Greedy Galois Games

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August 26, 2013

Abstract

We show that two duelers with similar, lousy shooting skills (a.k.a. *Galois duelers*) will choose to take turns firing in accordance with the famous Thue-Morse sequence if they greedily demand their chances to fire as soon as the other's *a priori* probability of winning exceeds their own. This contrasts with a result from the approximation theory of complex functions that says what more patient duelers would do, if they *really* cared about being as fair as possible. We note a consequent interpretation of the Thue-Morse sequence in terms of certain expansions in fractional bases close to, but greater than, 1.

Two players, Alice and Bob, are in a duel. They take turns firing at each other. However, both are *Galois*[1] duelers, i.e., terrible shots, and equally so. On the other hand, they are deeply committed to fairness, and therefore they make the following deal. Before a single firearm is discharged, they draw up a firing sequence, i.e., the sequence of turns they will take, according to the following "greedy" rules. Alice shoots first. Bob then shoots as many times as he needs to obtain a probability of winning that exceeds the probability that Alice has won so far. Then Alice shoots again, until her *a priori* probability of having won exceeds Bob's. Bob shoots next following the same rule, and so on until someone finally shuffles off his/her mortal coil.

To illustrate, suppose the duelers' hitting probability is 1/3. Alice shoots first, so her probability of winning by the end of round 0 is 1/3. Bob's probability of winning so far is zero, so he shoots next. For Bob to win in round 1, Alice has to have missed in round 0, and Bob has to hit. Therefore, Bob's probability of having won by the end of round 1 is $(2/3)(1/3) = 2/9$. This is still less than 1/3, so Bob shoots again in round 2. For Bob to win in round 2, he must survive Alice's initial shot, miss in round 1, and hit in round 2. Hence his probability of

[*]This work was funded in part by NSF grant DMS-1001370.

[1]Famously, the prodigal algebraist and Republican Radical Évariste Galois lost a duel over a lover on May 30, 1832, dying the next day.

winning by the end of round 2 is $(1/3)(2/3)+(1/3)(2/3)^2 = 10/27$. This is more than Alice's probability of $1/3$ $(= 9/27)$, so Alice gets to go next. In round three, Alice adds $(1/3)(2/3)^3$ to her probability of winning, since $1/3$ is the probability she succeeds in round 3, and $(2/3)^3$ is the probability that everyone missed in the previous 3 rounds. If we define $S_{n,X} = \{i \le n \mid \text{player } X \text{ shoots in round } i\}$, then the probability of player $X = A(\text{lice})$ or $B(\text{ob})$ winning by the end of round n is given by

$$\frac{1}{3}\sum_{i\in S_{n,X}}\left(\frac{2}{3}\right)^i.$$

The following is a table for $p = 1/3$, showing the probability of success for each player as well as the sequence of shooters.

Round	$\mathbb{P}(A)$	$\mathbb{P}(B)$	Shooter
0	1/3	0	A
1	1/3	2/9	B
2	1/3	10/27	B
3	35/81	10/27	A
4	35/81	106/243	B
5	347/729	106/243	A
6	347/729	1018/2187	B
7	347/729	3182/6561	B
8	9625/19683	3182/6561	A
9	9625/19683	29150/59049	B
10	87649/177147	29150/59049	A

For arbitrary probability p, we can determine the sequence $\{a_i\}_{i=0}^n$ of players inductively. Let $q = 1-p$, let $a_n = 1$ mean that Alice shoots in round n, and $a_n = -1$ means that Bob shoots in round n. Let A_n be the event that Alice wins by round n, and define B_n similarly. Since Alice shoots first, $a_0 = 1$. Write

$$f_n(q) = a_n\left(\sum_{j=0}^n a_j q^j\right).$$

Then

$$\begin{aligned} f_n(q) &= a_n\left(\sum_{j=0}^n a_j q^j\right) \\ &= \frac{a_n}{p}\left(p\sum_{i\in S_{n,A}} q^i - p\sum_{i\in S_{n,B}} q^i\right) \\ &= \frac{a_n}{p}\cdot\left(\mathbb{P}(A_n) - \mathbb{P}(B_n)\right) \qquad (1) \end{aligned}$$

