A Harary-Sachs Theorem for Hypergraphs

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Abstract

We generalize the Harary-Sachs theorem to k-uniform hypergraphs: the codegree-d coefficient of the characteristic polynomial of a uniform hypergraph $\mathcal{H}$ can be expressed as a weighted sum of subgraph counts over certain multi-hypergraphs with d edges. This includes a detailed description of said multi-hypergraphs and a formula for their corresponding weights, which we use to describe the low codegree coefficients of the characteristic polynomial of a 3-uniform hypergraph. We further provide explicit and asymptotic formulas for the contribution from k-uniform simplices and conclude by showing that the Harary-Sachs theorem for graphs is indeed a special case of our main theorem.

1 Introduction

An early, seminal result in spectral graph theory of Harary [7] (and later, more explicitly, Sachs [11]) showed how to express the coefficients of a graph's characteristic polynomial as a certain weighted sum of the counts of various subgraphs of G (a thorough treatment of the subject is given in [1], Chapter 7).

Theorem 1 *([7],[11]) Let G be a labeled simple graph on n vertices. If H_i denotes the collection of i-vertex graphs whose components are edges or cycles, and c_i denotes the coefficient of λ^{n-i} in the characteristic polynomial of G, then*

$$c_i = \sum_{H \in H_i} (-1)^{c(H)} 2^{z(H)} [\# H \subseteq G]$$

where $c(H)$ is the number of components of H, $z(H)$ is the number of components which are cycles, and $[\# H \subseteq G]$ denotes the number of (labeled) subgraphs of G which are isomorphic to H.

The goal of the present paper is to provide an analogous result for the characteristic polynomial of a hypergraph. The full result is given in Theorem 14, but to state here simply: fix $k \geq 2$ and let H_d denote the set of k-valent (i.e., k divides the degree of each vertex) k-uniform multi-hypergraphs on d edges. For a k-uniform hypergraph $\mathcal{H}$ the codegree-d coefficient (i.e., the coefficient of $x^{\deg -d}$) of the characteristic polynomial of the n-vertex hypergraph $\mathcal{H}$ can be written

$$c_d = \sum_{H \in H_d} (-(k-1)^n)^{c(H)} C_H (\#H \subseteq \mathcal{H})$$

where $c(H)$ is the number of components of H, C_H is a constant depending only on H, and $(\#H \subseteq \mathcal{H})$ is the number of times H occurs (in a certain sense that is a minor generalization of the subgraph relation) in $\mathcal{H}$.

The quantity $(\#H \subseteq \mathcal{H})$ is straightforward to compute. However, computing C_H is more complicated. This notion of an *associated coefficient of a hypergraph* first appeared in [4], where the authors provide a combinatorial description of the codegree k and codegree $k+1$ coefficient, denoted c_k and c_{k+1} respectively, for the normalized adjacency characteristic polynomial of a k-uniform hypergraph.

Theorem 2 *[4] Let $\mathcal{H}$ be a k-uniform hypergraph. Then*

$$c_k = -k^{k-2}(k-1)^{n-k}|E(\mathcal{H})|$$

and

$$c_{k+1} = -C_k(k-1)^{n-k}(\#\textit{ of simplices in } \mathcal{H}),$$

where C_k is some constant depending on k.

This idea was further studied by Shao, Qi, and Hu where the authors prove (restating Theorem 4.1 of [12]),

$$c_d = (k-1)^{n-1} \sum_{D \in \mathbf{D}} f_D |\mathfrak{E}(D)|$$

where $\mathbf{D}$ is a certain large family of digraphs, f_D is a function of D and $\mathfrak{E}(D)$ is the set of Euler circuits in D. The authors then use their formula to provide a general description of $\mathrm{Tr}_2(T)$ and $\mathrm{Tr}_3(T)$ for a general tensor T. Our first few results are similar to that of [12] (as described in more detail below), and we use them to provide an explicit combinatorial description of H_D and the resulting C_H which yields a Harary-Sachs type formula for hypergraphs which is amenable to computation. We also answer a question of [4] by providing a computationally efficient formula for C_H in the case where H is the k-uniform simplex, i.e., $K_{k+1}^{(k)}$. We conclude by answering a question of [8] by applying our explicit formula for higher order traces to the computation of the characteristic polynomial of a hypergraph.

Here we present some requisite background maintaining the notation of [4]. A *(cubical) hypermatrix* $\mathcal{A}$ over a set $\mathbb{S}$ of *dimension* n and *order* k is a collection of n^k elements $a_{i_1 i_2 \ldots i_k} \in \mathbb{S}$ where $i_j \in [n]$. A hypermatrix is said to be *symmetric* if entries with identical multisets of indices are the same. That is, $\mathcal{A}$ is symmetric if

$a_{i_1 i_2 \ldots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}}$ for all permutations σ of $[k]$. An order k dimension n symmetric hypermatrix $\mathcal{A}$ uniquely defines a homogeneous degree k polynomial in n variables (a.k.a. a "k-form") by

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

If we write $\mathbf{x}^{\otimes r}$ for the order r dimension n hypermatrix with $i_1, i_2, \ldots, i_k$ entry $x_{1_1} x_{i_2} \ldots x_{i_r}$ and x^r for the vector with i-th entry x_i^r then the above expression can be written as

$$\mathcal{A}\mathbf{x}^{\otimes k-1} = \lambda \mathbf{x}^{k-1}$$

where the multiplication denoted by concatenation is tensor contraction. Call $\lambda \in \mathbb{C}$ an *eigenvalue* of $\mathcal{A}$ if there is a non-zero vector $\mathbf{x} \in \mathbb{C}^n$, which we call an *eigenvector*, satisfying

$$\sum_{i_2, i_3, \ldots, i_k = 1}^{n} a_{j i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} = \lambda x_j^{k-1}.$$

Next we offer an important result from commutative algebra to proceed the definition of the adjacency characteristic polynomial of a hypergraph.

Theorem 3 *(The Resultant, [6]) Fix degrees $d_1, d_2, \ldots, d_n$. For $i \in [n]$, consider all monomials $\mathbf{x}^\alpha$ (where α is itself a vector) of total degree d_i in $x_1, \ldots, x_n$. For each such monomial, define a variable $u_{i,\alpha}$. Then there is a unique polynomial* RES $\in \mathbb{Z}[\{u_{i,\alpha}\}]$ *with the following three properties:*

1. *If $F_1, \ldots, F_n \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of degrees $d_1, \ldots, d_n$ respectively, then the polynomials have a non-trivial common root in $\mathbb{C}^n$ exactly when* $\mathrm{RES}(F_1, \ldots, F_n) = 0$*. Here,* $\mathrm{RES}(F_1, \ldots, F_n)$ *is interpreted to mean substituting the coefficient of $\mathbf{x}^\alpha$ in F_i for the variable $u_{i,\alpha}$ in* RES.
2. $\mathrm{RES}(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$.
3. RES *is irreducible, even in $\mathbb{C}[\{u_{i,\alpha}\}]$.*

Moreover, for $i \in [n]$, RES *is homogeneous in the variable $\{u_{i,\alpha}\}$ with degree $\prod_{j \in [n], j \neq i} d_i$.*

Definition 1 *([10]) The symmetric hyperdeterminant of $\mathcal{A}$, denoted* $\det(\mathcal{A})$ *is the resultant of the polynomials which comprise the coordinates of $\mathcal{A}x^{\otimes k-1}$. Let λ be an indeterminate. The characteristic polynomial $\phi_{\mathcal{A}}(\lambda)$ of a hypermatrix $\mathcal{A}$ is* $\phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I} - \mathcal{A})$.

We consider the normalized adjacency matrix of a k-uniform hypergraph, $\mathcal{H} = (V, E)$. We refer to such hypergraphs as *k-graphs* and we reserve the language of *graph* for the case of $k = 2$. For a k-graph $\mathcal{H} = ([n], E)$ we denote the *(normalized) adjacency hypermatrix* $\mathcal{A}_{\mathcal{H}}$ to be the order k dimension n hypermatrix with entries

$$a_{i_1, i_2, \ldots, i_k} = \frac{1}{(k-1)!} \begin{cases} 1 & : \{i_1, i_2, \ldots, i_k\} \in E(H) \\ 0 & : \text{otherwise.} \end{cases}$$

For simplicity, we denote $\phi(\mathcal{H}) = \phi_{\mathcal{A}_\mathcal{H}}(\lambda)$ and write

$$\phi(\mathcal{H}) = \sum_{i=0}^{t} c_i \lambda^{t-i}$$

where $t = n(k-1)^{n-1}$ by Theorem 3. Throughout, we make use of the notation $\phi_d(\mathcal{H}) = c_d$ for the codegree-d coefficient of $\phi(\mathcal{H})$.

Our approach relies on the following trace formula for the hyperdeterminant of a tensor. In [9], Morozov and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using Schur polynomials in the generalized traces of the order k, dimension n hypermatrix $\mathcal{A}$. Let $f : \mathbb{C}^n \to C^n$ be a linear map and let I be the unity map, $I = (x_1, x_2, \ldots, x_n)^T \to (x_1, x_2, \ldots, x_n)^T$. Famously,

$$\log\det(I - f) = \operatorname{tr}\log(I - f) = -\sum_{k=1}^{\infty} \frac{\operatorname{tr}(f^k)}{k}.$$

The characteristic polynomial is defined as the resultant of a certain system of equations, so calculating the characteristic polynomial requires computation of the resultant. Moroz and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using *Schur polynomials* in the generalized traces of the order k, dimension n hypermatrix $\mathcal{A}$.

