $\pi_0^{\mathbf{q}}$. Indeed, the sequence $(L_1, \ldots, L_b)$ clearly satisfies (2), since each L_m is consecutive, and since $(L_1, L_2, \ldots, L_b)$ precisely mimics the block structure of $\pi_0^{\mathbf{q}} = (A_1, -, A_2, -, \ldots, -, A_b)$ (cf. (3)). It remains to check, therefore, that $(L_1, \ldots, L_b)$ is an occurrence of the classical pattern π_0 in π, i.e., that $(L_1, \ldots, L_b)$ satisfies (1).

Indeed, rewrite the sequence $(L_1, \ldots, L_b)$ as

$$(\ell_1, \ldots, \ell_k) = \big(\ell_1, \ldots, \ell_{q_1}, \ell_{q_1+1}, \ldots, \ell_{q_1+q_2}, \ldots\ldots, \ell_{k-q_b+1}, \ldots, \ell_k\big)$$
$$\text{so that for } x \in [b], \qquad L_x = \big(\ell_{q_1+\cdots+q_{x-1}+1}, \ldots, \ell_{q_1+\cdots+q_x}\big). \tag{20}$$

Comparing (19) and (20), we see that a term of the sequence $(L_1, \ldots, L_b)$ is determined by a choice of indices $1 \leq x \leq b$ and $1 \leq w \leq q_x$, and written simultaneously as

$$\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w}. \tag{21}$$

(Such a term necessarily belongs to the block L_x.) Observe from (11) and (18) that

$$\pi(\hat{\ell}_x + w - 1) \in I_{i(x,w)}, \quad \text{where} \quad i(x,w) = i_{\pi_0(q_1+\cdots+q_{x-1}+w)}. \tag{22}$$

Now, fix two terms (cf. (21)) of the sequence $(L_1, \dots, L_b)$:

$$\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w} \qquad \text{and} \qquad \hat{\ell}_y + z - 1 = \ell_{q_1+\cdots+q_{y-1}+z},$$

where $1 \le x, y \le b$, $1 \le w \le q_x$ and $1 \le z \le q_y$. From (22), we conclude

$$\pi\big(\ell_{q_1+\cdots+q_{x-1}+w}\big) < \pi\big(\ell_{q_1+\cdots+q_{y-1}+z}\big) \iff \max I_{i(x,w)} < \min I_{i(y,z)} \iff i(x,w) < i(y,z)$$
$$\overset{(22)}{\iff} i_{\pi_0(q_1+\cdots+q_{x-1}+w)} < i_{\pi_0(q_1+\cdots+q_{y-1}+z)} \iff \pi_0(q_1+\cdots+q_{x-1}+w) < \pi_0(q_1+\cdots+q_{y-1}+z),$$

as required by (1). (For the last step, recall the ordering $1 \le i_1 < \cdots < i_k \le s$ of the fixed indices.)

Now, the discussion above implies that

$$f_{\pi_0^{\mathbf{q}}}(\pi) \ge \sum\sum \left\{ \prod_{x=1}^{b} \big|E_{\mathbf{i}_x j_x}(\pi)\big| : 1 \le i_1 < \cdots < i_k \le s,\, 1 \le j_1 < \cdots < j_b \le t \right\}. \tag{23}$$

Since $\pi \in S_n$ is (ζ, q)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$ for every $q \in \{q_1, \dots, q_b\}$, we have, for each $x \in [b]$,

$$\big|E_{\mathbf{i}_x j_x}(\pi)\big| \ge (1-\zeta)\frac{1}{(s)_{q_x}}\left\lfloor \frac{n}{t} \right\rfloor \ge (1-2\zeta)\frac{n}{t(s)_{q_x}} \ge (1-2\zeta)\frac{n}{ts^{q_x}}.$$

Returning to (23),

$$\frac{f_{\pi_0^{\mathbf{q}}}(\pi)}{\binom{s}{k}\binom{t}{b}} \ge (1-2\zeta)^b \left(\frac{n}{t}\right)^b \prod_{x=1}^{b} \frac{1}{s^{q_x}} = (1-2\zeta)^b \left(\frac{n}{t}\right)^b \frac{1}{s^{q_1+\cdots+q_b}} = (1-2\zeta)^b \left(\frac{n}{t}\right)^b \frac{1}{s^k} \ge (1-2\zeta)^k \left(\frac{n}{t}\right)^b \frac{1}{s^k},$$

and so (17) follows from

$$f_{\pi_0^{\mathbf{q}}}(\pi) \ge \binom{s}{k}\binom{t}{b}(1-2\zeta)^k \left(\frac{n}{t}\right)^b \frac{1}{s^k} \overset{(14),\,(15)}{\ge} (1-2\zeta)^{k+2}\frac{1}{k!}\left(\frac{n^b}{b!}\right) \overset{(13)}{\ge} (1-\delta)\frac{1}{k!}\binom{n}{b}.$$

