Critical Sets for Sudoku and General Graph Colorings

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Abstract

We discuss the problem of finding *critical sets* in graphs, a concept which has appeared in a number of guises in the combinatorics and graph theory literature. The case of the Sudoku graph receives particular attention, because critical sets correspond to minimal fair puzzles. We define four parameters associated with the sizes of extremal critical sets and (a) prove several general results about these parameters' properties, including their computational intractability, (b) compute their values exactly for some classes of graphs, (c) obtain bounds for generalized Sudoku graphs, and (d) offer a number of open questions regarding critical sets and the aforementioned parameters.

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A recent announcement due to McGuire, et al. ([8]), surprised many in the community of Sudoku researchers, amateur and professional alike, with its complete resolution of the "minimum number of clues" (MNC) problem. *Sudoku* is a single-player game in which one completes a partial 9×9 matrix M all of whose entries are drawn from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by appealing to the *rules*: no number may appear twice in any row, any column, or any of the nine "blocks", each a 3×3 submatrix with indices $\{3a + 1, 3a + 2, 3a + 3\} \times \{3b + 1, 3b + 2, 3b + 3\}$ for $a, b \in \{0, 1, 2\}$. A "board" is a matrix adhering to these rules; a "puzzle" is a partially filled-in board; the nonempty entries of a puzzle are called "clues" or "givens". A puzzle is said to be "fair" if it can be completed to a valid board in precisely one way. The MNC problem asks: what is the fewest number of clues in a fair Sudoku puzzle? While it was long suspected that the answer is 17, a proof seemed out of reach until [8].

However, the sense in which the authors of [8] "proved" that the solution is indeed 17 arguably does not meet modern standards of mathematical rigor. (The

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paper briefly acknowledges this deficiency, although popular press' wide reporting of the result typically did not address this important, if subtle, issue. See [9] for more discussion of philosophical aspects of computer-assisted proof.) There are several interesting ideas presented in the aforementioned manuscript – mostly careful case reductions and very clever search strategies – but, in the end, the result relied on a *year-long* computation, amounting to 7.1 million core hours on an SGI Altic ICE 8200EX cluster with 320 nodes, each of which consisted of two Intel Xeon E5650 hex-core processors with 24GB of RAM. Even if one sets aside well-worn (and important) philosophical critiques of computer-assisted proofs that appeal to uncheckability by humans and the social nature of proof, it is almost inconceivable that this enormous computation on an extremely complicated configuration of networked and nested devices did not experience hardware errors (due to manufacturing defects, cosmic rays, background radiation, the inherent stochasticity of quantum mechanics, etc.) *and* software errors (bugs in the various operating systems, firmware, algorithmic code, GUIs, etc.). While such errors may not have produced an incorrect answer, they certainly undermine the definitiveness of the result. Therefore, we wish to draw attention to the subject matter of "critical sets" for graph colorings, a concept that neatly generalizes the MNC problem as well as several other questions scattered throughout the discrete mathematics literature, in the hopes that greater visibility might eventually lead to human-readable solutions to questions like the MNC.

We begin by defining "determining sets" for graph colorings: a set S of vertices with the property that the coloring, restricted to S, can be completed in precisely one way (i.e., back to the original coloring).

Definition 1. *A "determining set" of vertices in a graph $G = (V, E)$ with respect to a proper vertex coloring $c : V \to [\chi(G)]$ is a set $S \subseteq V$ with the property that, for any proper vertex coloring c' of G, if $c'|_S \equiv c|_S$, then $c' \equiv c$ on all of V.*

If a determining set is minimal with respect to this property, we call it "critical".

Definition 2. *A "critical set" of vertices in a graph $G = (V, E)$ with respect to a proper vertex coloring $c : V \to [\chi(G)]$ is a minimal determining set for the pair (G, c).*

The cardinality of the largest and smallest critical sets in various graphs have appeared in a number of guises. Latin squares (and, of course, the special subclass of them that comprise Sudoku boards), matching theory, design theory, and the study of dominating vertex sets all feature variants of this idea. (See [4] for a more comprehensive list of related topics and references.) Therefore, we define the following parameters.

Definition 3. *For a graph $G = (V, E)$ and a vertex coloring $c : V \to [\chi(G)]$, define*

$$\textsc{scs}(G, c) = \min\{|X| : X \text{ is a critical set for } (G, c)\}$$

and

$$\textsc{lcs}(G, c) = \max\{|X| : X \text{ is a critical set for } (G, c)\}.$$

Definition 4. *For a graph $G = (V, E)$, define*

$$\underline{\text{SCS}}(G) = \min_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \text{SCS}(G, c)$$

$$\underline{\text{LCS}}(G) = \min_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \text{LCS}(G, c).$$

Similarly, define

$$\overline{\text{SCS}}(G) = \max_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \text{SCS}(G, c)$$

$$\overline{\text{LCS}}(G) = \max_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \text{LCS}(G, c).$$