Definition 2 *Define the d-th Schur polynomial $P_d \in \mathbb{Z}[t_1, \ldots, t_d]$ by $P_0 = 1$ and, for $d > 0$,*

$$P_d(t_1, \ldots, t_d) = \sum_{m=1}^{d} \sum_{d_1 + \cdots + d_m = d} \frac{t_{d_1} \cdots t_{d_m}}{m!}.$$

More compactly, one may define P_d by

$$\exp\left(\sum_{d=1}^{\infty} t_d z^d\right) = \sum_{d=1}^{\infty} P_d(t_1, \ldots, t_d) z^d.$$

Let f_i denote the ith coordinate of $\mathcal{A}\mathbf{x}^{\otimes k-1}$. Define A to be an auxiliary $n \times n$ matrix with distinct variables A_{ij} as entries. For each I, we define the differential operator

$$\hat{f}_i = f_i\left(\frac{\partial}{\partial A_{i1}}, \frac{\partial}{\partial A_{i2}}, \ldots, \frac{\partial}{\partial A_{in}}\right)$$

in the natural way. In [4], Cooper and Dutle use the aforementioned Morozov-Shakirov formula to show that the d-th trace of $\mathcal{A}_\mathcal{H}$,

$$\operatorname{Tr}_d(\mathcal{A}_\mathcal{H}) = (k-1)^{n-1} \sum_{d_1 + \cdots + d_n = d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{\,d_i}}{(d_i(k-1))!} \operatorname{tr}(A^{d(k-1)})\right) \qquad (1)$$

where $\operatorname{tr}(A^{d(k-1)})$ is the standard matrix trace (for a more detailed explanation, see [4]). We prove Theorem 14 with the aid of the following reformulation of Equation 1:

$$\operatorname{Tr}_d(\mathcal{A}_\mathcal{H}) = (k-1)^n \sum_{H \in H_d} C_H(\#H \subseteq \mathcal{H}).$$

The paper is arranged as follows. In the following section we define the associated digraph of an operator in $\mathrm{Tr}_d(\mathcal{H})$, as in Equation 1. In particular, we provide a formula for a summand of $\mathrm{Tr}_d(\mathcal{H})$ in terms of associated digraphs. In Section 3 we give a combinatorial description of the differential operators which have non-zero contribution to $\mathrm{Tr}_d(\mathcal{H})$ and characterize these operators in terms of associated digraphs. In Section 4 we show that associated digraphs correspond to a particular type of hypergraph, termed Veblen hypergraphs. We motivate our main result by defining the associated coefficient of a Veblen hypergraph. Our main result is then applied to address a question of [8] in Section 6 by expressing low codegree coefficients of the characteristic polynomial of a 3-uniform hypergraph in terms of particular subgraphs. In Section 7 we answer a question of [4] by providing a formula for the associated coefficient of the k-uniform simplex. We conclude by showing that our main result is indeed a generalization of the Harary-Sachs theorem to hypergraphs. We leave the reader with a conjecture concerning the multiplicity of the zero-eigenvalue.

2 The associated digraph of an operator

Recall the d-th trace of $\mathcal{A}_{\mathcal{H}}$ from Equation 1,

$$\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}}) = (k-1)^{n-1} \sum_{d_1+\cdots+d_n=d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{\,d_i}}{(d_i(k-1))!} \,\mathrm{tr}(A^{d(k-1)}) \right)$$

where $\mathrm{tr}(A^{d(k-1)})$ is the standard matrix operation. Let $\hat{f}_{d_1,d_2,\ldots,d_n}$ be an addend of $\prod_{i=1}^{n} \hat{f}_i^{d_i}$ in $\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}})$. When the context is clear we suppress the subscript and simply write $\hat{f}$. Given $\alpha = (i_1, i_2, \ldots, i_{d(k-1)})$ let

$$A_\alpha := A_{i_1,i_2} A_{i_2,i_3} \ldots A_{i_{d(k-1)-1}, i_{d(k-1)}} A_{i_{d(k-1)}, i_1} \tag{2}$$

and recall that

$$\mathrm{tr}(A^{d(k-1)}) = \sum_{\alpha} A_\alpha$$

where the factors of A_α are commutative. Adhering to the terminology of [4] we say A_α is k-valent if k divides the number of times i occurs in a subscript of A_α. We utilize divisibility notation for monomials in $\mathrm{tr}(A^{d(k-1)})$, e.g., using $g|h$ to denote that g occurs as a factor of the formal product h. We say that A_α *survives* $\hat{f}$ if $\hat{f} A_\alpha \neq 0$.

Definition 3 *For a differential operator $\hat{f}_{d_1,d_2,\ldots,d_n}$ the associated digraph of $\hat{f}$, denoted $D_{\hat{f}}$, is the directed multigraph where there are d_i distinguishable edges directed from i to j given $\left(\frac{\partial}{\partial A_{i,j}}\right)^{d_i} \mid \hat{f}$ and isolated vertices are ignored.*

We suppress the subscript and write D when $\hat{f}$ is understood. We recall the following graph theoretic definitions according to [5].

Definition 4 *A* walk *in a graph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. A walk is* closed *if*

$v_0 = v_k$. *A closed walk in a graph is an* Euler tour *if it traverses every edge of the graph exactly once. An* Euler circuit *is an Euler tour up to cyclic permutation of its edges, i.e., an Euler tour with no distinguished beginning.*

We denote the set of Euler tours of a graph G which begin at the edge $e \in E(G)$ by $\mathfrak{E}_e(G)$ and we denote the set of Euler circuits of G by $\mathfrak{E}(G)$. Recall that a digraph D has an Euler circuit if and only if $\deg^+(v) = \deg^-(v)$ for all $v \in V(D)$ and D is weakly connected.

Definition 5 *Let $w = (v_i)_{i=0}^m$ be a sequence of (not necessarily distinct) vertices of D. We say that w* describes an Euler tour *in D if there exist distinct edges $e_0, \ldots, e_{m-1}$ such that $v_0 e_0 v_1 e_1 \ldots e_{m-1} v_m$ is an Euler tour in D. Moreover we say that such Euler tours are* described by *w.*

Note that the use of Euler tour in the previous definition is well-founded as e_0 is distinguished as the first edge.

Lemma 4 *Consider $\mathrm{Tr}_d(\mathcal{H})$, $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ if and only if D is Eulerian. In this case*

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = |E(D)||\mathfrak{E}(D)|. \tag{3}$$

Proof: Consider $\mathrm{Tr}_d(\mathcal{H})$. Fix a term A_α of $\mathrm{tr}(A^{d(k-1)})$ and a differential operator $\hat{f}$ of $\prod_{i=1}^n \hat{f}_i^{d_i}$. Suppose $\hat{f} A_\alpha \neq 0$. Whence $\hat{f} A_\alpha \neq 0$ the factors of $\hat{f}$ are in one-to-one correspondence with the factors of A_α. It follows that the edges of $D_{\hat{f}}$ are in one-to-one correspondence with the factors of A_α. Notice that for $i \in V(D)$, $\deg^+(i) = \deg^-(i)$ by Equation 2. Further, D is strongly connected as the sequence of indices from i to j (cyclically, if necessary) is a walk from vertex i to vertex j. Therefore, D is Eulerian.

Suppose now that D is Eulerian and let $\alpha = (v_i)_{i=0}$ describe an Euler tour in D. We claim that $\hat{f} A_\alpha$ is equal to the number of Euler tours in $D_{\hat{f}}$ described by α. Let $m(i,j)$ denote the number of edges from i to j in D. Notice that

$$A_\alpha = \prod_{i,j \in V(D)} A_{i,j}^{m(i,j)} \text{ and } \hat{f} = \prod_{i,j \in V(D)} \frac{\partial^{m(i,j)}}{\partial A_{i,j}^{m(i,j)}}.$$

Clearly

$$\hat{f} A_\alpha = \prod_{i,j \in V(D)} m(i,j)!.$$

Moreover, α describes $\prod_{i,j \in V(D)} m(i,j)!$ Euler tours in D by straightforward enumeration. Observe that there are $|E(D)||\mathfrak{E}(D)|$ Euler tours in D as we may distinguish any edge from D as the first edge of an Euler circuit. Thus,

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = \sum_\alpha \hat{f} A_\alpha = |E(D)||\mathfrak{E}(D)|$$

as every Euler tour is described by exactly one α. ■

We conclude this section with a remark about the evaluation of Equation 3. Conveniently, $|\mathfrak{E}(D)|$ can be computed using the BEST theorem, originally appearing in [15] as a variation of a result of [14].

Theorem 5 *(BEST Theorem) The number of Euler circuits in a connected Eulerian graph G is*

$$|\mathfrak{E}(G)| = \tau(G) \prod_{v \in V} (\deg(v) - 1)!$$

where $\tau(G)$ is the number of arborescences (i.e., the number of rooted subtrees of G with a specified root).

For simplicity we abbreviate $\tau(f) = \tau(D_{\hat{f}})$. Combining the BEST theorem and the observation that $|E(D)| = d(k-1)$ yields the following.

Corollary 6

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = d(k-1)\tau(f) \prod_{v \in V(D)} (\deg^-(v) - 1)!$$

As a final note, recall that $\tau(G)$ can be computed using the Matrix Tree Theorem, which makes the computation of the right-hand side of the equality in Corollary 6 efficient.

Theorem 7 *(Matrix Tree Theorem/Kirchhoff's Theorem) For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \ldots \lambda_{n-1}$ be the non-zero eigenvalues of $\mathcal{L}(G) = D(G) - A(G)$. Then*

$$\tau(G) = \frac{\lambda_1 \lambda_2 \ldots \lambda_{n-1}}{n}.$$

3 Euler operators and Veblen hypergraphs

In Lemma 4 we showed that the only differential operators $\hat{f}$ for which $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ are the operators whose associated digraphs are Eulerian. The question remains: which $\hat{f}$ have an Eulerian associated digraph? We answer this question with the following graph decoration.

Definition 6 *We define the* u-rooted directed star of a k-uniform edge *e to be*

$$S_e(u) = (e, \{uv : v \in e, u \neq v\}).$$

A rooting *of a k-graph $\mathcal{H}$ is an ordering $R = (S_{e_1}(v_1), S_{e_2}(v_2), \ldots, S_{e_m}(v_m))$ such that $E(\mathcal{H}) = \{e_1, \ldots, e_m\}$ and $v_i \leq v_{i+1}$. Given a rooting of $\mathcal{H}$ we define the* rooted multi-digraph *of R to be*

$$D_R = \bigcup_{i=1}^{m} S_{e_i}(v_i)$$

where the union sums edge multiplicities. We say that a rooting R is an Euler rooting *if D_R is Eulerian. We denote the mutli-set of rooted digraphs of $\mathcal{H}$ as $S(\mathcal{H})$.*

Note that two distinct rootings can yield the same rooted digraph. We suppress the subscript D_R and write D when the context is clear. We further refer to D as a rooted digraph of $\mathcal{H}$ for convenience.