Since a_n is negative whenever Bob is the shooter, we see that (up to the positive factor $1/p$) the polynomial $f_n(q)$ records the current player's probability of success minus the opposing player's probability of success. Therefore, the next player is completely determined by the value of $f_n(q)$. Specifically,

$$a_{n+1} = \begin{cases} -a_n & \text{if } f_n(q) > 0 \\ a_n & \text{otherwise.} \end{cases} \quad (2)$$

It is easy to see that regardless of the value of p, the first three terms of the sequence $\{a_i\}$ are $1, -1, -1$. To determine the fourth term, we consider $f_2(q) = q^2 + q - 1$. The unique positive root of this polynomial is $\frac{-1+\sqrt{5}}{2} = \frac{1}{\phi} \approx 0.618$ where ϕ is the Golden ratio. Since $f_2(q)$ is increasing after this, we have that for any $q \geq 1/\phi$, the fourth term of the sequence is $a_3 = 1$.

The above is a special case of the following.

Proposition 1.1. For each $n \in \mathbb{N}$, there is an $\epsilon > 0$ so that the sequence $\{a_i\}_{i=0}^n$ is the same for all $q \in (1-\epsilon, 1)$.

Proof. We proceed by induction, noting that the base case is trivial. Assume by induction that for all $q \in (1-\epsilon_0, 1)$, the sequence $\{a_i\}_{i=0}^n$ is the same. Recall that a_{n+1} is determined by the sign of $f_n(q)$, which is now a fixed polynomial, since the coefficients are exactly the a_i. Since f_n has degree n, it has at most n roots. Thus we can find $\epsilon_1 > 0$ so that none of the roots occur in $(1-\epsilon_1, 1)$. Setting $\epsilon = \min\{\epsilon_0, \epsilon_1\}$, we have that $f_n(q)$ does not change sign or become zero for $q \in (1-\epsilon, 1)$. Therefore, a_{n+1} does not depend on q inside this interval, completing the induction, and proving the proposition. □

One could continue along the lines above, and for each n, attempt to find the threshold value of q so that the first n terms of the sequence stabilize. Indeed, the authors have done this for some small values of n, although none of the threshold values other than $1/\phi$ appear to be numbers of independent interest. Computer experimentation reveals that some of the $f_n(q)$ have no roots inside $[0, 1]$, some have a single root, and some have a root at $q = 1$ and for some other $q \in (0, 1)$. For those $f_n(q)$ with a single root, and with this root lying in $(0, 1)$, let α_n denote the root. The following table shows the first thirty values of n and α_n.

n	α_n	n	α_n	n	α_n
2	0.61803	42	0.88482	82	0.92119
4	0.66099	44	0.88631	84	0.92176
8	0.73564	50	0.89543	88	0.92383
14	0.80016	52	0.89660	94	0.92724
16	0.80650	56	0.90071	98	0.92910
22	0.83787	62	0.90720	100	0.92954
26	0.85202	64	0.90805	104	0.93115
28	0.85493	70	0.91341	110	0.93382
32	0.86435	74	0.91629	112	0.93418
38	0.87798	76	0.91695	118	0.93657

Based on our computational data, we are willing to conjecture that $(1 - \alpha_n)$ behaves essentially like $n^{-1/2}$. More formally, we conjecture the following.