The corresponding upper bound $f_{\pi_0^{\mathbf{q}}}(\pi) \le (1+\mu)\frac{1}{k!}\binom{n}{b}$ follows, in fact, from the lower bound. Indeed, first conclude (17) for every permutation $p \in S_k$ and $p^* = p^{\mathbf{q}}$. Thus, all but

$$k! \exp\left\{-\frac{c_0}{2}n\right\} n! < \exp\left\{-\frac{c_0}{4}n\right\} n! \overset{(16)}{=} c_1^n n!$$

permutations $\pi \in S_n$ satisfy, for every $p \in S_k$, $f_{p^{\mathbf{q}}}(\pi) \ge (1-\delta)\frac{1}{k!}\binom{n}{b}$. Fix such a $\pi \in S_n$. Observe that every $1 \le \ell_1 < \cdots < \ell_k \le n$ of the form in (19) and (20) defines a generalized pattern $p^{\mathbf{q}}$ of some $p \in S_k$. (Indeed, if $\pi(\{\ell_1, \dots, \ell_k\}) = \{\lambda_1, \dots, \lambda_k\}$, define $p(i) = j$ if and only if $\pi(\ell_i) = \lambda_j$.) Thus,

$$\binom{n}{b} \ge \sum_{p \in S_k} f_{p^{\mathbf{q}}}(\pi) = f_{\pi_0^{\mathbf{q}}}(\pi) + \sum_{\pi_0 \ne p \in S_k} f_{p^{\mathbf{q}}}(\pi) \ge f_{\pi_0^{\mathbf{q}}}(\pi) + (k!-1)(1-\delta)\frac{1}{k!}\binom{n}{b}$$
$$\implies \qquad f_{\pi_0^{\mathbf{q}}}(\pi) \le \left(\frac{1}{k!} + \delta - \frac{\delta}{k!}\right)\binom{n}{b} \le (1+\delta k!)\frac{1}{k!}\binom{n}{b} \overset{(13)}{=} (1+\mu)\frac{1}{k!}\binom{n}{b}.$$

3. Proof of Propositions 1.2 and 1.3

We prove Propositions 1.2 and 1.3 in chronological order.

Proof of Proposition 1.2. Fix a non-identity permutation $\pi_0 = (a_1, \dots, a_k) \in S_k$. Fix a sequence π_0^*, as in (3), satisfying the property in (5). In particular, let $a_i > a_j$, where $i < j$, belong to blocks $A_{i'} \ne A_{j'}$. Let $g = \lfloor 1/\gamma \rfloor$, and take n to be divisible by g and sufficiently large. Partition $[n] = I_1 \cup I_2 \cup \cdots \cup I_g$ into consecutive intervals $I_s = [(s-1)(n/g)+1, s(n/g)]$, $1 \le s \le g$. Define $S'_n = \{\pi \in S_n : \pi(I_s) = I_s, \forall\, 1 \le s \le g\}$. By Stirling's formula,

$$|S'_n| = \left(\left(\frac{n}{g}\right)!\right)^g > \frac{1}{2}\left(\sqrt{2\pi(n/g)}\left(\frac{n}{eg}\right)^{(n/g)}\right)^g > \frac{\sqrt{\gamma}}{2}(2\pi\gamma n)^{\frac{g-1}{2}} \times \gamma^n \sqrt{2\pi n}\left(\frac{n}{e}\right)^n > \gamma^n n!.$$

Now, fix $\pi \in S'_n$ and let $1 \le \ell_1 < \cdots < \ell_k \le n$ be an occurrence of the generalized pattern π_0^* in π. Then, for some $1 \le s \le g$, we have $\ell_i, \ell_j \in I_s$. Indeed, if $\ell_i \in I_{s_i}$ and $\ell_j \in I_{s_j}$ for some $s_i < s_j$, then