Definition 7 *Given a rooted digraph $D \in S(\mathcal{H})$, we define the* rooted operator of D *to be*

$$\hat{f}_D = \prod_{uv \in E(D)} \frac{\partial}{\partial A_{u,v}}.$$

Moreover, we denote

$$\hat{S}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

In the case when D is Eulerian we refer to $\hat{f}_D$ as an Euler operator.

The notation of $\hat{f}_D$ is consistent with our usage of $\hat{f}$ whence

$$\hat{f}_D \mid \prod_{i=1}^{n} \hat{f}_i^{d_i}$$

where d_i is equal to the number of times vertex i is appears as a root of D. If $\hat{f}$ is a rooted operator then it is understood that there exists a (not necessarily unique) rooting R such that, with a slight abuse of notation, $\hat{f} = \hat{f}_{D_R}$. We call such a rooting an *underlying rooting* of a differential operator.

Lemma 8 *The associated digraph of an operator $\hat{f}$ is Eulerian if and only if $\hat{f}$ is an Euler operator.*

Remark 1 *By Lemma 8 the only operators which have non-zero contribution to* $\mathrm{Tr}_d(\mathcal{H})$ *are Euler operators. We denote $R(\mathcal{H}) \subseteq S(\mathcal{H})$ to be the multi-set of* Euler rooted digraphs of $\mathcal{H}$. *We further denote*

$$\hat{R}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

Remark 2 *One can deduce Theorem 4.1 of [12] from Lemma 8 by a change of notation: our $\hat{f}_D$ is their F, our set of Eulerian associated digraphs arising from $\hat{f}_R$ is their $\mathbf{E}_{d,m-1}(n)$, and our $\mathfrak{E}(D)$ is their $\mathbf{W}(E)$.*

We now show that an Euler rooting is a rooting of a special type of hypergraph.

Definition 8 *A* Veblen hypergraph[1] *is a k-uniform, k-valent multi-hypergraph.*

Lemma 9 *An Euler rooting R is a rooting of precisely one labeled Veblen hypergraph.*

Proof: Suppose $S = (S_i)_{i=1}^m$ is a rooting of a connected k-graph $\mathcal{H}$. Since D_S is Eulerian we have for all $j \in V(\mathcal{H})$

$$\deg^+(j) = (k-1)|\{i : v_i = j\}| = |\{i : v_i \neq j, j \in e_i\}| = \deg^-(j).$$

Fix a vertex $v \in V(\mathcal{H})$. We compute

$$\begin{aligned}
\deg_{\mathcal{H}}(v) &= \deg^+_D(v) + \deg^-_D(v) \\
&= |\{i : v_i = v\}| + |\{i : v_i \neq v, v \in e_i\}| \\
&= |\{i : v_i = v\}| + (k-1)|\{i : v_i = v\}| \\
&= k|\{i : v_i = v\}|.
\end{aligned}$$

[1] The nomenclature is a reference to Oswald Veblen (1880-1960) who proved an extension of Euler's theorem in 1912. We present a brief note about Veblen's namesake theorem at the conclusion of this section.

Observe that $k \mid \deg_{\mathcal{H}}(v)$; it follows that $\mathcal{H}$ is Veblen by definition. Now suppose that $\mathcal{H}_0$ is a connected Veblen graph such that S is an Euler rooting of $\mathcal{H}_0$. As S is a rooting of H_0, $E(\mathcal{H}_0) = E(\mathcal{H})$ and since both hypergraphs are connected $V(\mathcal{H}_0) = V(\mathcal{H})$. It follows that $\mathcal{H}$ is unique. ■

We combine Lemmas 8 and 9 into the following Lemma.

Lemma 10 *We have*

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$$

if and only if $\hat{f} = \hat{f}_D$ is a rooted operator. Moreover, the underlying rooting of $\hat{f}$ is necessarily an Euler rooting of precisely one connected, labeled Veblen hypergraph.

In the following section we use Lemma 10 to express the codegree-d coefficient of a k-graph as a function of Veblen hypergraphs. Here we conclude with a note about Veblen's theorem.

Theorem 11 *(Veblen's theorem [16]) The set of edges of a finite graph can be written as a union of disjoint simple cycles if and only if every vertex has even degree.*

Unfortunately, Veblen's theorem does not extend to higher uniformity: the set of edges of a finite k-graph $\mathcal{H}$ can not always be written as a union of disjoint simple k-regular k-graphs if and only if $\mathcal{H}$ is k-valent. Consider the Veblen 3-graph $\mathcal{T}$ which consists of three bottomless tetrahedrons each sharing a common base. To be precise,

$$\mathcal{T} = \left(\{a,b,c,1,2,3\}, \bigcup_{i=1}^{3} \left(\binom{\{a,b,c,i\}}{3} \setminus \{a,b,c\} \right) \right).$$

A drawing of $\mathcal{T}$ is given in Figure 1. Since there are only three edges containing

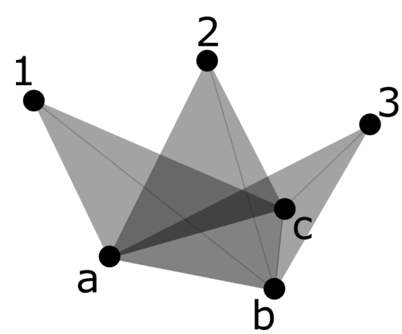

Figure 1: The Veblen 3-graph $\mathcal{T}$ where edges are drawn as triangular faces.

$i \in [3]$ it must be the case that any partition into Veblen graphs places each edge containing i into the same class. Observe that for each i the vertices a, b, and c each have degree 2. Therefore, the only 3-valent edge partition is the trivial one.

4 The associated coefficient of a Veblen hypergraph

We now turn our attention to computing the codegree-d coefficient of a k-graph $\mathcal{H}$ via Equation 1. From Lemma 10 we know that the only operators which satisfy $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ are rooted operators. Furthermore, as a differential operator of $\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}})$ is of degree d, the underlying Euler rooting of $\hat{f}$ is a rooting of precisely one connected, labeled Veblen hypergraph with d edges. We equate $\mathrm{Tr}_d(\mathcal{H})$ to a weighted sum over Euler rootings of connected Veblen graphs with d edges which "appear" in $\mathcal{H}$. Consider the following generalization of the notion of subgraph.

Definition 9 *For a labeled multi-hypergraph H, we call the simple k-graph formed by removing duplicate edges of H the* flattening *of H and denote it $\underline{H}$. We say that H is an* infragraph *of $\mathcal{H}$ if $\underline{H} \subseteq \mathcal{H}$. Let $\mathcal{V}_d(\mathcal{H})$ denote the set of isomorphism classes of connected, labeled Veblen infragraphs with d edges of $\mathcal{H}$.*

Definition 10 *The* associated coefficient of a connected Veblen hypergraph *H is*

$$C_H = \sum_{D \in R(H)} \left(\frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right).$$

The associated coefficient *of a (possibly disconnected) Veblen hypergraph $H = \bigcup_{i=1}^m G_i$ is*

$$C_H = \prod_{i=1}^m C_{G_i}.$$

Definition 11 *For a k-graph $\mathcal{H}$ and a Veblen k-graph $H = \bigcup_{i=1}^m G_i$ we define*

$$(\#H \subseteq \mathcal{H}) = \frac{1}{|\mathrm{Aut}(H)|} \prod_{i=1}^m |\mathrm{Aut}(\underline{G_i})| |\{S \subseteq \mathcal{H} : S \cong \underline{G_i}\}|.$$

In the case when H is connected this simplifies to

$$(\#H \subseteq \mathcal{H}) = \frac{|\mathrm{Aut}(\underline{H})|}{|\mathrm{Aut}(H)|} |\{S \subseteq \mathcal{H} : S \cong \underline{H}\}| = |\mathrm{Aut}(\underline{H})/\mathrm{Aut}(H)| \cdot |\{S \subseteq \mathcal{H} : S \cong \underline{H}\}|.$$

Note that for $H = \bigcup_{i=1}^m G_i$, $(H \subseteq \mathcal{H})$ is not multiplicative over the components of H as

$$\prod_{i=1}^m (\#G_i \subseteq \mathcal{H}) = \frac{(\#H \subseteq \mathcal{H})|\mathrm{Aut}(H)|}{\prod_{i=1}^m |\mathrm{Aut}(G_i)|}.$$

However, we have the following identity.