Conjecture 1.2. The sequence α_n is increasing, and furthermore,

$$\lim_{n\to\infty} \frac{\log(1-\alpha_n)}{\log n} = -\frac{1}{2}.$$

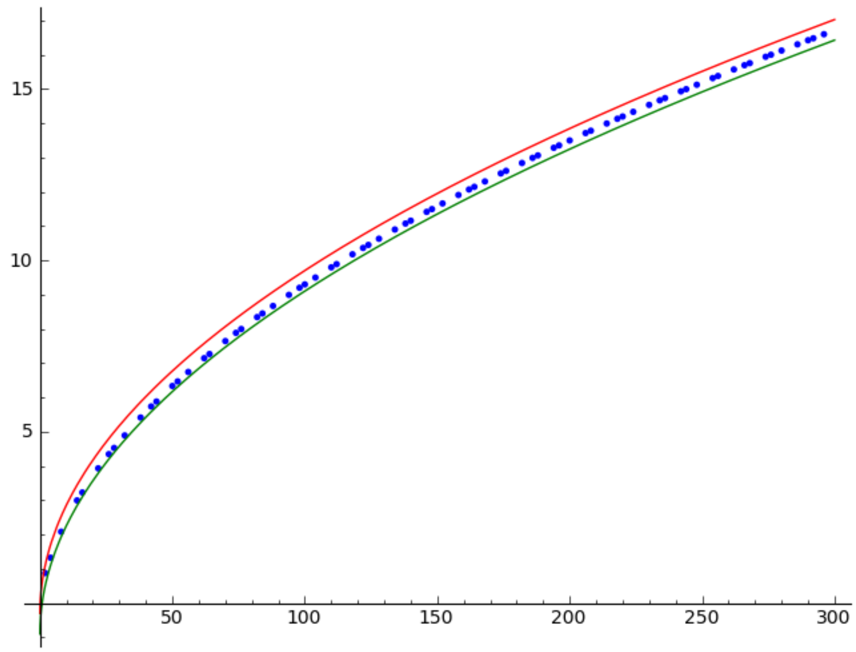

Figure 1: Support for Conjecture 1.2. Blue dots indicate points of the form $(n, n^{1/2}(1-\alpha_n))$. The red curve is $x^{1/2} - .3$, and the green curve is $x^{1/2} - .9$.

2 Convergence of the Firing Sequence

While these thresholds are interesting, we are concerned here with a different question. As q tends to 1 (i.e., $p \to 0$), what does the sequence of players tend to? A quick calculation with $q = 0.9$ reveals the following 21 turns in the sequence of players.

$$ABBABAABBAABABBABAABB.$$

At first glance, this appears to be the same as the famous (Prouhet-)Thue-Morse(-Euwe)[2] sequence, one definition of which is the sequence of parities of the number of 1's in the binary expansions of n, $n = 1, 2, \ldots$. In fact, the

[2]Prouhet used this sequence in 1851 to solve what is now known as the Prouhet-Tarry-Escott problem, although he did not make the sequence explicit. Thue introduced it in 1906 to devise cube-free words, and Morse applied it to differential geometry in 1921. Euwe, not

sequence above differs only in the last position. This disagreement can be fixed by raising the value of q very slightly (setting $q = 0.902$ is sufficient). That our sequence bears such close resemblance to the Thue-Morse sequence is no coincidence, as evidenced by the following.

Theorem 2.1. The sequence $\{a_i\}_{i=0}^{\infty}$ tends to the Thue-Morse sequence (on the alphabet $\{1, -1\}$) as $q \to 1^-$.

Our proof will use the following well-known facts about the Thue-Morse sequence, which can found, for example, in [3].

Proposition 2.2. The Thue-Morse sequence $\{t_i\}_{i=0}^{\infty}$ on alphabet $\{1, -1\}$ is defined by the following recurrences.

$$\begin{aligned} t_0 &= 1 \\ t_{2i} &= t_i \\ t_{2i+1} &= (-1)t_{2i}. \end{aligned}$$

Proposition 2.3. The sequence $\{(t_{2i}, t_{2i+1})\}_{i=0}^{\infty}$ is the Thue-Morse sequence on alphabet $\{(1, -1), (-1, 1)\}$.

