Geometric vs Algebraic Nullity for Hyperpaths

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Abstract

We consider the question of how the eigenvarieties of a hypergraph relate to the algebraic multiplicities of their corresponding eigenvalues. Specifically, we (1) fully describe the irreducible components of the zero-eigenvariety of a loose 3-hyperpath (its "nullvariety"), (2) use recent results of Bao-Fan-Wang-Zhu to compute the corresponding algebraic multiplicity of zero (its "nullity"), and then (3) for this special class of hypergraphs, verify a conjecture of Hu-Ye about the relationship between the geometric (multi-)dimension of the nullvariety and the nullity.

1 Introduction

We begin with two questions:

1. What is the combinatorial meaning of the multiplicity of the zero eigenvalue of a (hyper)graph?
2. What is the relationship between the various notions of "multiplicity" for an eigenvalue?

One may combine these two questions by asking, "What is the combinatorial meaning of *each notion* of the multiplicity of the zero eigenvalue of (hyper)graphs?" For the Laplacian matrix $L(G) = D(G) - A(G)$ of a graph, in the 1970s, Fiedler showed that the multiplicity – in both the algebraic and geometric senses – of the zero eigenvalue is equal to the number of components of G. Thus it is natural to ask this same question about the seemingly simpler adjacency matrix $A(G)$, and indeed considerable attention has been given to Question 1 (e.g., [4, 6, 7, 10, 12]). Because $A(G)$ is real symmetric and therefore diagonalizable, the answer to Question 2 is simple for a graph, however: they agree.

In contrast, these questions are nearly untouched for hypergraphs. The first question has been investigated for some special graphs – for example, [1] implicitly provides an algorithm for computing the algebraic multiplicity of zero as an eigenvalue of a hypergraph. In a related vein, [2] analyzes which eigenvectors corresponding to the zero eigenvalue of a subgraph of G are also such "null eigenvectors" for G. The second question is also almost entirely unexplored for hypergraphs, and Sturmfels observed (see [8]) that the relatively straightforward linear eigenspaces of matrices become complicated "eigenvarieties" when one passes to adjacency tensors/hypermatrices to study hypergraphs. Hu and Ye [8] take up this matter in earnest and pose a conjecture about the relationship between the (multi-)dimension of such varieties and their multiplicities as roots of a hypermatrix's characteristic polynomial; these are natural choices for analogizing "geometric" and "algebraic" multiplicity, respectively, and the conjecture is an attempt to generalize the fact that the geometric multiplicity of a matrix eigenvalue is bounded above by its algebraic multiplicity. Another notable contribution [5] by Fan-Bao-Huang investigated properties of the eigenvariety associated with the spectral radius of a hypergraph (and, more generally, certain hypermatrices/tensors).

The aforementioned Hu-Ye Conjecture can be stated as follows; definitions follow below. Let $\mathrm{am}(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of the hypermatrix M. Let $V_\lambda^1, \ldots, V_\lambda^\kappa$ denote the irreducible components of V_λ, the eigenvariety correponding to λ.

Conjecture 1.1 ([8])**.** *For any order-k hypermatrix M, define*

$$\mathrm{gm}(\lambda) := \sum_{j=1}^{\kappa} \dim(V_\lambda^j)(k-1)^{\dim(V_\lambda^j)-1}$$

Then $\mathrm{gm}(\lambda) \leq \mathrm{am}(\lambda)$*.*

Here we verify this for the zero eigenvalue of a simple class of 3-uniform hypergraphs – sometimes called "loose paths" or "linear hyperpaths" – by obtaining an explicit description of the irreducible components of their nullvarieties, using this to obtain a generating function that encodes said irreducible components' dimensions, using results from [1] to obtain an explicit expression for the multiplicity of zero as a root of their characteristic polynomials, and comparing the resulting quantities to confirm the conjecture in this special case.