Lemma 12 *Let $H = \bigcup_{i=1}^m G_i$ be a Veblen k-graph. If μ_H denotes the number of linear orderings of the components of H (where two components are indistinguishable if they are isomorphic) then*

$$(\#H \subseteq \mathcal{H}) = \frac{\mu_H}{m!} \prod_{i=1}^m (\#G_i \subseteq \mathcal{H}).$$

Proof: Suppose there are t isomorphism classes of components of H with representatives $H_1, H_2, \ldots H_t$. Denote the number of components of H which are isomorphic to H_i as μ_i. Fix an ordering of the components which are isomorphic to H_i, $\{G_{1,i}, G_{2,i}, \ldots, G_{\mu_i,i}\}$. The number of distinct linear orderings of the components of H where G_i and G_j are indistinguishable when $G_{r,i} \cong G_{s,i}$ is

$$\mu_H = \binom{m}{\mu_1, \mu_2, \ldots, \mu_t}$$

so that

$$\frac{m!}{\mu_H} = \prod_{i=1}^{t} \mu_i!.$$

Note that for $a \in \mathrm{Aut}(H)$, there exists $\sigma \in \mathfrak{S}_{\mu_i}$ such that $a(G_{j,i}) = G_{\sigma(j),i}$. In this way, $\mathrm{Aut}(H)$ induces a permutation on the isomorphism classes of the components of H (note that this map is well-defined since the components are labeled). Let $\psi : \mathrm{Aut}(H) \to \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_t}$ be such a map. Notice $\ker(\psi)$ is the group of automorphisms of H which maps each component to itself. Appealing to the First Isomorphism Theorem we have

$$\frac{|\mathrm{Aut}(H)|}{\prod_{i=1}^{m} |\mathrm{Aut}(G_i)|} = \prod_{i=1}^{t} \mu_i!.$$

The desired equality follows by substitution. ∎

Remark 3 *The equation in Lemma 12 implies that $(\#H \subseteq \mathcal{H})$ is multiplicative over its components if and only if the components of H are pairwise non-isomorphic.*

Let

$$A(d,n) = \left\{ (d_1, \ldots, d_n) : \sum d_i = d, d_i \geq 0 \right\}$$

be the set of *arrangements of d into n non-negative parts* and further let $A^+(d,n) \subset A(d,n)$ be the set of *arrangements of d into n positive parts.* For a k-graph $\mathcal{H}$ and $a \in A^+(d,|E|)$ let $R^a(\mathcal{H})$ be the set of Euler rootings of all labeled, connected Veblen infragraphs of $\mathcal{H}$ which have the property that vertex v_i is the root of exactly d_i edges. (N.B. We take $a \in A^+(d,|E|)$ as it is necessary that $d_i > 0$ for $D \in R^a(\mathcal{H})$ to be Eulerian.)

Remark 4 *Let $\mathcal{V}_d^*(\mathcal{H})$ denote the set of (possibly disconnected) Veblen infragraphs of $\mathcal{H}$ with d edges up to isomorphism. Further let $\mathcal{V}_d(\mathcal{H}) \subseteq \mathcal{V}_d^*(\mathcal{H})$ denote the set of connected Veblen infragraphs of $\mathcal{H}$ with d edges up to isomorphism.*

We now present a formula for $\mathrm{Tr}_d(\mathcal{H})$ as a weighted sum over its Veblen infragraphs.

Lemma 13 *For a k-graph $\mathcal{H}$*

$$\mathrm{Tr}_d(\mathcal{H}) = d(k-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}).$$

Proof: For convenience let $|E| = |E(\mathcal{H})|$. We equate

$$\sum_{H\in\mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}) = \sum_{H\in\mathcal{V}_d(\mathcal{H})} \left(\left(\sum_{D\in R(H)} \frac{\tau_D}{\prod_{v\in V(D))} \deg^-(v)} \right) (\#H \subseteq \mathcal{H}) \right)$$
$$= \sum_{a\in A^+(d,|E|)} \left(\sum_{D\in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v\in V(D)} \deg^-(v)} \right).$$

Recall

$$\mathrm{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{d_1+\cdots+d_n=d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{d_i}}{(d_i(k-1))!} \operatorname{tr}(A^{d(k-1)}) \right).$$

Applying Lemma 10 we have

$$\mathrm{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{a\in A^+(d,|E|)} \left(\sum_{D\in R^a(\mathcal{H})} \frac{\hat{f}_D \operatorname{tr}(A^{d(k-1)})}{\prod_{i=1}^{n}(d_i(k-1))!} \right).$$

By Corollary 6 ,

$$\hat{f}_D \operatorname{tr}(A^{d(k-1)}) = d(k-1)\tau_D \prod_{v\in V(D_R)} (\deg^-(v)-1)!$$

When $D \in R^a(\mathcal{H})$ with $a = (d_1, \ldots, d_n)$, we have $\deg^-(v_i) = d_i(k-1)$. By substitution we have

$$\mathrm{Tr}_d(\mathcal{H}) = d(k-1)^n \sum_{a\in A^+(d,|E|)} \left(\sum_{D\in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v\in V(D)} \deg^-(v)} \right)$$
$$= d(k-1)^n \sum_{H\in\mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}).$$

■

We are now prove a generalization of the Harary-Sachs formula for k-graphs.

Theorem 14 *For a simple k-graph $\mathcal{H}$,*

$$\phi_d(\mathcal{H}) = \sum_{H\in\mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H(\#H \subseteq \mathcal{H}).$$

Proof: Fix a k-graph $\mathcal{H}$ and $d \geq 1$. From [4] we have by Equation 1

$$\phi_d(\mathcal{H}) = P_d\left(\frac{-\mathrm{Tr}_1(\mathcal{H})}{1}, \frac{-\mathrm{Tr}_2(\mathcal{H})}{2}, \ldots, \frac{-\mathrm{Tr}_d(\mathcal{H})}{d} \right)$$

where

$$P_d(t_1, t_2, \ldots, t_d) = \sum_{m=1}^{d} \sum_{d_1+\cdots+d_m=d} \frac{t_{d_1}t_{d_2}\cdots t_{d_m}}{m!}.$$

Fix an $1 \le m \le d$ and an arrangement $a = (d_1, d_2, \ldots, d_m) \in A^+(d, m)$. By Lemma 13,

$$\frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} = -(k-1)^n \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}).$$

Let $\mathcal{V}^a(\mathcal{H})$ denote the set of m-tuples of connected unlabeled Veblen infragraphs of $\mathcal{H}$ whose i-th coordinate has d_i edges for $i \in [m]$, such that $\sum_i d_i = d$. We have

$$\begin{aligned}\prod_{i=1}^{m} \frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} &= (-1)^m (k-1)^{mn} \prod_{i=1}^{m} \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}) \\ &= (-(k-1)^n)^m \sum_{\substack{H = G_1 \cup \cdots \cup G_m \\ (G_1, G_2, \ldots, G_m) \in \mathcal{V}^a(\mathcal{H})}} C_H \prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H})\end{aligned}$$

For $m \in \mathbb{N}$, let $\mathcal{V}_d^m(\mathcal{H})$ be the set of unlabeled Veblen infragraphs of $\mathcal{H}$ with d edges and m components. Appealing to Lemma 12 we may write

$$\begin{aligned}\phi_d &= P_d\left(-\frac{\operatorname{Tr}_1(\mathcal{H})}{1}, -\frac{\operatorname{Tr}_2(\mathcal{H})}{2}, \ldots, -\frac{\operatorname{Tr}_d(\mathcal{H})}{d}\right) \\ &= \sum_{m=1}^{d} \sum_{d_1 + \cdots + d_m = d} \frac{1}{m!} \prod_{i=1}^{m} \frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} \\ &= \sum_{m=1}^{d} \left(\sum_{a \in A^+(m,d)} (-(k-1)^n)^m \sum_{H \in \mathcal{V}^a(\mathcal{H})} C_H \frac{\left(\prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H})\right)}{m!} \right) \\ &= \sum_{m=1}^{d} (-(k-1)^n)^m \sum_{H \in \mathcal{V}_d^m(\mathcal{H})} C_H \left(\frac{\mu_H}{m!} \prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H}) \right) \\ &= \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H (\#H \subseteq \mathcal{H}).\end{aligned}$$

■

In the following sections we apply Theorem 14 to provide a combinatorial description for the first six proper leading coefficients of the characteristic polynomial of a 3-graph, determine the associated coefficient of the k-uniform simplex, and show that the Harary-Sachs Theorem for graphs is indeed a special case of Theorem 14.

5 Low Codegree Coefficients of 3-Graphs

We provide an explicit formula for the first six codegree coefficients of the characteristic polynomial of a 3-graph. Let $\mathcal{H}$ be a simple 3-graph with n vertices. We write $(\# H \in \mathcal{H}) = |\{S \subseteq \mathcal{H} : S \cong \underline{H})|$. From [4] we have that $c_1 = 0, c_2 = 0$,

$$c_3 = -3 \cdot 2^{n-3} (\# e \in \mathcal{H}),$$

where $e = K_3^{(3)}$ is the single-edge hypergraph and

$$c_4 = 21 \cdot 2^{n-3}(\# K_4^{(3)} \in \mathcal{H}).$$

We use Theorem 14 to situation these results and further provide an analogous description of c_5 and c_6. Clearly there are no Veblen 3-graphs with one or two edges as each vertex must have degree at least three. It follows that $c_1, c_2 = 0$. There is only one Veblen 3-graph with three edges, the single edge with multiplicity three. By Theorem 14, then, we have

$$c_3 = -2^n \cdot \frac{3}{8}(\# e \in \mathcal{H}).$$

Similarly, [4] observed that the only Veblen 3-graph with four edges is the complete graph, which implies

$$c_4 = -2^n \cdot \frac{21}{8}(\# K_4^{(3)} \in \mathcal{H}).$$

For the case of c_5 we have

$$c_5 = -2^n \left(\frac{51}{16}(\# \Gamma_{5,1} \in \mathcal{H}) + \frac{27}{16}(\# \Gamma_{5,2} \in \mathcal{H}) \right)$$

where $\Gamma_{5,1}$ is the tight 5-cycle and $\Gamma_{5,2}$ is the 3-pointed crown (given explicitly in Figure 2). We further compute

$$\begin{aligned} c_6 = 2^{2n} \cdot \frac{9}{64} \left(\frac{(\# e \in \mathcal{H})^2}{2} \right) - 2^n \biggl(&\frac{3}{16}(\# e \in \mathcal{H}) + \frac{9}{8}(\# \Gamma_{6,1} \in \mathcal{H}) + \frac{9}{32}(\# \Gamma_{6,2} \in \mathcal{H}) \\ &+ \frac{99}{32} \cdot 2(\# \Gamma_{6,3} \in \mathcal{H}) + \frac{213}{16}(\# \Gamma_{6,4} \in \mathcal{H}) + \frac{69}{16}(\# \Gamma_{6,5} \in \mathcal{H}) + \frac{63}{32}(\# \Gamma_{6,6} \in \mathcal{H}) \\ &+ \frac{129}{32}(\# \Gamma_{6,7} \in \mathcal{H}) + \frac{27}{32} \cdot 2(\# \Gamma_{6,8} \in \mathcal{H}) + \frac{63}{16}(\# \Gamma_{6,9} \in \mathcal{H}) + \frac{117}{32}(\# \Gamma_{6,10} \in \mathcal{H}) \biggr) \end{aligned}$$

where $\Gamma_{6,i}$ are provided in Figure 2.