A few words on notation. For graph-theoretic concepts, unless stated explicitly, we generally rely on the conventions of [3]; in particular, we sometimes write the edge $\{x, y\}$ simply as xy. For a vertex $v \in V(G)$, we define the "neighborhood" of v to be $N_G(v) = \{w \in V(G) : vw \in E(G)\}$, where the subscript may be omitted if it is obvious. The symbol "$\square$" denotes Cartesian product, i.e., given two graphs $G = (U, E)$ and $H = (V, F)$, the vertex set of $G\square H$ is $U \times V$, and its edge set is given by all pairs of the form $\{(u, v), (u', v')\}$ with $[(u = u') \wedge (vv' \in E(H))] \vee [(v = v') \wedge (uu' \in E(G))]$. We denote the set $\{1, \ldots, n\}$ by $[n]$. Finally, let SUD_n denote the Sudoku graph, i.e., $V(\text{SUD}_n) = [n^2] \times [n^2]$ and $\{(a, b), (c, d)\} \in E(\text{SUD}_n)$ for $(a, b) \neq (c, d)$ iff $a = c$, $b = d$, or $(\lceil a/n\rceil = \lceil c/n\rceil) \wedge (\lceil b/n\rceil = \lceil d/n\rceil)$.

1. General Bounds and Observations

In this section, we introduce a few important definitions and prove some general facts about critical sets.

Definition 5. *A coloring $c : V(G) \to [k]$ of a graph G is "optimal" if $k = \chi(G)$.*

Definition 6. *A graph G is "uniquely colorable" if it has exactly one optimal coloring up to permutation of the colors.*

Definition 7. *A graph G is "critically k-uniform" if every critical set $S \subset V(G)$ satisfies $|S| = k$; it is "critically uniform" if it is critically k-uniform for some k.*

Note that the condition of G being critically k-uniform is equivalent to the statement that

$$k = \overline{\text{LCS}}(G) = \underline{\text{LCS}}(G) = \overline{\text{SCS}}(G) = \underline{\text{SCS}}(G).$$

Proposition 1. *If a graph G is uniquely colorable, then G is critically $(\chi(G) - 1)$-uniform.*

Proof. First, every critical set S must have cardinality at least $\chi(G) - 1$; if $|S| < \chi(G)-1$, then, given any coloring $c : V(G) \to [\chi(G)]$, there are at least two extensions of $c|_S$ to a proper coloring: c itself and c', where

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \in c(S) \\ \pi(c(v)) & \text{if } c(v) \notin c(S), \end{cases}$$

where π is any permutation of $[\chi(G)]$ so that $\pi|_{[\chi(G)]\setminus c(S)}$ is not the identity function.

Next, suppose S is a critical set for the coloring $c : V(G) \to [\chi(G)]$. Note that, if $c(v) = c(w)$, then at most one of v or w is in S, as the partition into color classes is uniquely determined; therefore, $|S| = |c(S)|$. If $|S| = |c(S)| = \chi(G)$, then let $S' = S \setminus \{c^{-1}(\chi(G))\}$. Since the partition $\Pi = \{c^{-1}(1), \ldots, c^{-1}(\chi(G))\}$ is unique, and $\tilde{c} = c|_{S'}$ determines the color of all blocks but one, the last block must be colored $\chi(G)$, and the only proper coloring extending $\tilde{c}$ is c itself. Therefore, $|S| \leq \chi(G) - 1$. □

Note that it is not true that, if a graph is critically uniform, then it is uniquely colorable, as evidenced by the graph obtained by joining a pendant edge to each vertex of a K_3. (In this case, all four parameters equal 4.) However, we have been unable to determine the status of the full converse of Proposition 1: whether a graph being $(\chi(G) - 1)$-uniform implies unique colorability.

Corollary 2. *If G is bipartite and consists of k components, then G is critically k-uniform.*

Proof. Since $\chi(G) = 2$, every proper coloring c gives the same bipartition of the vertex set of each component. Therefore, specifying the color of one vertex of each component completely determines the rest, i.e., a set $S \subseteq V(G)$ is determining if it contains a vertex of each component. Furthermore, one must specify the color of at least one vertex of each component, or else the colors within the omitted component could be swapped in any extension of $c|_S$ to a proper coloring. The only critical sets in G, then, have cardinality k. □

Proposition 3. *For any graph G, $\underline{\text{SCS}}(G) \leq \overline{\text{SCS}}(G)$, $\underline{\text{SCS}}(G) \leq \underline{\text{LCS}}(G)$, $\underline{\text{LCS}}(G) \leq \overline{\text{LCS}}(G)$, and $\overline{\text{SCS}}(G) \leq \overline{\text{LCS}}(G)$.*

Proof. Obvious. □

Proposition 4. *For any graph G, every critical set S satisfies $|S| \leq |V(G)| - 1$.*

Proof. Suppose $t = \chi(G)$ and $V(G)$ is a critical set for some coloring $c : V(G) \to [\chi(G)]$. Let $\mathcal{N}(v) = c(N(v))$. Note that, for each vertex $v \in V(G)$, $|\mathcal{N}(v)| \leq t - 2$. Therefore, for each $v \in c^{-1}(\chi(G))$, the set $\mathcal{C}(v) = [\chi(G) - 1] \setminus \mathcal{N}(v)$ is nonempty; let α_v be an arbitrary element of $\mathcal{C}(v)$, and define a new coloring $c' : V(G) \to [\chi(G)]$ by

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \in [\chi(G) - 1] \\ \alpha_v & \text{otherwise.} \end{cases}$$

Note that $\chi(G) \notin c'(V(G))$. Furthermore, $c^{-1}(\chi(G))$ is an independent set, so c' is a proper coloring of G. Therefore, c' is actually a $(t - 1)$-coloring of G, contradicting the definition of t. □

2. Specific Graphs

For even cycles, Corollary 2 gives the values of the four parameters.