$\pi(\ell_i) < \pi(\ell_j)$, which disagrees with $\pi_0(i) = a_i > a_j = \pi_0(j)$. Now, by (4), ℓ_i belongs to block $L_{i'}$ and ℓ_j belongs to $L_{j'}$, where necessarily, $\hat{\ell}_{i'} \in I_{s-1} \cup I_s$ and $\hat{\ell}_{j'} \in I_s$. It is easy to check that there are fewer than n/g choices for $\hat{\ell}_{i'}$. Clearly, there are at most $|I_f| = n/g$ choices for $\hat{\ell}_{j'}$, and at most n^{b-2} choices for any remaining $\hat{\ell}_1, \ldots, \hat{\ell}_b$ in (4). Thus, $f_{\pi_0^*}(\pi) < n^{b-2} \sum_{f=1}^{g} (n/g)^2 \leq \gamma n^b$. □

Proof of Proposition 1.3. Fix an integer k. We begin the proof by observing an equivalent formulation of the property in (5). To that end, fix a permutation $\pi_0 = (a_1, \ldots, a_k) \in S_k$ and a sequence $\pi_0^* = \pi_0^{\mathbf{q}} = (A_1, \ldots, A_b)$ as in (3), where $\mathbf{q} = (q_1, \ldots, q_b)$ (recall the notation in (3)). Suppose π_0^* satisfies that

$$\forall\ 1 \leq h \leq b, \quad \left\{a_{q_1+\cdots+q_{h-1}+1}, \ldots, a_{q_1+\cdots+q_h}\right\} = \left\{q_1 + \cdots + q_{h-1} + 1, \ldots, q_1 + \cdots + q_h\right\}. \tag{24}$$

In other words, suppose each block A_h of π_0^*, $1 \leq h \leq b$, consists of a permutation of the elements $\{q_1 + \cdots + q_{h-1} + 1, \ldots, q_1 + \cdots + q_h\}$. We claim that π_0^* does not satisfy the property in (5) if, and only if, π_0^* satisfies the property in (24), which we abbreviate by

$$\neg(5) \iff (24).$$

Indeed, it is easy to see that $(24) \Rightarrow \neg(5)$. To see that $\neg(24) \Rightarrow (5)$, let $1 \leq h \leq b$ be the minimum integer for which

$$\left\{a_{q_1+\cdots+q_{h-1}+1}, \ldots, a_{q_1+\cdots+q_h}\right\} \neq \left\{q_1 + \cdots + q_{h-1} + 1, \ldots, q_1 + \cdots + q_h\right\}.$$

Since h is the minimum integer for which the inequality above is true, the following two properties must hold:

(i) there exists $i \in \{q_1 + \cdots + q_{h-1} + 1, \ldots, q_1 + \cdots + q_h\}$ for which
$$a_i > q_1 + \cdots + q_h;$$
(ii) there exists $q_1 + \cdots + q_h < j \leq k$ for which
$$a_j \in \{q_1 + \cdots + q_{h-1} + 1, \ldots, q_1 + \cdots + q_h\}.$$

But now, (i) and (ii) imply $i < j$ and $a_i > a_j$, and clearly, a_i and a_j belong to different blocks of π_0^*. Thus, π_0^* satisfies the property in (5).

We now prove Proposition 1.3, and to that end, assume k is somewhat large. Let π_0^* be a sequence selected uniformly at random among all sequences of the form (3), of which there are $2^{k-1}k!$ many. Then,

$$\mathbb{P}\left[\pi_0^* \text{ does not satisfy (5)}\right] = \mathbb{P}\left[\pi_0^* \text{ satisfies (24)}\right]$$
$$= \frac{1}{2^{k-1}k!} \sum_{b=1}^{k} \sum \left\{q_1! \cdots q_b! : (q_1, \cdots + q_b) = \mathbf{q} \text{ is a composition of } k\right\}. \tag{25}$$

Fix a composition $\mathbf{q} = (q_1, \ldots, q_b)$ of k. We claim that

$$\prod_{i=1}^{b} q_i! \leq \left(\frac{k+b}{2b}\right)^k. \tag{26}$$

Indeed, $\prod_{i=1}^{k} q_i!$ is log-concave, and so we shall (twice) apply Jensen's inequality. In particular, since $\log x = \log_e x$ is concave, a first application of Jensen's inequality provides

$$\log \prod_{i=1}^{b} q_i! = \sum_{i=1}^{b} \sum_{j=1}^{q_i} \log j \leq \sum_{i=1}^{b} q_i \log\left(\frac{q_i + 1}{2}\right).$$

Since $x \log((x+1)/2)$ is concave, a second application of Jensen's inequality provides

$$\sum_{i=1}^{b} q_i \log\left(\frac{q_i+1}{2}\right) \leq b\left(\frac{k}{b}\right) \log\left(\frac{\frac{k}{b}+1}{2}\right),$$

which implies (26).