Recall that $\mathcal{V}_d$ and $\mathcal{V}_d^*$ denote the number of connected and (possibly) disconnected Veblen 3-graphs with d edges. We have computed

$$(|\mathcal{V}_d|)_{d=1}^{\infty} = (0, 0, 1, 1, 2, 11, 26, 122, 781, \dots)$$

and further

$$(|\mathcal{V}_d^*|)_{d=1}^{\infty} = (0, 0, 1, 1, 2, 12, 27, 125, 795, \dots)$$

see A320648 [13].

Remark 5 *The Fano Plane is a Veblen 3-graph with seven edges. The associated coefficient of the Fano Plane is* $87/16 = 5.4375$.

Note that, for the classical Harary-Sachs Theorem, the number of elementary graphs (explained in the next section) one needs to sum over for the codegree-d coefficient

Γ	$E(\Gamma)$	C_Γ	$\lvert\mathrm{Aut}(\underline{\Gamma})\rvert/\lvert\mathrm{Aut}(\Gamma)\rvert$
$\Gamma_{5,1}$	$(123)(125)(145)(234)(345)$	$51/16$	1
$\Gamma_{5,2}$	$(123)(145)(145)(234)(235)$	$27/16$	1
$\Gamma_{6,1}$	$(123)^3(124)^3$	$9/8$	1
$\Gamma_{6,2}$	$(123)^3(145)^3$	$9/32$	1
$\Gamma_{6,3}$	$(123)^2(124)(135)(145)^2$	$99/32$	2
$\Gamma_{6,4}$	$(123)(124)(125)(134)(135)(145)$	$213/16$	1
$\Gamma_{6,5}$	$(123)(124)(156)(256)(345)(346)$	$69/16$	1
$\Gamma_{6,6}$	$(123)(124)(145)(246)^3$	$63/32$	1
$\Gamma_{6,7}$	$(123)(134)(145)(246)(256)^2$	$129/32$	1
$\Gamma_{6,8}$	$(123)^2(124)(356)(456)^2$	$27/32$	2
$\Gamma_{6,9}$	$(123)(124)(134)(256)(356)(456)$	$63/16$	1
$\Gamma_{6,10}$	$(123)(124)(135)(246)(356)(456)$	$117/16$	1
$\Gamma_{9,2}$	$(123)^6(145)^3$	$9/32$	2
$\Gamma_{9,3}$	$(123)^3(145)^3(246)^3$	$9/8$	1
$\Gamma_{9,4}$	$(123)^3(145)^3(167)^3$	$81/128$	1
$\Gamma_{12,1}$	$(123)^9(145)^3$	$9/32$	2
$\Gamma_{12,2}$	$(123)^6(145)^6$	$27/64$	1
$\Gamma_{12,3}$	$(123)^6(145)^3(167)^3$	$81/128$	3
$\Gamma_{12,4}$	$(123)^6(145)^3(246)^3$	$63/32$	3
$\Gamma_{12,5}$	$(123)^3(145)^3(167)^3(246)^3$	$459/64$	1
$\Gamma_{12,6}$	$(123)^3(145)^3(246)^3(356)^3$	$255/16$	1

Figure 2: Some connected Veblen 3-graphs and corresponding values

is simply the number of partitions of d into positive parts. The number of Veblen 3-graphs is exponentially larger.

To demonstrate Theorem 14 we have computed the first sixteen coefficients of the characteristic polynomial of the Fano Plane with two edges removed (called the Rowling hypergraph in [3]), the Fano Plane with one edge removed, and the Fano Plane in Figure 3. Aside from the novelty of performing these computations, the coefficients can be used to determine the characteristic polynomial of hypergraph [3]. we provide a numerically stable algorithm for computing the characteristic polynomial of a hypergraph given its set spectrum and an appropriate number of leading coefficients. Consider the Rowling hypergraph

$$\mathcal{R} = ([7], \{1,2,3\}, \{\{1,4,5\}, \{1,6,7\}, \{2,5,6\}, \{3,5,7\}\}).$$

Using the aforementioned algorithm we were able to compute

$$\begin{aligned}\phi(\mathcal{R}) =& x^{133}(x^3-1)^{27}(x^{15}-13x^{12}+65x^9-147x^6+157x^3-64)^{12}\\ &\cdot (x^6-x^3+2)^6(x^6-17x^3+64)^3\end{aligned}$$

when traditional methods (i.e., direct computation of the resultant) failed to do so.

In this case, we can determine $\phi(\mathcal{R})$ if we know c_3, c_6, c_9, c_{12}. We show

$$\begin{aligned}
c_3 &= -240 \\
c_6 &= 28320 \\
c_9 &= -2190860 \\
c_{12} &= 125012034
\end{aligned}$$

One can show that every Veblen infragraph of $\mathcal{R}$ has edge multiplicity congruent to 0 modulo 3 (this is immediate as the roots of $\phi(\mathcal{R})$ are invariant under multiplication by third roots of unity). Appealing to our previous formulae we have

$$c_3 = -2^7 \cdot \frac{3}{8}(\# e \in \mathcal{H}) = -2^n \cdot \frac{3}{8} \cdot 5 = -240$$

and

$$\begin{aligned}
c_6 &= \frac{c_3^2}{2!} - 2^n \left(\frac{3}{16}(\# e \in \mathcal{H}) + \frac{9}{32}(\# \Gamma_{6,2} \in \mathcal{H}) \right) \\
&= \frac{(-240)^2}{2} - 2^7 \left(\frac{3}{16} \cdot 5 + \frac{9}{32}\binom{5}{2} \right) = 28320.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
c_9 &= \frac{c_3^3}{3!} + c_3c_6 - 2^7 \left(\frac{1}{8}(\# e \in \mathcal{H}) + \frac{9}{32} \cdot 2(\# \Gamma_{9,2} \in \mathcal{H}) + \frac{9}{8}(\# \Gamma_{9,3} \in \mathcal{H}) + \frac{81}{128}(\# \Gamma_{9,4} \in \mathcal{H}) \right) \\
&= -2190860
\end{aligned}$$

and

$$\begin{aligned}
c_{12} &= \frac{c_3^4}{4!} + \frac{c_3^2 c_6}{2!} + c_3 c_9 + 2^7 \Bigg(\frac{3}{32}(\# e \in \mathcal{H}) + \frac{9}{32} \cdot 2(\#\Gamma_{12,1} \in \mathcal{H})\frac{27}{64}(\#\Gamma_{12,2} \in \mathcal{H}) \\
&\quad + \frac{81}{128} \cdot 3(\#\Gamma_{12,3} \in \mathcal{H}) + \frac{63}{32} \cdot 3(\#\Gamma_{12,4} \in \mathcal{H}) + \frac{459}{64}(\#\Gamma_{12,5} \in \mathcal{H}) + \frac{255}{16}(\#\Gamma_{12,6} \in \mathcal{H}) \Bigg) \\
&= 125012034.
\end{aligned}$$

6 The Associated Coefficient of a k-uniform Simplex

Fix $k \geq 2$ and let

$$K_{k+1}^{(k)} = \left([k+1], \binom{[k+1]}{k} \right)$$

be the k-uniform simplex. It was shown in [4] that the k-uniform simplex is the only connected Veblen hypergraph with $k+1$ edges, up to isomorphism. Moreover, it was shown for a k-graph $\mathcal{H}$

$$\phi_{k+1}(K_{k+1}^{(k)}) = -(k-1)^{n-k} C_k(\# H \subseteq \mathcal{H})$$

	FP-2	FP-1	FP
c_0	1	1	1
c_1	0	0	0
c_2	0	0	0
c_3	-240	-288	-336
c_4	0	0	0
c_5	0	0	0
c_6	28320	40788	55524
c_7	0	0	-696
c_8	0	0	0
c_9	-2190860	-3788016	-6017746
c_{10}	0	0	220038
c_{11}	0	0	0
c_{12}	125012034	259553826	481293561
c_{13}	0	0	-34237560
c_{14}	0	0	-122004
c_{15}	5612445168	-13997317932	-30303162330

Figure 3: Leading coefficients of the Fano plane and its subgraphs.

for some constant C_k depending only on k. The authors of [4] were able to show that $C_2 = 2$, $C_3 = 21$, $C_4 = 588$, $C_5 = 28230$ via laborious use of resultants. In this section we provide an explicit, efficient formula for C_k and use it to compute C_{100}. To that end, we first describe the set of rootings of the k-uniform simplex. The following lemma imposes a necessary condition on rootings.

Lemma 15 *Let H be a Veblen k-graph. Fix $t \in \mathbb{N}$ and d such that $tk \leq d < (t+1)k$. Let $\hat{f}_{d_1,d_2,\ldots,d_n}$ be a differential operator of $\mathrm{Tr}_d(H)$. If $d_i > t$ then*

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = 0.$$

In particular, if $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ then $d_i \leq \lfloor \frac{d}{k} \rfloor$ and this bound is sharp.

Proof: Consider $\mathrm{Tr}_d(H)$ and fix a differential operator $\hat{f}_{d_1,d_2,\ldots,d_n}$. Observe that for $D = D_{\hat{f}}$ we have $\deg^+_D(i) = (k-1)d_i$ and $\deg^-_D(i) \leq d - d_i$. Without loss of generality, suppose $d_1 > t$. Then

$$\deg^-_D(1) \leq d - d_1 \leq d - (t+1) < (t+1)k - (t+1) = (t+1)(k-1) \leq \deg^+_D(1).$$

By Euler's theorem D does not have an Euler circuit. Appealing to Lemma 4,

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = 0$$

and the first statement follows.