Corollary 5. *For n even, C_n is critically 1-uniform.*

We need only consider the case of n odd now. Throughout the sequel, we label the vertices of C_n in cyclic order as $v_0, \ldots, v_{n-1}$; the indices are interpreted $\bmod n$. The following result is a corollary of a result in [7]; for completeness, in order to frame it in our notation, and because the argument is somewhat different, we include the proof here.

Theorem 6. *For n odd,*

$$\underline{\mathrm{SCS}}(C_n) = \frac{n+1}{2}.$$

Proof. Since C_n is not bipartite, any determining set S cannot exclude any two consecutive vertices. Indeed, suppose $v_0, v_1 \notin S$. Fix a proper vertex coloring $c : V \to \{0, 1, 2\}$. If $c(v_{n-1}) = c(v_2)$ ($= 0$ without loss of generality), then there are two ways to color v_0 and v_1: $c(v_0) = 1$ and $c(v_1) = 2$ or $c(v_0) = 2$ and $c(v_1) = 1$. If $c(v_{n-1}) \neq c(v_2)$, without loss of generality $c(v_{n-1}) = 0$ and $c(v_2) = 1$, then either $c(v_0) = 1$ and $c(v_1) = 2$ or $c(v_0) = 2$ and $c(v_1) = 0$. In either case, there is more than one way to complete $c|_S$ to a coloring, a contradiction.

Now, if $|S| < n/2$, then S excludes two consecutive vertices, a contradiction. Therefore, $|S| \geq n/2$, i.e., $|S| \geq (n+1)/2$ (since $|S| \in \mathbb{Z}$). On the other hand, we must exhibit 3-colorings of C_n, n odd, along with sets of $(n+1)/2$ vertices which determine the coloring. We consider three cases:

$n \equiv 0 \pmod 3$: Let $c(v_j) = j \pmod 3$, and $S = \{v_0, v_2, v_4, \ldots, v_{n-1}\}$. Then, since $c(v_i) \neq c(v_{i+2})$ for each $i \in \mathbb{Z}_n$, the color of each vertex in $V \setminus S = \{v_1, \ldots, v_{n-2}\}$ is uniquely determined.

$n \equiv 1 \pmod 3$: Let $c(v_j) = j \pmod 3$ for $j < n-1$ and $c(v_{n-1}) = 1$, and $S = \{v_0,\ v_2,\ v_4,\ \ldots,\ v_{n-5},\ v_{n-3},\ v_{n-2}\}$. Then, since $c(v_i) \neq c(v_{i+2})$ for each $i \in \{0, \ldots, n-4\}$ and $c(v_{n-2}) = 2 \neq 0 = c(v_0)$, the color of each vertex in $V \setminus S = \{v_1, \ldots, v_{n-6}, v_{n-4}, v_{n-1}\}$ is uniquely determined.

$n \equiv 2 \pmod 3$: Let $c(v_j) = j \pmod 3$, and $S = \{v_0, v_2, v_4, \ldots, v_{n-5}, v_{n-3}, v_{n-1}\}$. Then, since $c(v_i) \neq c(v_{i+2})$ for each $i \in \{0, \ldots, n-2\}$, the color of each vertex in $V \setminus S = \{v_1,\ \ldots,\ v_{n-6},\ v_{n-4},\ v_{n-2}\}$ is uniquely determined.

□

Theorem 7. *For n odd,*

$$\overline{\mathrm{LCS}}(C_n) = n - 1.$$

Proof. Color C_n by

$$c_0(v_i) = \begin{cases} i \pmod 2 & \text{if } 0 \leq i < n-1 \\ 2 & \text{if } i = n-1. \end{cases}$$

Let S consist of all vertices but v_{n-1}. Then S determines the coloring, since v_{n-1} has one 1-colored vertex v_{n-2} and one 0-colored vertex v_0 from S adjacent to it. On the other hand, it is not possible to remove a vertex from S and have it still determine the coloring: removing either v_0 or v_{n-2} would leave a gap of two adjacent vertices, which we noted above is not possible in a determining set, and removing any other vertex allows for a recoloring of that vertex with 2. Therefore, S is critical, and $\overline{\text{LCS}}(C_n) \geq n-1$. To conclude that $\overline{\text{LCS}}(C_n) \leq n-1$, we need only invoke Proposition 4. □

For the next few proofs, we find the following definition useful.