We note a simple consequence of Proposition 2.3 which we will also use.

Corollary 2.4. For any $n \in \mathbb{N}$, we have $\sum_{i=0}^{2n+1} t_i = 0$.

Proof of Theorem 2.1. In light of Proposition 1.1, q can be taken arbitrarily close to 1. We proceed by induction. We have already shown that $a_i = t_i$ for $i = 0, 1, 2, 3$, so the base cases hold. We assume $n > 2$, and by induction that the two sequences agree for all $i \leq n$.

Case 1: $n = 2m$ is even. Consider $g(q) = \sum_{i=0}^{n-1} a_i q^i$. Since the a_i are the Thue-Morse sequence, Corollary 2.4 implies that $g(1) = 0$. Since q can be taken arbitrarily close to 1 and g is continuous, we can ensure $-1/2 < g(q) < 1/2$ for all q under consideration. We may also assume that $q > (1/2)^{1/n}$. Then note that $f_n(q) = q^n \pm g(q)$, so that for all of our q,

$$f_n(q) = q^n \pm g(q) > 1/2 - 1/2 \geq 0.$$

Thus (2) gives that $a_{n+1} = (-1)a_n$. Since n is even, induction and the recurrence for Thue-Morse give that

$$a_{n+1} = (-1)a_n = (-1)t_n = t_{n+1}.$$

Case 2: $n = 2m+1$ is odd. Since n is odd, Corollary 2.4 implies that $f_n(1) = 0$. Hence we can write

$$f_n(q) = (q - 1)g(q)$$

knowing about these previous works, used the sequence in 1929 to show the existence of infinitely long chess games, despite the rule designed to prevent this: any three-fold repetition of a sequence of moves ends the game in a draw. The reader is directed to the delightful survey [1] for more of this sequence's interesting history.

for some monic degree $2m$ polynomial g.

We claim that $g(q) = f_m(q^2)$. We know by induction that sequence $\{a_i\}$ matches Thue-Morse up to n, whence $a_{2i+1} = (-1)a_{2i}$ and $a_{2i} = a_i$ for all of the coefficients in our polynomial. We can write

$$\begin{aligned} f_n(q) &= a_{2m+1} \sum_{i=0}^{2m+1} a_i q^i \\ &= a_{2m+1} \sum_{i=0}^{m} \left(a_{2i} q^{2i} + a_{2i+1} q^{2i+1} \right) \\ &= (-1) a_{2m} \sum_{i=0}^{m} \left(a_{2i} q^{2i} - a_{2i} q^{2i+1} \right) \\ &= (-1) a_{2m} (1-q) \sum_{i=0}^{m} a_{2i} q^{2i} \\ &= (q-1) a_m \sum_{i=0}^{m} a_i (q^2)^i \\ &= (q-1) f_m(q^2), \end{aligned}$$

proving the claim.

As q can be taken arbitrarily close to 1, we can assume that q and q^2 are past the threshold for stabilizing all coefficients up to a_{n+1}. Note also that $(q-1)$ is negative, so one of $f_n(q)$ and $f_m(q^2)$ is positive, and the other is negative. Therefore, (2) says that for some $j \in \{0, 1\}$, $a_{n+1} = (-1)^j a_n$ and $a_{m+1} = (-1)^{j+1} a_m$. Then since we know the Thue-Morse relations hold up to n, we have

$$a_{n+1} = (-1)^j a_n = (-1)^j a_{2m+1} = (-1)^{j+1} a_{2m} = (-1)^{j+1} a_m = a_{m+1}.$$

By the inductive hypothesis and Proposition 2.2, $a_{m+1} = t_{m+1} = t_{2m+2} = t_{n+1}$, completing the proof. $\square$

3 Approximation and β-Expansions

Not all of this gun violence is fun and games. Indeed, it bears on some serious business in the approximation theory of complex functions. Specifically, Güntürk ([4]) recounts the following question of S. Konyagin[3]:

> There are two duelists A and B who will shoot at each other ... using a given ± 1 sequence $\mathbf{b} = (b_i)_{i \geq 0}$ which specifies whose turn it is to shoot at time i. The shots are independent and identically distributed random variables with outcomes hit or miss. Each shot

[3]Some of the notation has been slightly modified to align the text more closely with our own exposition.

hits (and therefore kills) its target with small unknown probability ϵ, which is arbitrary but fixed throughout the duel. The "fair duel" problem is to find an ordering $\mathbf{b}$, which is independent of ϵ, and is as fair as possible in the sense that the probability of survival for each is as close to 1/2 as possible. We measure the fairness of an ordering $\mathbf{b}$ by its bias function $D_{\mathbf{b}}(\epsilon)$, defined to be

$$D_{\mathbf{b}}(\epsilon) = \mathbb{P}(A \text{ survives}) - \mathbb{P}(B \text{ survives}),$$

and ask that $D_{\mathbf{b}}(\epsilon) \to 0$ as $\epsilon \to 0$ as fast as possible.

As illustrated by the discussion above, this question amounts to finding a sequence $\mathbf{b} = \{b_i\}_{i\geq 0}$ of ± 1's so that the power series $\sum_{i=0}^{\infty} b_i z^i$ closely approximates the zero function in the vicinity of $z = 1$. Indeed, by interpreting (1) for the shooting sequence $\mathbf{b}$, setting $z = q = 1 - p$ yields

$$\begin{aligned} p\sum_{i=0}^{\infty} b_i q^i &= p \lim_{n\to\infty} b_n f_n(q) \\ &= \lim_{n\to\infty} [\mathbb{P}(A_n) - \mathbb{P}(B_n)] \\ &= \lim_{n\to\infty} [\mathbb{P}(\text{Bob dies by round } n) - \mathbb{P}(\text{Alice dies by round } n)] \\ &= \lim_{n\to\infty} [(1 - \mathbb{P}(\text{Bob survives through round } n)) \\ &\qquad\qquad -(1 - \mathbb{P}(\text{Alice survives through round } n)] \\ &= \lim_{n\to\infty} [\mathbb{P}(\text{Alice survives through round } n) \\ &\qquad\qquad -\mathbb{P}(\text{Bob survives through round } n)] \\ &= \mathbb{P}(\text{Alice survives forever}) - \mathbb{P}(\text{Bob survives forever}) \\ &= D_{\mathbf{b}}(p). \end{aligned} \tag{3}$$

Güntürk showed ([4]) that, in a sense, there is an even "fairer" sequence than the Thue-Morse sequence, if only the shooters were not so greedy. Indeed, his much more general result is the following, which says that, near (but to the 'left' of) $z = 1$ in the complex plane, it is possible to approximate surprisingly well *any* analytic function whose Taylor coefficients lie in a small (real) interval around zero.

Theorem 3.1. Let $0 \leq \mu < 1 \leq M < \infty$ be arbitrary and $\mathcal{R}_M = \{z \in \mathbb{C} : |1 - z| \leq M(1 - |z|)\}$. There exist constants $C_1 = C_1(\mu, M) > 0$ and $C_2 = C_2(\mu, M) > 0$ such that, for any power series

$$f(z) = \sum_{n=0}^{\infty} r_n z^n, \qquad r_n \in [-\mu, \mu], \ \forall n,$$

there exists a power series with ± 1 coefficients, i.e.,

$$Q(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad b_n \in \{-1, +1\}, \ \forall n,$$

which satisfies

$$|f(z) - Q(z)| < C_1 e^{-C_2/|1-z|}$$

for all $z \in \mathcal{R}_m \setminus \{1\}$.