We briefly define the multilinear algebra and spectral hypergraph theory terminology and notation used throughout the paper. More detailed information and references can be found in [3, 5]. An order-k *hypermatrix*[1] M over a ring R is a k-dimensional array of values $M_{i_1 \cdots i_k} \in R$ (usually $R = \mathbb{C}$), which we often identify with the function $M : (i_1, \ldots, i_k) \mapsto M_{i_1 \cdots i_k}$. A hypermatrix is *cubical* if the i_j, $j = 1, \ldots, k$, all belong to the same index set $\mathcal{I}$, in which case we say that its *dimension* is $|\mathcal{I}|$, and a cubical hypermatrix is *symmetric* if, for every permutation σ of $\mathcal{I}$ and $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$, $M_\mathbf{i} = M_{\sigma(\mathbf{i})}$, where $\sigma(\mathbf{i}) = (\sigma(i_1), \ldots, \sigma(i_k))$. An order-$k$ cubical hypermatrix M of dimension n over R gives rise to a homogeneous k-form Mx^k, where $x = (x_1, \ldots, x_n)$, given by $\sum_{\mathbf{i} \in [n]^k} M_\mathbf{i} x^\mathbf{i}$, where $x^\mathbf{i}$ denotes $\prod_{j=1}^k x_{i_j}$ if $\mathbf{i} = (i_1, \ldots, i_k)$. The *symmetric hyperdeterminant* $\det(M)$ of a symmetric hypermatrix M over $R = \mathbb{C}[\{x_\mathbf{i}\}_{\mathbf{i} \in [n]^k}]$ is the unique monic irreducible polynomial over R which vanishes if and only if $\nabla(Mx^k) = \mathbf{0}$ for some nonzero vector $x \in \mathbb{C}^n$. The *identity* hypermatrix I of rank k and order n is the function so that $I(i_1, \ldots, i_k) = 1$ if $i_1 = \cdots = i_k$ and 0 otherwise. Write λM for the hypermatrix whose $\mathbf{i}$ entry is $\lambda M_\mathbf{i}$ for each valid multi-index $\mathbf{i}$. Then the *characteristic polynomial* of M is $\phi_M(\lambda) := \det(\lambda I - \lambda M) \in \mathbb{C}[\lambda]$. The (homogeneous)

[1] Variously known as a "tensor" in some literature.

spectrum of M is the multiset of roots of $\phi_M(\lambda)$; the elements λ of the adjacency spectrum of M are referred to as *eigenvalues* of M, and any nonzero x so that $\nabla[(M - \lambda I)x^k] = 0$ is a *corresponding eigenvector*. The set of all eigenvectors corresponding to an eigenvalue λ of a hypermatrix M of dimension n is its λ-eigenvariety $\mathcal{V}_\lambda$. Then $\mathcal{V}_\lambda$ is an affine algebraic variety in $\mathbb{C}^n$; indeed, since the equations defining eigenvectors are homogeneous, $\mathcal{V}_\lambda$ can also be viewed as a projective variety, although we adhere to the affine perspective presently. The multiplicity of λ as a root of $\phi_M(\lambda)$ is its *algebraic muliplicity*, and the dimension of the variety $\mathcal{V}_\lambda$ is its *geometric multiplicity*. **Since the 0-eigenvariety of a matrix M – i.e., a hypermatrix of order $k = 2$ – is its nullspace, we refer to the 0-eigenvariety as the *nullvariety* of M. We also refer to the algebraic multiplicity of 0 as the *nullity* of M.**