Fix $tk \leq d < (t+1)k$ and suppose A_α survives $\hat{f}_{d_1,d_2,\ldots,d_n}$. As $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ we have by our first statement $d_i \leq t \leq \lfloor d/k \rfloor$. To see that this bound is sharp, fix

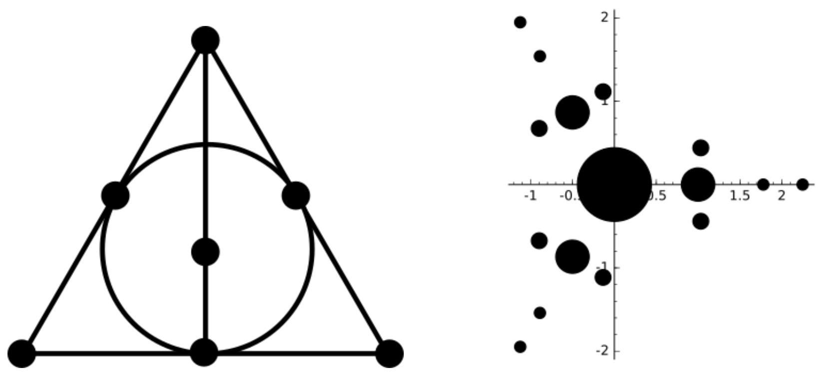

Figure 4: The Rowling hypergraph and its spectrum, visualized in the complex plane by centering a disk at each eigenvalues whose area is proportional to its multiplicity.

$t \in \mathbb{N}$ and $k \geq 2$. Set $d := kt$ and consider $\mathrm{Tr}_d(H)$ where $H = ([k], \{d \times [k]\})$ is the k-uniform edge with multiplicity d. Choosing $d_i = t$ for each $i \in [k]$ yields precisely one differential operator $\hat{f}$ as

$$f_i = d\frac{x_1 x_2 \ldots x_k}{x_i}.$$

Observe that D is the complete multi-digraph on k vertices where each pair of vertices has t edges oriented in both directions. By Euler's theorem D has an Euler circuit. Let α describe such a circuit. By our previous claim A_α survives $\hat{f}$ so $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ as desired. ■

Remark 6 *Lemma 15 is most useful when d is close to k in value. In particular, if $d < 2k$ then $d_i = 1$ for all i.*

For the remainder of this section let H denote the k-uniform simplex $K_k^{(k+1)}$. Observe that the i-th coordinate polynomial of $\mathcal{A}_H \mathbf{x}^{\otimes k-1}$ can be written

$$f_i = \sum_{j \neq i} \frac{x_1 x_2 \ldots x_{k+1}}{x_i x_j}$$

for $i \in [k+1]$. Let $\mathfrak{D}_{k+1}$ be the set of derangements of $[k+1]$, i.e., permutations without any fixed points. Recall that $S_e(u)$ is the u-rooted directed star of e.

Lemma 16 *For $\sigma \in \mathfrak{D}_{k+1}$ define*

$$D_\sigma = \bigcup_{i=1}^{k+1} S_{[k+1]\setminus\{\sigma(i)\}}(i)$$

Then

$$R(H) = \{D_\sigma : \sigma \in \mathfrak{D}_{k+1}\}.$$

Proof: Let $\sigma \in \mathfrak{D}_{k+1}$ and consider D_σ. We suppress the subscript and write D for convenience. Note that D is Eulerian whence

$$\deg_D^+(j) = (k-1)|\{i : v_i = j\}| = k - 1 = |\{i : v_i \neq j, j \in e_i\}| = \deg_D^-(j)$$

for all $j \in [k+1]$ as there are exactly k edges which contain j and σ is a derangement. Suppose $D \in R(H)$. As D is Eulerian we have by Lemma 15

$$D = \bigcup_{i=1}^{k+1} S_{[k+1]\setminus\{v_i\}}(i).$$

In particular, $\{v_1, \ldots, v_{k+1}\} = [k+1]$. Observe that $\sigma = \{(i, v_i)\}_{i=1}^{k+1} \in \mathfrak{D}_{k+1}$ is a derangement. The conclusion follows from the fact that $D = D_\sigma$. ■

This immediately implies the following.

Lemma 17 *For the k-uniform simplex H,*

$$C_H = \sum_{\sigma \in \mathfrak{D}_{k+1}} \frac{\tau_\sigma}{\prod_{v \in V(D_\sigma)} \deg^-(v)}.$$

We now show that a summand in the aformentioned formula of C_H depends only on the cycle type of the derangement.

Theorem 18 *Let $\sigma \in \mathfrak{D}_{k+1}$ with cycle decomposition $c_1c_2\ldots c_t$ where cycle c_i has length ℓ_i. Then*

$$C_H = \frac{1}{(k-1)^{k+1}(k+1)} \sum_{\sigma = c_1c_2\ldots c_t \in \mathfrak{D}_{k+1}} \prod_{i=1}^{t} \left(k^{\ell_i} + (-1)^{\ell_i + 1}\right).$$

We first prove a technical lemma. The notation spec(M) denotes the ordinary (multiset) spectrum of a matrix M.

Lemma 19 *For $\sigma \in S_{n+1}$, $spec(M_\sigma - J) = (spec(M_\sigma) \setminus \{1\}) \cup \{-n\}$ where M_σ is the permutation matrix associated with σ.*

Proof: Let σ be a permutation of $[n+1]$ with cycles $c_1, c_2, \ldots, c_t$ of length $l_1, l_2, \ldots, l_t$, respectively. Recall that the spectrum of M_σ is given by

$$\text{spec}(M_\sigma) = \bigcup_{i=1}^{t} \{\zeta_{l_i}^0, \zeta_{l_i}^1, \ldots, \zeta_{l_i}^{l_i - 1}\}$$

and note that the spectrum of M_σ depends only on the cycle type of σ. Without loss of generality, suppose that the cycles of σ are increasing (i.e., $[1, \ell_1], [\ell_1 + 1, \ell_2 + \ell_1], \ldots$). Consider the following block partition

$$M_\sigma - J_{n+1} = \begin{pmatrix} B_1 & -J_{\ell_2} & \ldots & -J_{\ell_t} \\ -J_{\ell_1} & B_2 & \ldots & -J_{\ell_n} \\ \vdots & \vdots & \ddots & \vdots \\ -J_{\ell_1} & -J_{\ell_2} & \ldots & B_t \end{pmatrix}$$

where B_i is the $l_i \times l_i$ square circulant matrix corresponding to c_i,

$$B_i = \begin{pmatrix} -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 \\ 0 & -1 & -1 & \dots & -1 \end{pmatrix}.$$

Let $j > 0$ and consider the eigenpair $(\zeta_{\ell_i}^j, \mathbf{x})$ of M_σ. We compute

$$(M_\sigma - J)\mathbf{x} = M_\sigma \mathbf{x} - J\mathbf{x} = \zeta_{\ell_i}^j \mathbf{x} - \mathbf{0} = \zeta_{\ell_i}^j \mathbf{x}$$

where $J\mathbf{x} = 0$ because the coordinates of x corresponding to C_i are the complete set of ℓ_i-th roots of unity.

We now show $1 \in \operatorname{spec}(M_\sigma - J)$ has a geometric multiplicity of $t-1$. Fix $1 \le i \le t-1$ and consider $\mathbf{x} \in \mathbb{C}^{n+1}$ where

$$\mathbf{x}_j = \begin{cases} 1 & : j \in C_i \\ -\frac{\ell_i}{\ell_{i+1}} & : j \in C_{i+1} \\ 0 & : \text{otherwise.} \end{cases}$$

For $j \in C_i$ we have,

$$((M_\sigma - J)\mathbf{x})_j = -(\ell_i - 1) + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1}) = 1 = \mathbf{x}_j,$$

for $j \in C_{i+1}$ we have,

$$((M_\sigma - J)\mathbf{x})_j = -\ell_i + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1} - 1) = -\frac{\ell_i}{\ell_{i+1}} = \mathbf{x}_j,$$

and for $j \notin C_i, C_{i+1}$ we have

$$((M_\sigma - J)\mathbf{x})_j = -\ell_i + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1}) = 0 = \mathbf{x}_j.$$

Therefore, $(1, \mathbf{x})$ is an eigenpair for $1 \le i \le t$; moreover, these vectors are linearly independent.

Finally, consider the all-ones vector $\mathbf{1} \in \mathbb{C}^{n+1}$ where

$$(M_\sigma - J)\mathbf{1} = -n\mathbf{1}$$

so $(-n, \mathbf{1})$ is an eigenpair of $M_\sigma - J$. We have shown

$$\operatorname{spec}(M_\sigma - J) \supseteq (\operatorname{spec}(M_\sigma) \setminus \{1\}) \cup \{-n\}$$

and the reverse inclusion follows from the observation that the multiplicities on the right-hand side add up to (at least, and therefore exactly) $n+1$. ■

We now prove Theorem 18.

Proof: Consider
$$C_H = \sum_{\sigma \in \mathfrak{D}_{k+1}} \frac{\tau_\sigma}{\prod_{v \in V(D_\sigma)} \deg^-(v)}.$$
Applying Lemma 15 (choosing $d = k+1$) implies that each vertex of H is a root of exactly one edge in an Euler rooting of H. It follows that, for all $\sigma \in \mathfrak{D}_{k+1}$,
$$\prod_{v \in V(D_\sigma)} \deg^-(v) = (k-1)^{k+1}.$$
Thus,
$$C_H = \frac{1}{(k-1)^{k+1}} \sum_{\sigma \in \mathfrak{D}_{k+1}} \tau_\sigma.$$
Consider $\sigma = c_1 c_2 \dots c_t$ where cycle c_i has length ℓ_i. Observe that
$$\mathcal{L}(D_\sigma) = kI + M_\sigma - J.$$
Since $\operatorname{spec}(kI + M_\sigma - J) = k + \operatorname{spec}(M_\sigma - J)$ we have by Lemma 19 and Kirchoff's theorem
$$\tau_\sigma = \frac{\prod_{i=1}^{t} \left(k^{l_i} + (-1)^{l_i+1}\right)}{k+1}.$$
The desired equality follows by substitution. ∎

Remark 7 *It was shown in [4] that*
$$\phi_{k+1}(H) = -C_k (k-1)^{n-k}$$
and by Theorem 14 we have
$$\phi_{k+1}(H) = -C_H (k-1)^n.$$
In our notation, we can write $C_k = (k-1)^k C_H$*.*

For ease of computation we consider C_k instead of C_H. As it is stated, Theorem 18 is slow to compute as $|\mathfrak{D}_{n+1}| \sim n!/e$. However, summing over all derangements of $[k+1]$ is wasteful, as we have shown C_k is a function only of the cycle structure of σ. In fact,
$$\prod_{i=1}^{t} \left(k^{l_i} + (-1)^{l_i+1}\right)$$
is constant for derangements with the same cycle type. We present a reformulation of Theorem 18 which has the advantage of considering a smaller search space.