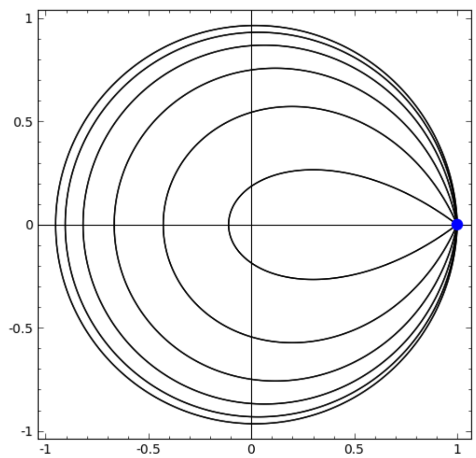

Figure 2: Graph showing the boundary of $\mathcal{R}_M$ from Theorem 3.1 in the complex plane, for $M = 5/4$, $5/2$, 5, 10, 20, and 40 in order from inner-most to outer-most. The point $z = 1$ is labeled with a blue point.

Furthermore, this result is the best possible in a certain precise sense by a theorem of Borwein-Erdélyi-Kós ([2]). Setting $\mu = 0$ (so that $r_n = 0$ for all n and $f(z) \equiv 0$) and $M = 2$ (for example), we obtain the corollary that one can approximate the constant 0 function (of p) within $\exp(-c/p)$ by a power series with coefficients in $\{1, -1\}$. The author goes on to show that the Thue-Morse sequence only obtains an approximation of $\exp(-c(\log p)^2)$. It pays to have patience!

By (3), $p\sum_{n=0}^{\infty} b_n q^n = D_{\mathbf{b}}(p)$. Suppose that instead of simply approaching zero as $p \to 0$, we required the right-hand side of this equation to be *exactly* zero. Letting $S \subseteq \mathbb{N}$ be the set of $i \in \mathbb{N}$ for which $b_i = 1$ and T be the set of $i \in \mathbb{N}$ for which $b_i = -1$, we would have

$$p\sum_{i \in S} q^i = p\sum_{i \in T} q^i.$$

The following lemma shows that our greedy dueling sequence has this property, provided the hitting probability is at most 50%.

Lemma 3.2. For any probability $p \leq 1/2$, the sequence of coefficients $\{a_i\}_{i=0}^{\infty}$ obtained from the greedy Galois duel with hitting probability p satisfies

$$\sum_{i=0}^{\infty} a_i q^i = 0.$$

Proof. Let $p \leq 1/2$, and so $q \geq p$. We note that

$$\begin{aligned}
\lim_{n\to\infty} (\mathbb{P}(A_n) + \mathbb{P}(B_n)) &= \lim_{n\to\infty} \left(p \sum_{i \in S_{n,A}} q^i + p \sum_{i \in S_{n,B}} q^i \right) \\
&= p \sum_{i=0}^{\infty} q^i \\
&= 1.
\end{aligned}$$

Both $\mathbb{P}(A_n)$ and $\mathbb{P}(B_n)$ are monotone nondecreasing with n, and their sum tends to 1. So to prove the lemma, it suffices to prove that neither player's probability exceeds $1/2$ at any round.

Suppose toward a contradiction that some player's probability first exceeds $1/2$ in round n. The argument will be symmetric with respect to the player, so without loss of generality we assume this player is Alice. Write $\mathbb{P}(A_{n-1}) = a$, and $\mathbb{P}(B_{n-1}) = b$. Since Alice has to shoot in round n to have increased her probability, the dueling rules imply $b \geq a$. Then by assumption, $\mathbb{P}(A_n) = a + pq^n > 1/2$. Bob's probability can never exceed this, as the sum of the two probabilities is 1. Hence Bob will shoot in round $n+1$ and every round afterward.