A (uniform) *hypergraph* $\mathcal{H}$ of rank k is a pair (V, E), where $E \subset \binom{V}{k}$. The *adjacency hypermatrix* of a hypergraph $\mathcal{H}$ is the symmetric hypermatrix $A(\mathcal{H}) : V^k \to \mathbb{C}$ so that $A(\mathcal{H})_{v_1 \cdots v_k}$ is $1/(k-1)!$ if $\{v_1, \ldots, v_k\} \in E(\mathcal{H})$ and 0 otherwise. We will often abuse notation slightly and refer to the multilinear algebraic properties of $A(\mathcal{H})$ by describing them as properties of $\mathcal{H}$ instead. For example, the (adjacency) *spectrum* of a hypergraph $\mathcal{H}$ is the spectrum of $A(\mathcal{H})$, the *nullvariety* of $\mathcal{H}$ is the nullvariety of $A(\mathcal{H})$, and $\phi_{\mathcal{H}}(\lambda) := \phi_{A(\mathcal{H})}(\lambda)$. A loose hyperpath P_n^k is the k-uniform hypergraph on n edges $\{e_1, \ldots, e_n\}$ so that, for $i \neq j$, $|e_i \cap e_j|$ is 1 if $|i - j| = 1$ and 0 otherwise. We label the vertex set $V(P_n^k)$ with $\{v_1, \ldots, v_{(k-1)n+1}\}$ so that $e_j = \{v_{(k-1)(j-1)+1}, \ldots, v_{(k-1)j+1}\}$ for $j \in [n]$.

Throughout, we also write $\mathcal{V}(S)$ for the affine variety over $\mathbb{C}$ defined as the zero locus of the set of polynomials S, and $\mathcal{V}(p)$ for $\mathcal{V}(\{p\})$. Also, given $p \in \mathbb{C}[x_1, \ldots, x_m]$ and a vector $\mathbf{c} \in \mathbb{C}^m$, we will sometimes say "$\mathbf{c}$ satisfies p" if $p(\mathbf{c}) = 0$.

In the next section, we enumerate the irreducible components of the nullvariety of P_n^3 and capture their count and the quantity $\mathrm{gm}(0)$ as a generating function. The following section repeats this exercise, but for the nullity $\mathrm{am}(0)$ of P_n^3 – in fact, more generally P_n^k for $k \geq 3$. The last section compares these two functions of n, verifying the Hu-Ye Conjecture for the zero eigenvalue of P_n^3.

2 Null Variety for Rank-3 Loose Hyperpaths

We examine the "geometric multiplicity" of the zero eigenvalue for a hypergraph $\mathcal{H}$, or more accurately, the multiset of dimensions of irreducible components of the corresponding nullvariety. As a warm-up, and for completeness, we start with the one-edge and two-edge hyperpaths.

2.1 Small Cases

Proposition 2.1. *The 3-uniform hyperedge $\mathcal{H} = P_1^3$ has three irreducible components of dimension 1, and* $\mathrm{gm}(0) = 3$.

Proof. Let the vertices of $\mathcal{H}$ be v_1, v_2, v_3. Given a null vector x, if the adjacency tensor of $\mathcal{H}$ is A, then the ith component of $Ax^{\otimes 2}$ is given by $\sum_{\{i,j,k\}\in E(\mathcal{H})} x_j x_k$. Since x is a null vector, we have $x_1x_2 = x_1x_3 = x_2x_3 = 0$, and we consider the variety $V_0 \subset \mathbb{C}^3$ in three-dimensional affine space defined by these equations. If p, q are polynomials, then $\mathcal{V}(p, q) = \mathcal{V}(p) \cap \mathcal{V}(q)$ and $\mathcal{V}(pq) = \mathcal{V}(p) \cup \mathcal{V}(q)$. Therefore, we have the following.

$$\begin{aligned}\mathcal{V}(x_1x_2, x_1x_3, x_2x_3) &= \mathcal{V}(x_1x_2) \cap \mathcal{V}(x_1x_3) \cap \mathcal{V}(x_2x_3)\\ &= [\mathcal{V}(x_1) \cup \mathcal{V}(x_2)] \cap [\mathcal{V}(x_1) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_2) \cup \mathcal{V}(x_3)]\end{aligned}$$