Applying (26) to (25), and recalling the well-known fact that there are $\binom{k-1}{b-1}$ compositions of k into b parts, we obtain

$$\mathbb{P}[\pi_0^* \text{ does not satisfy (5)}] \le \frac{1}{2^{k-1}k!}\sum_{b=1}^{k}\binom{k-1}{b-1}\left(\frac{k+b}{2b}\right)^k = \frac{1}{4^{k-1}k!}\sum_{b=1}^{k}\binom{k-1}{b-1}\left(1+\frac{k}{b}\right)^k.$$

Applying Stirling's formula to $k!$, we see that

$$\mathbb{P}[\pi_0^* \text{ does not satisfy (5)}] = O\left(\frac{1}{\sqrt{k}}\left(\frac{e}{4}\right)^k\sum_{b=1}^{k}\binom{k-1}{b-1}\left(\frac{1}{k}+\frac{1}{b}\right)^k\right).$$

Taking, in places, $k \ge 3$ and $k \ge 7$, we have

$$\sum_{b=1}^{k}\binom{k-1}{b-1}\left(\frac{1}{k}+\frac{1}{b}\right)^k \le \left(1+\frac{1}{k}\right)^k + (k-1)\left(\frac{5}{6}\right)^k + (k-2)2^{k-1}\left(\frac{10}{21}\right)^k = O(1),$$

which proves Proposition 1.3. □

4. Proof of Theorem 1.5

Consider the following concept, which has a clear resemblance to patterns. For $\pi \in S_n$, call a pair $1 < i < j < n$ a *stretching pair* if $\pi(i) < i < j < \pi(j)$. We shall use stretching pairs to prove Theorem 1.5, although stretching pairs are interesting in their own right, as we discuss in Section 3.2.

4.1. Stretching pairs and Theorem 1.5.

We establish a few initial considerations. First, let $C_{n+1} \subset S_{n+1}$ denote the set of $(n+1)$-cycles of S_{n+1}, and write each $\pi \in C_{n+1}$ in cyclic notation: $\pi = (n+1\ a_1\ \dots\ a_n)$, i.e., $\pi(a_i) = a_{i+1}$ for $0 \le i \le n$ and $a_0 = a_{n+1} = n+1$. Consider the bijection $\phi : C_{n+1} \to S_n$ given by, for each $\pi = (n+1\ a_1\ \dots\ a_n) \in C_{n+1}$,

$$p = \phi(\pi) = (a_1, \dots, a_n), \text{ that is, } p(i) = a_i \text{ for each } 1 \le i \le n. \tag{27}$$

We prove that

$$\pi \in C_{n+1} \textit{ admits a stretching pair } 1 \le \pi(i) < i < j < \pi(j) \ne n+1$$
$$\textit{if and only if } p = \phi(\pi) \textit{ admits } 21\text{–}34 \textit{ or } 34\text{–}21 \textit{ as a generalized pattern.} \tag{28}$$

Before we prove (28), we note that 21–34 is not the same as 12–34, which Theorem 1.5 considers. However, Elizalde proved (see Proposition 5.3 from [8]) that

$$\alpha_n(12\text{–}34) = \alpha_n(21\text{–}34), \tag{29}$$

and so we shall be able to use (28).

Proof of (28). Suppose first that $p = f(\pi) = (a_1, \dots, a_n) \in S_n$ admits 21–34 or 34–21 as a generalized pattern. If $a_k, a_{k+1}, a_\ell, a_{\ell+1}$ is a copy of 21–34, where $1 < k+1 < \ell < n$, then $a_{k+1} < a_k < a_\ell < a_{\ell+1}$, and so $\pi(i) = a_{k+1} < a_k = i < j = a_\ell < a_{\ell+1} = \pi(j) \le n$ is a stretching pair of π. If $a_k, a_{k+1}, a_\ell, a_{\ell+1}$ is a copy of 34–21, then $a_{\ell+1} < a_\ell < a_k < a_{k+1}$, and so $\pi(i) = a_{\ell+1} < a_\ell = i < j = a_k < a_{k+1} = \pi(j) \le n$ is a stretching pair of π. Assume now that $\pi = (n+1\ a_1\ \dots\ a_n) \in C_{n+1}$ admits a stretching pair $1 \le \pi(i) < i < j < \pi(j) \le n$. If $\pi = (n+1\ a_1\ \dots\ i\ \pi(i)\ \dots\ j\ \pi(j)\ \dots\ a_n)$, then for some $1 < k+1 < \ell < n$, $p = f(\pi)$ has $i = a_k$, $\pi(i) = a_{k+1}$, $j = a_\ell$ and $\pi(j) = a_{\ell+1}$, where $a_{k+1} < a_k < a_\ell < a_{\ell+1}$ gives a copy of 21–34. If $\pi = (n+1\ a_1\ \dots\ j\ \pi(j)\ \dots\ i\ \pi(i)\ \dots\ a_n)$, then for some $1 < k+1 < \ell < n$, $p = f(\pi)$ has $j = a_k$, $\pi(j) = a_{k+1}$, $i = a_\ell$ and $\pi(i) = a_{\ell+1}$, where $a_{\ell+1} < a_\ell < a_k < a_{k+1}$ gives a copy of 34–21. □