This gives that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(B_n) &= b + p \sum_{i=n+1}^{\infty} q^i \\
&= b + p \frac{q^{n+1}}{1-q} \\
&= b + q^{n+1} \\
&\geq a + pq^n,
\end{aligned}$$

where the inequality follows from $q \geq p$ and $b \geq a$. But then

$$1 = \lim_{n\to\infty} (\mathbb{P}(A_n) + \mathbb{P}(B_n)) \geq 2(a + pq^n) > 1.$$

This provides a contradiction, and proves the lemma. $\square$

Let $\{d_i\}_{i\geq 0}$ be the indicator function for Alice shooting in round n in the greedy Galois duel with probability $p \leq 1/2$. (That is, $d_i = 1$ if Alice shoots in round i, and $d_i = 0$ if Bob shoots.) Let $\overline{d}_i$ denote boolean complement, so that $\{\overline{d}_i\}_{i\geq 0}$ is the indicator for Bob shooting. The lemma says that

$$\sum_{i=0}^{\infty} d_i q^i = \sum_{i=0}^{\infty} \overline{d}_i q^i. \tag{4}$$

Since $\sum_{i=0}^{\infty} d_i q^i + \sum_{i=0}^{\infty} \overline{d}_i q^i = \sum_{i=0}^{\infty} q^i = 1/(1-q) = p^{-1}$, we can conclude that both sides of equation (4) equal $(2p)^{-1}$.

For example, if $p = 1/5$, then $q = 4/5$ and $(2p)^{-1} = 5/2$. The above discussion implies that

$$\sum_{i=0}^{\infty} d_i \left(\frac{4}{5}\right)^i = \frac{5}{2}.$$

In other words, the sequence $\mathbf{d} = \{d_i\}_{i\geq 0}$ is the positional numeral system expansion of the number 5/2 in "base" 5/4, using only the digits 0 and 1. The same is true for the sequence $\bar{\mathbf{d}} = \{\bar{d}_i\}_{i\geq 0}$. To be more precise: An *expansion of* $x \in \mathbb{R}^+$ *in the base* $\beta \geq 1$ is any right-infinite string of the form

$$c_n c_{n-1} \ldots c_1 c_0 . c_{-1} c_{-2} \ldots,$$

where n is some nonnegative integer, for each $k \leq n$, $c_n \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$, and

$$x = \sum_{k=0}^{\infty} c_{n-k} \beta^{n-k}.$$

If $c_j = 0$ for all $j < -N$, one may write

$$c_n c_{n-1} \ldots c_1 c_0 . c_{-1} c_{-2} \ldots c_{-N}$$

for short. Such expansions were introduced by Rényi under the name "β-expansions"[4] ([7]). In light of the above argument, for any $p \leq 1/2$, let $\mathbf{c} = (c_i)_{i\geq 0}$ be the digits of any expansion of $(2p)^{-1}$ in the base $1/q$, and let $\mathbf{a} = (a_i)_{i\geq 0}$ be defined as $a_i = (-1)^{c_i}$. Then the generating function $g(z)$ of the sequence $\mathbf{a}$ satisfies $g(q) = 0$. Of course, the sequence $\mathbf{a}$ depends on q, so this expansion does not answer Konyagin's original "Fair Duel" problem. However, given this connection with nonstandard radix systems, we may reinterpret Theorem 2.1 as follows.

Corollary 3.3. For each $k \geq 1$ there is an $N \geq 1$ so that the first k digits of the Thue-Morse sequence (or its complement), expressed over the alphabet $\{0, 1\}$, agrees with an expansion of $n/2$ in base $\beta = n/(n-1)$, for any $n \geq N$.

Proof. Proposition 1.1 and Theorem 2.1 together say that we can find $N \geq 0$ so that the indicator sequence $\{\bar{d}_i\}_{i\geq 0}$ for Bob shooting in the greedy Galois duel with $p = 1/n$ agrees with the Thue-Morse sequence over $\{0, 1\}$ for the first k digits, provided $n \geq N$. Set $q = (n-1)/n$, and $\beta = 1/q = n/(n-1)$. The discussion following Lemma 3.2 implies that

$$\sum_{i=0}^{-\infty} \bar{d}_i \beta^i = \sum_{i=0}^{\infty} \bar{d}_i q^i = (2p)^{-1} = n/2,$$

and that the same holds for Alice's indicator sequence $\{d_n\}$. □

[4] Actually, Rényi referred to them as "f-expansions".