Definition 8. *A vertex v of a graph G associated with a coloring $c : V(G) \to [k]$ is "colorful" if $|c(\{v\} \cup N(v))| = k$.*

Lemma 8. *Every proper optimal coloring of C_n, for n odd, admits a colorful vertex.*

Proof. Let $c : V(C_n) \to \mathbb{Z}_3$ be any proper vertex coloring of C_n. Towards a contradiction, suppose that every $v_i \in C_n$, has the property that $|\{c(v_{i-1}), c(v_i), c(v_{i+1})\}| = 2$. Without loss of generality, $c(v_1) = 0$ and $c(v_2) = 1$, so $c(v_3) = 0$, or else v_2 would yield a contradiction. But then $c(v_4) = 1$, or else v_3 would yield a contradiction. Proceeding around the cycle, we see that the colors alternate, which is not possible for an odd cycle. Therefore, every coloring has a colorful vertex v. □

Note that no critical set can contain a colorful vertex and all of its neighbors, as its neighbors already determine that vertex's color.

Theorem 9. *For n odd,*

$$\overline{\text{SCS}}(C_n) = n - 2.$$

Proof. First, we argue that every coloring admits a critical set of cardinality less than $n-1$. Suppose S were a critical set of cardinality $n-1$, and c is a coloring of C_n for which it is critical. Without loss of generality, we may assume that v_0 is the single element of $V(C_n) \setminus S$, $c(v_0) = 2$ and $c(v_1) = 1$. Clearly, $c(v_2) \in \{0, 2\}$. Suppose that $c(v_2) = 2$. Then the alternating sequence $c(v_0), c(v_1), c(v_2), \ldots$ must terminate with a vertex v_j, $j < n$, at some point, since the entire cycle cannot be 2-colored. But then, $c(v_{j-2})$ and $c(v_j)$ differ, making $c(v_{j-1})$ colorful, contradicting its membership in S. Therefore, $c(v_2) = 0$; again the sequence $c(v_1), c(v_2), c(v_3), \ldots$ must eventually contain the color 2, say, for the first time at v_j. If $j < n$, then v_{j-1} is colorful, contradicting its membership in S. The only remaining possibility is that $j = 0$ and $c(v_j) = j \pmod 2$ for all $j \neq 0$. However, then $V(C_n) \setminus \{v_1, v_{n-1}\}$ is a smaller critical set. Therefore, $\overline{\text{SCS}}(C_n) < n - 1$.

Consider any critical set S for the coloring c_0 considered in the preceding proof. If $v_i \in S$, then $v_{i+1} \in S$ or $v_{i+2} \in S$, because $V(C_n) \setminus S$ cannot have two consecutive vertices. If $v_{i+1} \notin S$ and $i \leq n-4$, then $c(v_{i+2}) = c(v_i)$, so $v_{i+1} \in S$ or else it could be recolored with the single element of $\mathbb{Z}_3 \setminus \{c(v_i), c(v_{i+1})\}$. Therefore, $v_i \in S$ for all $1 \leq i \leq n-3$, and there are only three vertices which could be omitted from S: v_{n-2}, v_{n-1}, and v_0. All three cannot be omitted, or else $V(C_n) \setminus S$ would contain two consecutive vertices. Therefore, $|S| \geq n-2$ and $\overline{\text{SCS}}(C_n) \geq n-2$. □

Theorem 10. *For n odd,*

$$\underline{\text{LCS}}(C_n) = \begin{cases} \frac{n+3}{2} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Color $V(C_n)$ as follows:

$$c(v_i) = \begin{cases} 0 & \text{if } 2|i \text{ and } i < n-1 \\ 1 & \text{if } i \equiv 1 \pmod 4 \\ 2 & \text{if } i \equiv 3 \pmod 4 \\ 3 - c(v_{n-2}) & \text{if } i = n-1 \end{cases}$$

It is easy to check that this is a proper coloring. Let S be a critical set for c. Then S must contain $v_1, v_3, \ldots, v_{n-4}$, as, even if the vertices on both sides of these v_i are included, the color of v_i is not determined. However, then it is unnecessary to include $v_2, v_4, \ldots, v_{n-5}$, since these vertices are colorful. Now, at most three of the colorful vertices v_{n-3}, v_{n-2}, v_{n-1}, and v_0 can be in S, or else S would contain three consecutive vertices the middle one of which is colorful. Therefore,

$$|S| \leq \frac{n-3}{2} + 3 = \frac{n+3}{2}.$$

Therefore, $\underline{\text{LCS}}(C_n) \leq (n+3)/2$. If $n \equiv 3 \pmod 4$, we claim that in fact, S can include at most two of the vertices v_{n-3}, v_{n-2}, v_{n-1}, and v_0. Note that $c(v_{n-3}) = 0$, $c(v_{n-2}) = 1$, $c(v_{n-1}) = 2$, and $c(v_0) = 0$, and *all four of these vertices are colorful.* Suppose S contained three of these vertices. Since $v_{n-4} \in S$ and $v_1 \in S$, either S contains v_{n-4}, v_{n-3}, and v_{n-2}, or it contains v_{n-1}, v_0, and v_1; in either case, the middle vertex is determined by the colors of its neighbors, a contradiction. We may conclude that $|S| \leq (n-3)/2 + 2 = (n+1)/2$. Note that $(n+1)/2 = \underline{\text{SCS}}(C_n) \leq \underline{\text{LCS}}(C_n)$. Therefore, if $n \equiv 3 \pmod 4$, $\underline{\text{LCS}}(C_n) = (n+1)/2$.