Now, define S'_{n+1} to be the family of $\pi \in S_{n+1}$ satisfying $(n+1)/2 < \pi(i) \le n+1$ if, and only if, $1 \le i \le (n+1)/2$. Clearly, S'_{n+1} admits no stretching pairs. Set $C'_{n+1} = C_{n+1} \cap S'_{n+1}$, and observe that $C'_{n+1} \ne \emptyset$ if, and only if, n is odd. As such, if n is both odd and sufficiently large, Stirling's formula implies

$$|C'_{n+1}| = \frac{2}{n+1}\left(\left(\frac{n+1}{2}\right)!\right)^2 \ge \left(\frac{1}{2} - o(1)\right)^n n!.$$

It then follows from (28) that $\phi(C'_{n+1})$ avoids 21–34 and 34–21, and so

$$\alpha_n(12\text{–}34) \overset{(29)}{=} \alpha_n(21\text{–}34) \geq |A_n(21\text{–}34) \cap A_n(34\text{–}21)| \geq \left|\phi(C'_{n+1})\right| = |C'_{n+1}| \geq \left(\frac{1}{2} - o(1)\right)^n n!,$$

which proves Theorem 1.5.

4.2. **A corollary of Theorem 1.1 for stretching pairs.** Stretching pairs are motivated by considerations in dynamical systems. Namely, the occurrence of a stretching pair within a periodic orbit of a continuous interval map implies what is called 'turbulence'. (see [3, 12] for details). These considerations are closely related to the celebrated theorem of Sharkovsky [14]. From this point of view, the second author [12] considered which n-cycles $\pi \in C_n$ admit stretching pairs, and proved that all but $o(n-1)!$ of them do. Theorem 1.1 allows us to sharpen this result in the following way.

Corollary 4.1. *For all $\delta > 0$, there exists $0 < c < 1$ so that for all sufficiently large integers n, all but $c^n(n-1)!$ cyclic permutations $\pi \in C_n$ admit $\frac{1}{12}\binom{n}{2}(1 \pm \delta)$ stretching pairs.*

Proof of Corollary 4.1. Let $\delta > 0$ be given. Set $k = 4$ and $\mu = \delta/2$, and let $0 < c_1 < 1$ be the constant guaranteed by Theorem 1.1. Define c to be any constant satisfying $c_1 < c < 1$, and let n be sufficiently large. For an n-cycle $\pi \in C_n$, write $\sigma(\pi)$ for the number of stretching pairs of π, and write $\sigma'(\pi)$ for the number of stretching pairs $1 \leq \pi(i) < i < j < \pi(j) \neq n$. Note that $\sigma'(\pi) \leq \sigma(\pi) \leq \sigma'(\pi) + n$, since if $1 \leq \pi(i) < i < j < \pi(j) = n$, then $j = \pi^{-1}(n)$ is fixed and there are at most $j - 1 \leq n$ choices for i. Note, moreover, that it follows from (28) that, for $p = \phi(\pi) \in S_{n-1}$, $\sigma'(\pi) = f_{21\text{–}34}(p) + f_{34\text{–}21}(p)$. Theorem 1.1 ensures that all but $2c_1^{n-1}(n-1)! < c^n(n-1)!$ permutations $p \in S_{n-1}$ satisfy

$$f_{21\text{-}34}(p) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} \quad \text{and} \quad f_{34\text{-}21}(p) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2}.$$