Definition 12 *Let $P(n)$ be the set of partitions of n and let $P_{\geq 2}(n) \subseteq P(n)$ be the set of partitions of n into parts of size at least 2. For $p \in P_{\geq 2}(k+1)$, let $\mathfrak{D}_{k+1}(p) \subseteq \mathfrak{D}_{k+1}$ be the set of derangements whose cycle lengths agree with the parts of p. Further, for a partition $p \in P(n)$ let $V_p : [n] \to [0, n]$ be the map $V_p(i) = |\{j : p_j = i\}|$.*

We reformulate Theorem 18 (for C_k) as follows.

Corollary 20

$$C_k = \frac{1}{(k-1)(k+1)} \sum_{p=(p_1,\ldots,p_t)\in P_{\geq 2}(k+1)} \left(|\mathfrak{D}_{k+1}(p)| \prod_{i=1}^{t} \left(k^{p_i} + (-1)^{p_i+1}\right) \right)$$

where

$$|\mathfrak{D}_{k+1}(p)| = \frac{(k+1)!}{\left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)\right)}.$$

Proof: It is sufficient to show

$$|\mathfrak{D}_{k+1}(p)| = \frac{(k+1)!}{\left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)!\right)}$$

for $p = (p_1, \ldots, p_t) \in P_{\geq 2}(k+1)$. Let $\Delta : \mathfrak{S}_{k+1} \to \mathfrak{D}_{k+1}(p)$ by

$$\Delta(\sigma) = (\sigma(1), \sigma(2), \ldots, \sigma(p_1))(\sigma(p_1+1), \ldots, \sigma(p_1+p_2)) \ldots.$$

Note that Δ is surjective. Since a cycle of $\Delta(\sigma)$ can be written with any one of its p_i elements first and there are $\prod_{i=2}^{k+1} V_p(i)!$ linear orderings of the cycles by non-increasing length we have for $\delta \in \mathfrak{D}_{k+1}$

$$|\Delta^{-1}(\delta)| = \left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)!\right).$$

In particular, $|\Delta^{-1}(\delta)|$ is constant for $\delta \in \mathfrak{D}_{k+1}(p)$ so that

$$|\mathfrak{S}_{k+1}| = \sum_{\delta\in\mathfrak{D}_{k+1}(p)} |\Delta^{-1}(\delta)|$$

which implies

$$|\mathfrak{D}_{k+1}(p)| = \frac{|\mathfrak{S}_{k+1}|}{|\Delta^{-1}(\delta)|}.$$

∎

Remark 8 *Corollary 20 reduces the number of summands in the computation of* C_k *exponentially because*

$$\log |P_{\geq(2)}(n)| \leq \log |P(n)| \approx \pi\sqrt{2n/3},$$

but $\log |\mathfrak{D}_{n+1}| \approx n \log n$.

We have computed the first few values of C_k to be

$$
\begin{aligned}
C_6 &= 2092206 \\
C_7 &= 220611384 \\
C_8 &= 31373370936 \\
C_9 &= 5785037767440 \\
C_{10} &= 1342136211324090 \\
&\vdots \\
C_{100} &= 343345241982479590844776717586346303452689609890358711139013913758 \\
&\quad 779957888170716788865639598053642953208929209278848309297069686374206 \\
&\quad 618031496101898485314300253248855334075609527915686375386625810970778 \\
&\quad 814182546067369319275314946445603388115577892354872286012782651661555 \\
&\quad 310652736903712206018668653541524263903668524799914172228056595466145 \\
&\quad 2080249009900
\end{aligned}
$$

so that $C_{100} \approx 3.433... \cdot 10^{343}$, see A320653 [13]. Note that $C_2 = 2$ gives the well-known result that, for a graph G

$$\phi_3(G) = -2(\#\text{ of triangles in } G).$$

as discussed in the previous section. We conclude by presenting the asymptotics of C_k.

Theorem 21 *$C_k \sim (k+1)!k^{k+1}$ so $C_k = \exp(k \log k(2 + o(1))$.*

Proof:

As σ is a derangement it follows that the length of each cycle is at least 2. Notice

$$|\mathfrak{D}_{k+1}|(k^2-1)^{\frac{k+1}{2}} \le \sum_{\sigma = c_1 c_2 \dots c_t \in \mathfrak{D}_{k+1}} \left(\prod_{i=1}^{t} k^{l_i} + (-1)^{l_i+1} \right) \le |\mathfrak{D}_{k+1}|(k^3-1)^{\frac{k+1}{e}}$$

which implies

$$\sum_{\sigma = c_1 c_2 \dots c_t \in \mathfrak{D}_{k+1}} \left(\prod_{i=1}^{t} k^{l_i} + (-1)^{l_i+1} \right) \sim \frac{(k+1)!k^{k+1}}{e}.$$

Thus,

$$\lim_{k\to\infty} \frac{C_k}{(k+1)!k^{k+1}} = \lim_{k\to\infty} \frac{|\mathfrak{D}_{k+1}|k^{k+1}}{(k+1)!k^{k+1}} = \frac{1}{e}$$

and we have $C_k \sim (k+1)!k^{k+1}$. ∎

7 Deducing the Harary-Sachs Theorem for Graphs

With Theorem 14 in hand we can express the codegree-d coefficient of the normalized adjacency characteristic polynomial of a hypergraph as a weighted sum of Veblen infragraphs with *d edges*. In the case when G is a 2-graph our theorem simplifies to

$$\phi_d(G) = \sum_{H \in \mathcal{V}_d^*} (-1)^{c(H)} C_H (\# H \subseteq G).$$

Recall that the Harary-Sachs theorem (Theorem 1) expresses the codegree-d coefficient as a weighted sum over certain subgraphs on *d vertices*, whereas Theorem 14 expresses the same quantity as a weighted sum over certain subgraphs with *d edges*. We now argue that these two sums are equal.

An *elementary subgraph* of a graph G, is a simple subgraph of G whose components are edges or cycles (see [1] for further details). In keeping with their notation, let $\Lambda_d(G)$ be the set of elementary subgraphs of G with d vertices. Notice that a connected elementary graph is the flattening of a cycle (e.g., the flattening of a 2-cycle is an edge). Recall that cycles (and disjoint unions of cycles) are the only two regular non-empty graphs which have an equal number of vertices and edges. Indeed

$$\Lambda_d(G) \subseteq \{\underline{H} : H \in \mathcal{V}_d^*(G)\}.$$

By straightforward computation we have that the associated coefficient of a 2-cycle is 1 and the associated coefficient of a simple cycle (i.e., any cycle which is not a 2-cycle) is 2. Restricting our attention to $\Lambda_d(G)$, we have by Theorem 14

$$\sum_{H \in \Lambda_d(G)} (-1) C_H (\# H \subseteq G) = (-1)^{c(H)} 2^{z(H)} (\# H \subseteq G) \tag{4}$$

where $z(H)$ is the number of cycles in H. Note that Equation 4 is the conclusion of the Harary-Sachs theorem. We deduce the Harary-Sachs theorem from Theorem 14 by showing that the summands of

$$\phi_d(G) = \sum_{H \in \mathcal{V}_d^*} (-1)^{c(H)} C_H (\# H \subseteq G)$$

which do not arise from elementary graphs sum to zero. We make this statement precise with the following.

Definition 13 *For a multigraph G and an edge $e \in E(G)$, write $m(e) = m_G(e)$ for the multiplicity of e in $E(G)$. Let G be a connected, labeled Veblen graph with distinguishable multi-edges. Given a multiset P of multigraphs whose multi-edges are indistinguishable, each on the vertex set $V(G)$, we write $P \vdash G$ if, for each $e \in E(G)$*

$$\sum_{P_i \in P} m_{P_i}(e) = m_G(e).$$

Lemma 22

$$\sum_{P \vdash G} (-1)^{c(P)} C_P = \begin{cases} 1 & : G \text{ is a 2-cycle} \\ 2 & : G \text{ is a simple cycle} \\ 0 & : \text{otherwise.} \end{cases}$$

Remark 9 *We refer to Veblen 2-graphs as Veblen graphs. An* Euler orientation *of a graph G is an orientation of the edges of G such that the resulting digraph is Eulerian.*

We first provide a combinatorial formula for the associated coefficient of a Veblen graph.