If $n \equiv 1 \pmod 4$, we claim that every coloring in fact admits a critical set of size at least $(n+3)/2$. Suppose not. Then, by the above bounds, every critical set S for some coloring c is size $(n+1)/2$. Since S cannot omit two consecutive vertices, S must contain every other vertex all the way around the cycle, except for exactly one location where it contains two consecutive vertices. Without loss of generality, we may assume these are v_0 and v_1, so $S = \{v_0, v_1, v_3, v_5, \ldots\}$. Note that v_k must be colorful for $k > 0$ even, or else it would have to be included in S. Furthermore, every v_k with $k > 1$ odd must be noncolorful, or else we could replace v_k with v_{k-1} and v_{k+1} in S, obtaining a critical set of size $> (n+1)/2$. Therefore, if v_1 is noncolorful, then, without loss of generality, $c(v_k) = 0$ if k is even, $c(v_k) = 1$ if $k \equiv 1 \pmod 4$ and $c(v_k) = 2$ if $k \equiv 3 \pmod 4$. However, then $c(v_{n-1}) = 0$, contradicting the fact that $c(v_0) = 0$ and c is a proper coloring. If v_1 is colorful, then, without loss of generality, $c(v_k) = 0$ if $k \neq 0$ is even, $c(v_k) = 1$ if $k \equiv 3 \pmod 4$ or $k = 0$, and $c(v_k) = 2$ otherwise. In that case, v_{n-1} is not colorful, contradicting the statement above that v_k is colorful for each even $k > 0$. Therefore, if $n \equiv 1 \pmod 4$, then $\underline{\text{LCS}}(C_n) = (n+3)/2$. □

The proof of the next result draws heavily on that of a similar statement for Latin squares occurring in [6].

Theorem 11.

$$\overline{\text{SCS}}(\text{SUD}_n) \leq n^4 - \Omega(n^{10/3}).$$

Proof. For each of the n^4 vertices $(a, b) \in [n^2] \times [n^2]$ of SUD_n, let $t_{(a,b)}$ denote a uniform random real number in the interval $[0, 1]$. Fix a proper vertex coloring $\phi : V(\text{SUD}_n) \to [n^2]$. We construct a determining set S by the following random process:

1. Let $j = 0$ and $S_0 = [n^2] \times [n^2]$, and start a timer at $t = 0$.
2. Wait until the next vertex $(a, b) \in V(\text{SUD}_n)$ is "born" at time $t_{(a,b)}$ or else $t = 1$, in which case go to (6).
3. If, for all $\gamma \in [n^2]$, there exists a $v \in V(\text{SUD}_n)$ so that $\phi(v) = \gamma$ and $v \in S_j$ and $\{v, (a, b)\} \in E(\text{SUD}_n)$, then $S_{j+1} = S_j \setminus \{(a, b)\}$.
4. $j \leftarrow j + 1$
5. If $t < 1$, go to (2).
6. Return S_{n^2}.

Note that the algorithm executes properly with probability 1, since almost surely $t_v \neq t_w$ for any $v, w \in V(\text{SUD}_n)$ with $v \neq w$. Also, it is easy to see that S_{n^2} is a determining set for ϕ.

Now, we compute $\mathbb{E}[|S_{n^2}|]$. When a vertex $v = (a, b)$ is born at time t to be considered for omission from S_j (to obtain S_{j+1}), the probability that any particular vertex w in its neighborhood is not contained in S_j is bounded above by the probability that it has been born, i.e., t. Fix $\gamma \in [n^2]$, $\gamma \neq \phi(v)$. There is exactly one vertex $c_v^\gamma = (\alpha, \beta) \in V(\text{SUD}_n)$ with $a = \alpha$ and $b \neq \beta$ and $\phi(c_v^\gamma) = \gamma$; there is exactly one vertex $r_v^\gamma = (\alpha, \beta) \in V(\text{SUD}_n)$ with $a \neq \alpha$ and $b = \beta$ and $\phi(r_v^\gamma) = \gamma$; there is exactly one vertex $b_v^\gamma = (\alpha, \beta) \in V(\text{SUD}_n)$ with $\lceil \alpha/n \rceil = \lceil a/n \rceil$ and $\lceil \beta/n \rceil = \lceil b/n \rceil$ and $\phi(b_v^\gamma) = \gamma$. Let $N_v^\gamma = \{c_v^\gamma, r_v^\gamma, b_v^\gamma\}$; $|N_v^\gamma| = 2$ for exactly $2n - 2$ colors γ and $|N_v^\gamma| = 3$ for exactly $n^2 - 2n + 1 = (n - 1)^2$ colors γ. For any particular γ, the probability that every element of N_v^γ has been removed from S_j is at most $1 - t^{|N_v^\gamma|}$. Therefore, the probability that, for every $\gamma \in [n^2] \setminus \{\phi(v)\}$, $S_j \cap N_v^\gamma \neq \emptyset$ is at least