For each such permutation $p \in S_{n-1}$, the corresponding n-cycle $\pi = \phi^{-1}(p) \in C_n$ satisfies

$$\sigma(\pi) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} + (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} \pm n = (1 \pm \mu \pm o(1))\frac{1}{12}\binom{n}{2} = (1 \pm \delta)\frac{1}{12}\binom{n}{2},$$

which proves Corollary 4.1. □

5. Proof of Lemma 2.1

Fix $\zeta > 0$ and integer $q \geq 1$. Define auxiliary constant

$$\zeta_0 = \zeta/4. \tag{30}$$

Define $s_0 = s_0(q, \zeta_0)$ to be the least integer s for which

$$(s)_q \geq (1 - 2\zeta_0)s^q. \tag{31}$$

Let $s \geq s_0$ be given. Define

$$t_0 = \left\lceil 4q8^q s^{2q} \zeta_0^{-2} \right\rceil. \tag{32}$$

Let integer $t \geq t_0$ be given. Define

$$c_0 = \frac{\zeta_0^2}{3qt2^{q+3}s^q}. \tag{33}$$

Let n be a sufficiently large integer, and fix $(\mathbf{i}_0, j_0) \in ([s])_q \times [t]$. We prove

$$\text{all but } \exp\{-2c_0 n\} n! \text{ permutations } \pi \in S_n \text{ are } (\mathbf{i}_0, j_0, \zeta, q)\text{-typical w.r.t. } (\boldsymbol{P}_s, \boldsymbol{P}_t). \tag{34}$$

Applying (34) to all $(\mathbf{i}, j) \in ([s])_q \times [t]$ and noting $s^q t \exp\{-2c_0 n\} < \exp\{-c_0 n\}$ yields Lemma 2.1.

We now outline our approach for proving (34) (and reduce the $\hat{\ell}$ notation in (11) to ℓ). Define equivalence relation $\sim$ on E_{j_0}: $\ell \sim \ell' \iff q \,|\, (\ell - \ell')$. Thus, for an integer $0 \leq r < q$, we may write

$$E_{j_0}^{(r)} = \left\{\ell \in E_{j_0} : \ell \sim (j_0 - 1)\left\lfloor \frac{n}{t} \right\rfloor + 1 + r\right\} \qquad \text{so that} \qquad E_j = E_j^{(0)} \cup \cdots \cup E_j^{(q-1)} \tag{35}$$

is a partition. A key observation for later in the proof (cf. Claim 5.1) will be that

$$[\ell, \ell+1-q] \cap [\ell', \ell'+q-1] = \emptyset \quad \text{whenever } \ell \neq \ell' \in E_j^{(r)}. \tag{36}$$

For some final notation, we shall write, for a permutation $\pi \in S_n$,

$$E_{\mathbf{i}_0 j_0}^{(r)}(\pi) = E_{\mathbf{i}_0 j_0}(\pi) \cap E_{j_0}^{(r)} \qquad \text{so that} \qquad E_{\mathbf{i}_0 j_0}(\pi) = E_{\mathbf{i}_0 j_0}^{(0)}(\pi) \cup \cdots \cup E_{\mathbf{i}_0 j_0}^{(q-1)}(\pi) \tag{37}$$

is a partition. We shall prove that, for a fixed $0 \leq r < q$,

$$\textit{all but } \exp\{-3c_0 n\} n! \textit{ permutations } \pi \in S_n \textit{ satisfy that } \left| E_{\mathbf{i}_0 j_0}^{(r)}(\pi) \right| \geq (1-\zeta) \frac{1}{q (s)_q} |E_{j_0}|. \tag{38}$$

Note that (38) implies (34) since then all but $q \exp\{-3c_0 n\} n! < \exp\{-2c_0 n\} n!$ many $\pi \in S_n$ satisfy

$$|E_{\mathbf{i}_0 j_0}(\pi)| \overset{(37)}{=} \sum_{r=0}^{q-1} \left| E_{\mathbf{i}_0 j_0}^{(r)}(\pi) \right| \geq (1-\zeta) \frac{1}{(s)_q} |E_{j_0}|.$$

To prove (38), let $\pi \in S_n$ be chosen uniformly at random. Then, $Y = Y_{\mathbf{i}_0 j_0}^{(r)} = |E_{\mathbf{i}_0 j_0}^{(r)}(\pi)|$ is a random variable whose mean we evaluate. To that end, recall from (11) that for an element $\ell \in E_{j_0}$ to be an element of $E_{\mathbf{i}_0 j_0}(\pi)$, we require that $\ell \leq j_0 \lfloor n/t \rfloor - q + 1$, where we will write $n_t = \lfloor n/t \rfloor$ and $n_s = \lfloor n/s \rfloor$. As such, delete the last $q-1$ elements from E_{j_0}, and write

$$\tilde{E}_{j_0} \overset{\text{def}}{=} \big[(j_0 - 1) n_t + 1,\ j_0 n_t - q + 1 \big], \quad \tilde{E}_{j_0}^{(r)} = E_{j_0}^{(r)} \cap \tilde{E}_{j_0},$$
$$\text{and} \quad n_{t,q} \overset{\text{def}}{=} \left| \tilde{E}_{j_0}^{(r)} \right| = \left\lfloor \frac{n_t - q + 1}{q} \right\rfloor = \left\lfloor \frac{n_t + 1}{q} \right\rfloor - 1. \tag{39}$$