Rényi referred to a greedily-constructed β-expansion as "the" β-expansion. Specifically, one constructs the digit sequence $\{c_j\}_{j=-\infty}^{n}$ of x inductively as follows: $n = \lfloor \log_\beta x \rfloor$, and, given an integer $m \leq n$,

$$c_m = \lfloor \beta^m x - \sum_{j=m+1}^{n} c_j \beta^{j+m} \rfloor.$$

Presently, we will refer to a greedily-constructed β-expansion that starts with digits $c_j = 0$ for $j > 0$ as the *infrapotent*[5] β-expansion[6]. It is not difficult to see now that the infrapotent $\beta = 1 + 1/(n-1)$ expansion of $n/2$ is exactly Alice and Bob's firing sequence, where Alice is associated with 1 and Bob with 0. So, in fact, Corollary 3.3 can be strengthened slightly by changing "an expansion" to "the infrapotent expansion." For example, applying the $p = 1/3$ situation in the introduction, the infrapotent (3/2)-expansion of 3/2 is

$$1.0010100101\ldots$$

This reinterpretation of the greedy firing sequence nicely illustrates the fact that β-expansions need not be unique if $\beta \notin \mathbb{N}$. By Corollary 3.3, switching 1 with 0 gives another representation of 3/2 in base 3/2. Furthermore, the string "10." is itself a (3/2)-representation of 3/2 (though not an infrapotent one)!

4 Conclusion

Our results add a new interpretation of the Thue-Morse sequence to the ever-growing collection of known characterizations, many of which appear in [1] and entry A010060 of the Online Encyclopedia of Integer Sequences ([6]). They also reveal a connection between greedily played stochastic games, approximation theory, and nonstandard radix representations. As a starting point for generalizations, we conclude with three questions.

1. What happens in a Galois *truel*, i.e., a three-way duel between equally terrible shots who are nonetheless fair-minded, optimally strategic, and cannot deliberately miss? It is not immediately clear what the fairest policy for turn-taking should be.

2. How does the game change if Alice and Bob make it fairer by imposing 'less greedy' demands on the turn sequence? For example, they might remember only the last 87 rounds, or they may allow their opponent's *a priori* probability of success to exceed their own by up to 1% before demanding a turn.

3. What is the connection of the present discussion with the following result of Komornik and Loreti ([5]), which is strikingly redolent of multiple topics discussed above?

[5] From L. *infra* ("below, underneath, beneath") + *potentia* ("power").

[6] Of course, such an expansion may not exist. In particular, it is possible to write x thusly if and only if $|x| \leq \lfloor \beta \rfloor \beta / (\beta - 1)$.

Theorem 4.1. There is a smallest number $1 < q < 2$ for which there is precisely one choice of digit sequence $\mathbf{d} = \{d_j\}_{j=1}^{\infty}$ so that $1 = \sum_{j=1}^{\infty} d_j q^{-j}$. This q ($= 1.787231650\ldots$) is the unique positive solution to the equation

$$1 = \sum_{j=0}^{\infty} a_j q^{-j},$$

where a_j is the j^{th} digit of the Thue-Morse sequence over $\{0, 1\}$, i.e., a_j is the parity of the number of 1's in the base 2 representation of j.

5 Acknowledgments

The authors wish to thank the referees for their careful reading and helpful comments, S. Güntürk for introducing the authors to the question of S. Konyagin, and the international collaboration that has produced the tremendously useful free and open source mathematical software suite Sage ([8]), which we used extensively for numerical experimentation.

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