$$(1 - t^2)^{2n-2}(1 - t^3)^{(n-1)^2},$$

and

$$
\begin{aligned}
\mathbb{E}[|S_{n^2}|] &\le n^4 \int_0^1 1-(1-t^2)^{2n-2}(1-t^3)^{(n-1)^2}\,dt \\
&= n^4 - n^4 \int_0^1 (1-t^2)^{2n-2}(1-t^3)^{(n-1)^2}\,dt \\
&\le n^4 - n^4 \int_0^{n^{-2/3}} (1-t^2)^{2n-2}(1-t^3)^{(n-1)^2}\,dt \\
&\le n^4 - n^{4-2/3}(1-n^{-4/3})^{2n-2}(1-n^{-2})^{(n-1)^2} \\
&= n^4 - n^{10/3}\exp\left(-O(n^{-4/3}n - n^{-2}n^2)\right) \\
&= n^4 - n^{10/3}\exp\left(1-O(n^{-1/3})\right) \\
&= n^4 - O(n^{10/3}),
\end{aligned}
$$

where the constant implicit in the 'O' is universal. Thus, for every optimal coloring ϕ of $\textsc{Sud}_n$, there exists a determining set of size at most $n^4 - O(n^{10/3})$, and therefore also a critical set of size at most $n^4 - O(n^{10/3})$. In particular, then,

$$\overline{\text{scs}}(\textsc{Sud}_n) \le n^4 - \Omega(n^{10/3}).$$

$\square$

We conclude this section with a table (Figure 1) of the four parameters and their values on all nonbipartite graphs on 5 vertices, obtained by computation using Sage ([10]). Every graph on at most 4 vertices is critically uniform, with parameters: 0 for K_1, $2K_1$, $3K_1$, and $4K_1$; 1 for K_2, P_3, $K_{1,3}$, P_4, and C_4; 2 for $K_1 \cup K_2$, K_3, $2K_2$, $K_1 \cup P_3$, $\overline{K_1 \cup P_3}$, and $\overline{2K_1 \cup K_2}$; and 3 for $2K_1 \cup K_2$, $K_1 \cup K_3$, and K_4. (The symbols $\overline{\cdot}$ and $\cup$ denote complementation and disjoint union, respectively.)

3. Complexity

Another important question in studying the graph parameters above is the determining of the difficulty of computing them. In fact, some aspects of this question have been considered in the recent literature. Indeed, under the rubric of "defining sets", the following result is almost immediate from Theorem 1 of Hatami-Tusserkani ([5]):

Theorem 12. *It is* NP*-hard to decide, for a given integer k and graph G, whether $\underline{\text{scs}}(G) \ge k$.*

While [5] remains unpublished, the authors of [4], which has been peer-reviewed, reproved – indeed, substantially strengthened – Theorem 1 of [5]. Theorem 2 of [5] implies our next result, but a proof has not appeared in a peer-reviewed publication to our knowledge, so we reproduce their proof (in our notation) here. To show that the "other" two parameters are indeed hard to compute, we follow the proofs in [5] closely and make a few necessary modifications; the arguments amount to reductions from

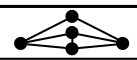
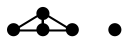
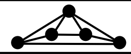
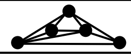
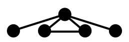
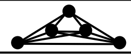
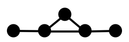
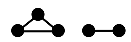
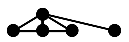
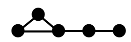
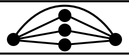
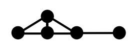
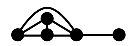
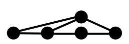
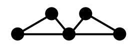
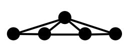
Graph	$\underline{\mathrm{SCS}}$	$\overline{\mathrm{SCS}}$	$\underline{\mathrm{LCS}}$	$\overline{\mathrm{LCS}}$	Graph	$\underline{\mathrm{SCS}}$	$\overline{\mathrm{SCS}}$	$\underline{\mathrm{LCS}}$	$\overline{\mathrm{LCS}}$
	4	4	4	4		2	2	3	3
	3	3	4	4		3	3	4	4
	3	3	3	3		2	2	2	2
	4	4	4	4		3	3	3	3
	3	3	3	4		4	4	4	4
	3	3	3	4		4	4	4	4
	2	2	3	3		3	3	3	4
	2	2	2	2		2	2	3	3
	3	3	4	4		3	3	4	4
	2	3	3	4		2	2	3	3
	2	2	2	2					

Figure 1: A table of the four parameters for all five vertex, nonbipartite graphs.

the well-known NP-hardness of vertex 3-colorability (see, for example, [1]). Because the constructions are somewhat complicated to relate in written form, we include Figure 2 for illustration.