Now, for $\ell \in \tilde{E}_{j_0}^{(r)}$, define indicator random variable Y_ℓ by (cf. $\mathbf{i}_0 = (i_1, \ldots, i_q)$)

$$Y_\ell = \begin{cases} 1 & \text{if } \pi(\ell + m - 1) \in I_{i_m} \ \forall\, m \in [q], \\ 0 & \text{otherwise}, \end{cases} \qquad \Longrightarrow \qquad Y = \sum \left\{ Y_\ell : \ell \in \tilde{E}_{j_0}^{(r)} \right\}$$

$$\text{so that} \quad \mathbb{E}[Y_\ell] = \frac{(n-q)! \prod_{m=1}^q |I_{i_m}|}{n!} = \frac{n_s^q}{(n)_q} \qquad \Longrightarrow \qquad \mathbb{E}[Y] = \frac{|\tilde{E}_{j_0}^{(r)}| \prod_{m=1}^q |I_{i_m}|}{(n)_q} = \frac{n_s^q n_{t,q}}{(n)_q}. \tag{40}$$

Following the method of Bernstein for the Chernoff inequality (cf. [10]), for $u = \log(1-\zeta_0) = \log_e(1-\zeta_0)$, the Markov inequality implies

$$\mathbb{P}[Y \leq \mathbb{E}[Y](1-\zeta_0)] = \mathbb{P}\left[e^{uY} \geq \exp\{u \mathbb{E}[Y](1-\zeta_0)\} \right]$$
$$\leq \exp\{-u \mathbb{E}[Y](1-\zeta_0)\} \mathbb{E}\left[e^{uY} \right] \overset{(40)}{=} \exp\left\{ -u \frac{n_s^q n_{t,q}}{(n)_q} (1-\zeta_0) \right\} \mathbb{E}\left[e^{uY} \right]. \tag{41}$$

While we do not have mutual independence among the Y_ℓ's, we will prove the following.

Claim 5.1.

$$\mathbb{E}\left[e^{uY} \right] = \mathbb{E}\left[\prod_{\ell \in \tilde{E}_{j_0}^{(r)}} e^{uY_\ell} \right] \leq \left(1 + q \frac{(4s)^q}{t} \right)^{n_{t,q}} \prod_{\ell \in \tilde{E}_{j_0}^{(r)}} \mathbb{E}\left[e^{uY_\ell} \right]$$
$$\overset{(40)}{=} \left(\left(1 + q \frac{(4s)^q}{t} \right) \left(1 + \frac{n_s^q}{(n)_q} (e^u - 1) \right) \right)^{n_{t,q}} \leq \exp\left\{ n_{t,q} \left(q \frac{(4s)^q}{t} + \frac{n_s^q}{(n)_q} (e^u - 1) \right) \right\}.$$

We shall defer the proof of Claim 5.1 in order first to finish the proof of (38).

Applying Claim 5.1 to (41), together with the Taylor series bound $-u(1-\zeta_0)+e^u-1 \le -\zeta_0^2/2$,

$$\begin{aligned}\mathbb{P}[Y \le \mathbb{E}[Y](1-\zeta_0)] &\le \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}+\frac{n_s^q}{(n)_q}\Big(-u(1-\zeta_0)+e^u-1\Big)\right)\right\} \\ &\le \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2 n_s^q}{2(n)_q}\right)\right\} \le \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2}{2}\left(\frac{n_s}{n}\right)^q\right)\right\} \\ &\le \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2}{2^{q+1}s^q}\right)\right\} \qquad (\text{since } n_s=\lfloor n/s\rfloor \ge n/(2s)) \\ &\overset{(32)}{\le} \exp\left\{n_{t,q}\left(-\frac{\zeta_0^2}{2^{q+2}s^q}\right)\right\} \le \exp\left\{-\left(\frac{\zeta_0^2}{qt2^{q+3}s^q}\right)n\right\} \qquad (\text{since } n_{t,q}=\lfloor (n_t+1)/q\rfloor -1 \ge n/(2tq)) \\ &\overset{(33)}{=} \exp\{-3c_0 n\}.\end{aligned}$$