Lemma 23 *Let G be a connected Veblen graph. We have*

$$C_G = \frac{|\mathfrak{E}(G)|}{\prod_{e \in E(G)} m(e)!}.$$

Proof: Let G be a Veblen graph. Since G is Eulerian, we write $\deg(v) = 2d_v$ for convenience. Because $\deg^-(v) = d_v$ for all $D \in R(G)$ we have

$$C_G = \frac{1}{\prod_{v \in V(G)} d_v} \sum_{D \in R(G)} \tau_D.$$

By the BEST theorem (i.e, Theorem 5) we have

$$\tau_D = \frac{|\mathfrak{E}(D)|}{\prod_{v \in V(G)} (d_v - 1)!}.$$

Let $N_D(v)$ denote the out-neighborhood of v in D and let $\deg_D(v, u)$ denote the number of edges directed from v to u in D. We denote

$$\binom{d_v}{N_D(v)} = \frac{d_v!}{\prod_{u \in N_D(v)} \deg_D(v, u)!}$$

which is the number of linear orderings of out-edges of v in an Euler orientation D. Consider the equivalence relation $\sim_{R(G)}$ where $R \sim R'$ if and only if $D_R = D_{R'}$. Note that $\sim$ identifies two Euler rootings if their associated digraphs are the same Euler orientation. Let $[R]$ denote the equivalence class of R under $\sim$. Suppose $R(G) = \bigcup_{i=1}^{t} [R_i]$ and note

$$|[R]| = \prod_{v \in V(G)} \binom{d_v}{N_{D_R}(v)}$$

as two rootings in $[R]$ differ only in the ordering of the u-rooted stars for $u \in V(G)$. Let $\mathcal{O}(G) = \{D_1, \ldots, D_t\}$, where $D_i = D_{R_i}$, denote the Euler orientations of G. For convenience, we write N_i for N_{D_i}. We equate

$$\sum_{D \in R(G)} \tau_D = \sum_{i=1}^{t} \tau_i |[R_i]| = \sum_{i=1}^{t} \left(\tau_i \prod_{v \in V(G)} \binom{d_v}{N_i(v)} \right).$$

Substitution and simplification yields

$$C_G = \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v \in V(G)} \left(\prod_{u \in N_i(v)} \deg_i(v,u)! \right)}.$$

Since $\deg_i(u,v) + \deg_i(v,u) = m(uv)$ we have

$$\binom{m(uv)}{\deg_i(u,v)} = \frac{m(uv)!}{\deg_i(u,v)! \deg_i(v,u)!}$$

so that

$$\begin{aligned}
\frac{\prod_{e \in E(G)} m(e)!}{\prod_{v \in V(G)} \prod_{u \in V(G)} \deg_i(v,u)!} &= \frac{\prod_{e \in E(G)} m(e)!}{\prod_{u<v \in V(G)} \deg_i(u,v)! \deg_i(v,u)!} \\
&= \prod_{uv \in E(G), u<v} \binom{m(uv)}{\deg_i(u,v)}.
\end{aligned}$$

Then

$$\begin{aligned}
C_G &= \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v \in V(G)} \left(\prod_{u \in N_i(v)} \deg_i(v,u)! \right)} \\
&= \frac{\prod_{e \in E(G)} m(e)!}{\prod_{e \in E(G)} m(e)!} \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v \in V(G)} \left(\prod_{u \in N_i(v)} \deg_i(v,u)! \right)} \\
&= \frac{1}{\prod_{e \in E(G)} m(e)!} \sum_{i=1}^{t} \left(\prod_{uv \in E(G), u<v} \binom{m_G(uv)}{\deg_i(u,v)} \right) |\mathfrak{E}(D_i)| \\
&= \frac{|\mathfrak{E}(G)|}{\prod_{e \in E(G)} m(e)!}
\end{aligned}$$

where the last equality follows from the observation that D_i has indistinguishable multi-edges. ■

We now prove Lemma 22.

Proof: Let G be a connected Veblen graph which is not a cycle. Further assume that the multi-edges of G are distinguishable. We aim to show

$$\sum_{P \vdash G} (-1)^{c(P)} C_P = 0.$$

By Lemma 23 we have for connected G,

$$C_G = \frac{|\mathfrak{E}(G)|}{\prod_{e \in E(G)} m(e)!}.$$

Let $P = \bigcup_{i=1}^{t} P_i$ be a disjoint union of Veblen graphs. We denote

$$\binom{m_G(e)}{P(e)} = \frac{m_G(e)!}{\prod_{i=1}^{t} m_{P_i}(e)!}$$

and

$$|\mathfrak{E}(P)| = \prod_{i=1}^{t} |\mathfrak{E}(P_i)|.$$

We equate

$$\left(\prod_{e\in E(G)} m(e)!\right) \sum_{P\vdash G} (-1)^{c(P)} C_P = \sum_{P\vdash G} (-1)^{c(P)} |\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)}.$$

Notice that $|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)}$ counts the number of partitions of $E(G)$ into edge-disjoint Euler circuits of graphs on $V(G)$ with unlabelled edges which are precisely the elements of P.

We say that an Euler circuit is *decomposable* if it can be written as a union of (at least) two edge disjoint Euler circuits, and *indecomposable* otherwise. The number of decompositions of Euler circuits of G into exactly t parts is

$$\sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right).$$

By Inclusion/Exclusion, the number of indecomposable Euler circuits of G is

$$\sum_{t=1} (-1)^t \sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right).$$

We assumed that G is a Veblen graph which is not a cycle. We have by Veblen's theorem (i.e., Theorem 11) that every Euler circuit in G is decomposable. It follows that G has no indecomposable Euler circuits; that is to say,

$$\sum_{t=1} (-1)^t \sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right) = 0,$$

from which the desired conclusion follows. ∎

A characterization of the multiplicity of the zero eigenvalue for the adjacency characteristic polynomial of a graph remains open. For convenience let m_0 denote the multiplicity of the zero eigenvalue for a given polynomial. Notice that one can provide an upper bound on m_0 by showing that a particular coefficient of $\phi(\mathcal{H})$ is non-zero. Combining this idea with the Harary-Sachs theorem gives the only known result in this direction for the adjacency characteristic polynomial: if T is a (2-uniform) tree then m_0 is the size of the largest matching of T. The authors ask if a similar result holds true for hypergraphs.

In our proof of Lemma 22 we showed that, for 2-graphs, the summands in $\phi_d(G)$ arising from Veblen graphs which are not elementary graphs necessarily summed to zero. We define the *coefficient threshold* of $\phi(\mathcal{H})$ as the least co-degree at which the coefficients of $\phi(\mathcal{H})$ cancel thusly:

Definition 14 *For an integer $v \geq 0$, the* coefficient v-threshold *of a k-graph $\mathcal{H}$, denoted $Th_v(\mathcal{H})$, is the least integer such that for $d > Th_v(\mathcal{H})$*

$$\sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-((k-1)^v)^{c(H)}) C_H(\#H \subseteq \mathcal{H}) = 0.$$

Note that the contribution of $\mathcal{H}$ to the codegree d coefficient of $\phi(\mathcal{G})$ for $\mathcal{H} \subseteq \mathcal{G}$ is zero if $d > \mathrm{Th}_v(\mathcal{H})$ where $v = |V(\mathcal{G})|$. As an example, we show that the v-threshold of the 3-uniform edge is $9 \cdot 2^{v-3}$.

Lemma 24 *Let e be the 3-uniform edge. Then $Th_v(e) = 9 \cdot 2^{v-3}$ for $v \geq 3$.*

Proof: For $n \geq 0$ define $f_n(t) : \mathbb{Z}^+ \to \mathbb{Q}$ by $f_n(0) = 1$ and

$$f_n(t) = \sum_{H \in \mathcal{V}_{3t}^*(e)} (-(2^n)^{c(H)}) C_H(\#H \subseteq e), t > 0.$$

Observe that $f_3(t) = \phi_{3t}(e)$ by Theorem 14. We conclude by showing $f_n(t) = (-1)^t \binom{3 \cdot 2^{n-3}}{t}$. Considering the characteristic polynomial of a single 3-uniform edge,

$$f_3(0) = 1,\ f_3(1) = -3,\ f_3(2) = 3,\ f_3(3) = -1,\ f_3(4) = 0$$

and $f_3(t) = 0$ for $t > 4$. Indeed $f_3(t) = (-1)^t \binom{3}{t}$ for all t. Suppose that for all n, up to some fixed n, we have $f_n(t) = (-1)^t \binom{3 \cdot 2^{n-3}}{t}$ for all t. Consider $f_{n+1}(t)$. We claim

$$f_{n+1}(t) = \sum_{j=0}^{t} f_n(j) f_n(t-j).$$

Recall that the associated coefficient is multiplicative over components. It follows that

$$\begin{aligned}\sum_{j=0}^{t} f_n(j) f_n(t-j) &= \sum_{H \in \mathcal{V}_3^*(e)} \left(\sum_{\substack{H = H_1 \cup H_2 \\ H_1 \cap H_2 = \emptyset}} (-(2^n))^{c(H_1)} C_{H_1} \cdot (-(2^n))^{c(H_2)} C_{H_2} \right) \\ &= \sum_{H \in \mathcal{V}_3^*(e)} 2^n (-(2^n))^{c(H)} C_H = f_{n+1}(t).\end{aligned}$$

We have

$$f_{n+1}(t) = (-1)^t \sum_{j=0}^{t} \binom{3 \cdot 2^{n-3}}{j} \binom{3 \cdot 2^{n-3}}{t-j} = (-1)^t \binom{3 \cdot 2^{n-2}}{t}$$

where the first equality follows from the inductive hypothesis and the second equality is given by the Chu-Vandermonde identity [2]. As $\mathrm{Th}_v(e) = f_v(t)$ we have that $\mathrm{Th}_v(e) = 9 \cdot 2^{n-3}$ as $f_v(3 \cdot 2^{n-3}) = \pm 1$ for $t = 3 \cdot 2^{n-2}$ and $f_v = 0$ for $t > 3 \cdot 2^{n-2}$. ■

Conjecture 1 *If $\mathcal{H} \subseteq \mathcal{G}$ are k-graphs, with $k > 2$ where $|V(\mathcal{G})| = n$ then $Th_n(\mathcal{H}) \leq Th_n(\mathcal{G})$.*

This conjecture implies that the multiplicity of the zero eigenvalue is at most $\deg(\phi(\mathcal{G})) - \mathrm{Th}_n(\mathcal{H})$. Note that the restriction of $k > 2$ is necessary as the conjecture is not true for graphs. For example, $C_6 \subseteq K_{6,6}$ and $\mathrm{Th}(K_{6,6}) = 2 < \mathrm{Th}(C_6) = 6$. More generally, it would help our understanding of the multiplicity of 0 to have a better understanding of $\mathrm{Th}_v(\mathcal{H})$, and so we ask:

Question 1 *Show how to compute or estimate $Th_v(\mathcal{H})$ for various hypergraphs $\mathcal{H}$.*

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