Theorem 13. *It is* NP*-hard to decide, for a given integer k and graph G, whether $\overline{\mathrm{SCS}}(G) \geq k$.*

Proof. Suppose H is a graph with n vertices and m edges. We show that determining whether or not H is 3-colorable is reducible to deciding whether $\overline{\mathrm{SCS}}(G) \geq k$ for $k = (2m+2)\sum_{v \in V(H)} \binom{\deg(v)}{2} + 2$ and some graph G, G to be defined below (but depending in size only polynomially on n and m). We define a graph H' as follows: $V(H')$ consists of vertices V_1 of the form $x_{v,e}$, where $v \in V(H)$ and $v \in e \in E(H)$ and vertices V_2 of the form $y_{v,e,f,j}$ with $v \in V(H)$, $v \in e \in E(H)$, $v \in f \in E(H)$, $e \neq f$, and $j \in [2m+2]$; $E(H')$ consists of edges of the form $x_{v,vw}x_{w,vw}$ for each $v, w \in V(H)$, $v \neq w$ and edges of the form $x_{v,e}y_{v,e,f,j}$ for $v \in V(H)$, $v \in e \in E(H)$, $v \in f \in E(H)$, and $j \in [2m+2]$. Define $G = H' \cup K_3$, and denote the vertices of K_3 by $V_3 = \{z_1, z_2, z_3\}$. We claim that H is 3-colorable if and only if $\overline{\mathrm{SCS}}(G) \geq k$. Suppose H is 3-colorable; let c be such a coloring, and define, for $x \in V(G)$

$$c'(x) = \begin{cases} c(v) & \text{if } x = x_{v,e} \in V_1 \\ \xi & \text{if } x = x_{v,e,f,j} \in V_2 \text{ where } \xi = \min([3] \setminus \{c(v)\}) \\ j & \text{if } x = z_j. \end{cases}$$

Let S be a critical set for c'. Note that $V_2 \subset S$, because the neighbors of each $y_{v,e,f,j}$ receive only one color. Furthermore, S must contain at least two vertices from V_3, so

$$|S| \geq (2m+2) \sum_{v \in V(H)} \binom{\deg(v)}{2} + 2 = k.$$

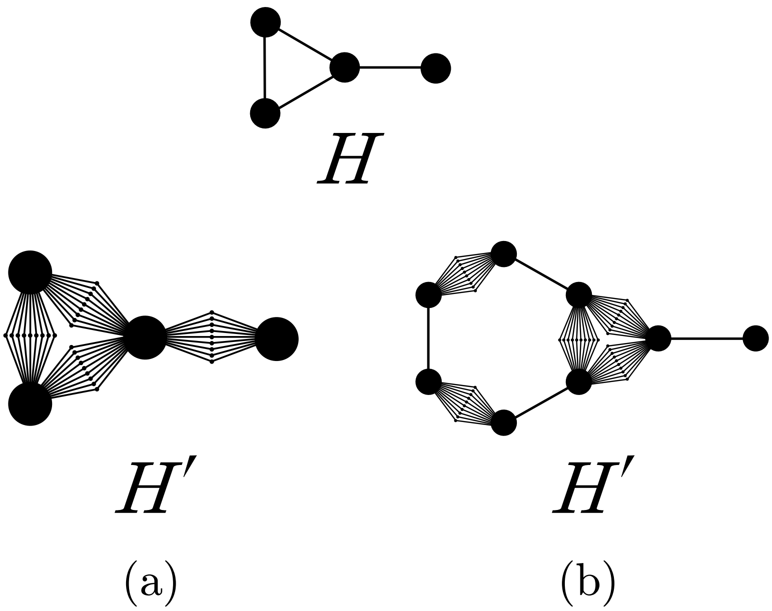


Figure 2: The constructions appearing in the proofs of Theorems 13, 14, and 15: graphs (b), (a), and (b), respectively.

Suppose H is not 3-colorable. Let c be a proper 3-coloring of G (which is clearly 3-colorable). Suppose there are two vertices $x_{w,e}$ and $x_{w,f}$, $e \neq f$, which receive distinct colors. Define S to be the set of all vertices of G except those of the form $y_{w,e,f,j}$; this is clearly a defining set, and

$$|S| = (2m+2) \sum_{v \in V(H)} \binom{\deg(v)}{2} + 2m + 3 - (2m+2) = k - 1.$$

Therefore, if the smallest critical set for c has size at least k, then $c(x_{v,e}) = c(x_{v,f})$ for every v, e, and f with $v \in e \cap f$; denote the color associated thusly with v by $c'(v)$. Then c' is a proper 3-coloring of H. $\square$

Theorem 14. *It is* NP*-hard to decide, for a given integer k and graph G, whether $\underline{\mathrm{LCS}}(G) \geq k$.*

Proof. Suppose H is a graph with n vertices and m edges. We show that determining whether or not H is 3-colorable is reducible to deciding whether $\underline{\mathrm{LCS}}(G) \geq k$ for $k = m + n + 3$ and some graph G, G to be defined below (but depending in size only polynomially on n and m). We construct a graph H' as follows: $V(H') = V_1 \cup V_2$, where $V_1 = V(H)$ and $V_2 = E(H) \times [m + n + 1]$, and $E(H')$ consists of all pairs of the form $\{v, (e, j)\}$ where $v \in V_1$, $(e, j) \in V_2$, and $v \in e$. Then let $G = H' \cup K_3$, and denote the set of vertices arising from the K_3 as $V_3 = \{x_1, x_2, x_3\}$. We claim that $\underline{\mathrm{LCS}}(G) \geq k$ if and only if H is not 3-colorable. If H is 3-colorable, then let $c : V(H) \to [3]$ be such a coloring, and define a 3-coloring of G by setting, for each