In other words, with probability $1-\exp\{-3c_0 n\}$, the randomly chosen permutation $\pi \in S_n$ satisfies

$$\begin{aligned}Y = \left|E^{(r)}_{\mathbf{i}_0 j_0}(\pi)\right| \ge \mathbb{E}[Y](1-\zeta_0) &\overset{(40)}{=} \frac{n_s^q n_{t,q}}{(n)_q}(1-\zeta_0) \overset{(39)}{=} (1-\zeta_0)(1-o(1))\frac{1}{qs^q}\left\lfloor\frac{n}{t}\right\rfloor \\ &\overset{(31)}{\ge} (1-2\zeta_0)^2\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor \ge (1-4\zeta_0)\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor \overset{(30)}{=} (1-\zeta)\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor.\end{aligned}$$

5.1. **Proof of Claim 5.1.** Write $\tilde{E}^{(r)}_{j_0}$ as $\ell_1 < \cdots < \ell_{n_{t,q}}$ (cf. (39)) so that

$$\mathbb{E}\left[\prod_{\ell\in\tilde{E}^{(r)}_{j_0}} e^{uY_\ell}\right] = \sum_{(y_1,\dots,y_{n_{t,q}})\in\{0,1\}^{n_{t,q}}} \mathbb{P}\left[\bigwedge_{i=1}^{n_{t,q}} Y_{\ell_i}=y_i\right]\prod_{i=1}^{n_{t,q}} e^{uy_i}. \tag{42}$$

Fix $(y_1,\dots,y_{n_{t,q}})\in\{0,1\}^{n_{t,q}}$ so that

$$\mathbb{P}\left[\bigwedge_{i=1}^{n_{t,q}} Y_{\ell_i}=y_i\right] = \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}}\Big|\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right]\cdot\mathbb{P}\left[\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right].$$

We claim that

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}}\Big|\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right] = \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}}\right]\left(1\pm q\frac{(4s)^q}{t}\right). \tag{43}$$

If so, iteratively applying (43) to (42) yields Claim 5.1.

To see (43), recall the observation in (36). Thus,

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\Big|\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right] \le \frac{(n-qn_{t,q})!\prod_{m=1}^q|I_{i_m}|}{(n-q(n_{t,q}-1))!} = \frac{n_s^q}{(n-qn_{t,q}+q)_q} \quad \text{and}$$

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\Big|\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right] \ge \frac{(n-qn_{t,q})!\prod_{m=1}^q(|I_{i_m}|-q(n_{t,q}-1))}{(n-q(n_{t,q}-1))!} = \frac{(n_s-qn_{t,q}+q)^q}{(n-qn_{t,q}+q)_q}.$$

For the upper bound, we use (40) (and $qn_{t,q}\le n_t$ (cf. (39)) to infer

$$\begin{aligned}\frac{n_s^q}{(n-qn_{t,q}+q)_q} &= \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\cdot\frac{(n)_q}{(n-qn_{t,q}+q)_q} \le \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(\frac{n}{n-n_t}\right)^q \\ &\le \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1-\frac{1}{t}\right)^{-q} \le \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1+\frac{2}{t}\right)^q \le \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1+q\frac{4^q}{t}\right).\end{aligned}$$

For the lower bound, we similarly infer

$$\frac{(n_s - qn_{t,q} + q)^q}{(n - qn_{t,q} + q)_q} \geq \frac{(n_s - qn_{t,q})^q}{(n)_q} = \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 1\right]\left(\frac{n_s - qn_{t,q}}{n_s}\right)^q$$

$$\geq \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 1\right]\left(\frac{n_s - n_t}{n_s}\right)^q \geq \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 1\right]\left(1 - 2\frac{s}{t}\right)^q \geq \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 1\right]\left(1 - q\frac{(4s)^q}{t}\right).$$

This proves (43) when $y_{n,t} = 1$. Otherwise, (using $\mathbb{P}[Y_{\ell_{n_{t,q}}} = 1] \leq 1/2 \leq \mathbb{P}[Y_{\ell_{n_{t,q}}} = 0]$) we have

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 0 \Big| \bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j} = y_j\right] = 1 - \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 1 \Big| \bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j} = y_j\right] = \mathbb{P}\left[Y_{\ell_{n_{t,q}}} = 0\right]\left(1 \pm q\frac{(4s)^q}{t}\right),$$

where we used $\mathbb{P}[Y_{\ell_{n_{t,q}}} = 1] \leq 1/2 \leq \mathbb{P}[Y_{\ell_{n_{t,q}}} = 0]$.

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