$v \in V(G)$,

$$c'(v) = \begin{cases} c(v) & \text{if } v \in V_1 \\ \xi & \text{if } v = (xy, j) \in V_2 \text{ where } \xi \notin \{c(x), c(y)\} \\ j & \text{if } v = x_j. \end{cases}$$

Then every critical set S for c' has cardinality $< k$, because

1. No two colorful vertices whose neighborhoods are identical are contained in the same critical set together. Therefore, if S contains a vertex $(e, j) \in V_2$, then it contains none of the vertices (e, j') with $j \neq j'$. This allows for at most m vertices from V_2 in S.
2. S will not contain all three vertices of V_3.

Therefore, $|S| \leq m + n + 2 < k$, and $\underline{\text{LCS}}(G) < k$.

In the other direction, suppose H is not 3-colorable. Therefore, every 3-coloring of H contains two adjacent vertices of the same color; in particular, any 3-coloring of G will have two vertices v and w from V_1 so that $vw \in E(H)$ to which it assigns the same color. One must therefore include every one of the $m + n + 1$ vertices (vw, j) in any critical set S. Additionally, S contains 2 elements of V_3, so $|S| \geq m + n + 1 + 2 = k$. □

Theorem 15. *It is* NP*-hard to decide, for a given integer k and graph G, whether $\overline{\text{LCS}}(G) \geq k$.*

Proof. Suppose H is a graph with n vertices and m edges. We show that determining whether or not H is 3-colorable is reducible to deciding whether $\overline{\text{LCS}}(G) \geq k$ for $k = (2m+2) \sum_{v \in V(H)} \binom{\deg(v)}{2} + 2$ and some graph G, G defined exactly as in Theorem 13. We claim that H is 3-colorable if and only if $\overline{\text{LCS}}(G) \geq k$. Suppose H is 3-colorable; let c be such a coloring, and define, for $x \in V(G)$

$$c'(x) = \begin{cases} c(v) & \text{if } x = x_{v,e} \in V_1 \\ \xi & \text{if } x = x_{v,e,f,j} \in V_2 \text{ where } \xi = \min([3] \setminus \{c(v)\}) \\ j & \text{if } x = z_j. \end{cases}$$

Let S be a critical set for c'. Note that $V_2 \subset S$, because the neighbors of each $y_{v,e,f,j}$ receive only one color. Furthermore, S must contain at least two vertices from V_3, so

$$|S| \geq (2m + 2) \sum_{v \in V(H)} \binom{\deg(v)}{2} + 2 = k.$$

Suppose H is not 3-colorable. Then, for every 3-coloring of G, there are two vertices $x_{v,e}$ and $x_{v,f}$, $e \neq f$, which receive distinct colors. Then S contains at most one of the vertices $y_{v,e,f,j}$ (since these vertices are colorful and have identical neighborhoods) and at most 2 vertices from V_3, so

$$|S| \leq (2m + 2) \sum_{v \in V(H)} \binom{\deg(v)}{2} + 2m + 2 - (2m + 1) = k - 1.$$

Therefore, $\overline{\text{LCS}}(G) < k$. □

4. Conclusion and Questions

We conclude with several open questions we find interesting, some of which have appeared in other guises in the literature. Of course, a central problem is to find a human-readable proof of the (seeming) fact that $\underline{\text{SCS}}(\text{SUD}_3) = 17$, as well as an understanding of the growth rate of $\underline{\text{SCS}}(\text{SUD}_k)$.

1. Is the converse of Proposition 1 true? That is, if every critical set of a graph G has cardinality $\chi(G) - 1$, must G be uniquely colorable? We have verified computationally that this statement is indeed true for every graph on at most 8 vertices.
2. What is the list-chromatic number of SUD_3? This is the smallest k so that, if every vertex of SUD_3 is associated with a list of k distinct colors, there will be an assignment of colors to vertices from their associated lists which is a proper coloring. Clearly, $\text{ch}(\text{SUD}_3) \geq 9$, but is it equal to 9?
3. One can define parametrized versions of the four functions $\underline{\text{SCS}}$, $\overline{\text{SCS}}$, $\underline{\text{LCS}}$, $\overline{\text{LCS}}$, by asking for the smallest/largest over all proper k-colorings – k not necessarily equal to the chromatic number – of the size of the smallest/largest critical set for that coloring. Are these parameters all monotone nondecreasing in k?
4. Is there a simple characterization of critically uniform graphs? By doing an extensive computation, we have verified that there are many graphs with this property which are *not* uniquely colorable.
5. What are $\underline{\text{SCS}}(G)$, $\overline{\text{SCS}}(G)$, $\underline{\text{LCS}}(G)$, and $\overline{\text{LCS}}(G)$ for $G = K_n \square K_n$, i.e., Latin squares? A superlinear lower bound on $\underline{\text{SCS}}(K_n \square K_n)$ was obtained only relatively recently ([2]), for example.

5. Acknowledgements

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