This is equal to the union over all choices of $\mathcal{V}(x_i) \cap \mathcal{V}(x_j) \cap \mathcal{V}(x_k) = \mathcal{V}(x_i, x_j, x_k)$ where $i \in \{1, 2\}$, $j \in \{1, 3\}$, and $k \in \{2, 3\}$. Thus, maximal subvarieties of V_0 correspond to minimal sets $\{i, j, k\}$ given these conditions, i.e.,

$$[\mathcal{V}(x_1) \cup \mathcal{V}(x_2)] \cap [\mathcal{V}(x_1) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_2) \cup \mathcal{V}(x_3)] = \mathcal{V}(x_1, x_2) \cup \mathcal{V}(x_1, x_3) \cup \mathcal{V}(x_2, x_3).$$

Since $\mathcal{V}(x_i, x_j)$ is the x_k-axis, V_0 is the union of three lines. □

Proposition 2.2. *If $\mathcal{H} = P_2^3$, then V_0 has one component of dimension 1 and another of dimension 3, so that* $\mathrm{gm}(0) = 13$.

Proof. Let the vertices of $\mathcal{H}$ be v_1, v_2, v_3, v_4, v_5. Let x be a null vector. The equations defining $V_0 \subset \mathbb{C}^5$ are $x_1x_3 = x_2x_3 = x_1x_2 + x_4x_5 = x_3x_4 = x_3x_5$. Decompose this system as follows:

$$V_0 = \mathcal{V}(x_1x_3, x_2x_3, x_3x_4, x_3x_5) \cap \mathcal{V}(x_1x_2 + x_4x_5).$$

In the first conjunct, we have intersections of unions, namely

$$\mathcal{V}(x_1x_3, x_2x_3, x_3x_4, x_3x_5) = [\mathcal{V}(x_1) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_2) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_3) \cup \mathcal{V}(x_4)] \cap [\mathcal{V}(x_3) \cup \mathcal{V}(x_5)].$$

Expand the expression on the right to obtain the union over all choices of $\mathcal{V}(x_i) \cap \mathcal{V}(x_j) \cap \mathcal{V}(x_k) \cap \mathcal{V}(x_\ell) = \mathcal{V}(x_i, x_j, x_k, x_\ell)$ where $i \in \{1, 3\}$, $j \in \{2, 3\}$, $k \in \{3, 4\}$, and $\ell \in \{3, 5\}$. The union $\bigcup_{\{i,j,k,\ell\}} \mathcal{V}(x_i, x_j, x_k, x_\ell)$ is the union over the minimal sets $\{i, j, k, \ell\}$ of this form, i.e.,

$$\begin{aligned}&[\mathcal{V}(x_1) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_2) \cup \mathcal{V}(x_3)] \cap [\mathcal{V}(x_3) \cup \mathcal{V}(x_4)] \cap [\mathcal{V}(x_3) \cup \mathcal{V}(x_5)]\\ &\qquad = \mathcal{V}(x_3) \cup \mathcal{V}(x_1, x_2, x_4, x_5).\end{aligned}$$

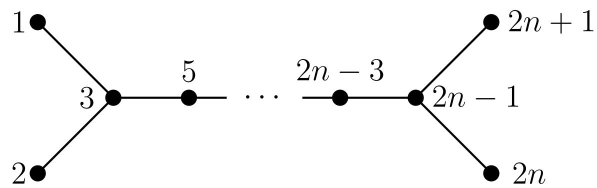
The second variety has dimension four, while the first variety is the x_3 axis. It remains to intersect each such set with $\mathcal{V}(x_1x_2 + x_3x_4)$. Note that $\mathcal{V}(x_1, x_2, x_4, x_5) \subseteq \mathcal{V}(x_1x_2 + x_4x_5)$, so that intersection yields $\mathcal{V}(x_1, x_2, x_4, x_5)$. The intersection of $\mathcal{V}(x_1x_2 + x_4x_5)$ and $\mathcal{V}(x_3)$ gives $\mathcal{V}(x_1x_2 + x_4x_5, x_3)$, which is a three-dimensional variety. Thus,

$$V_0 = \mathcal{V}(x_1, x_2, x_4, x_5) \cup \mathcal{V}(x_1x_2 + x_4x_5, x_3),$$

which is the union of a one-dimensional and a three-dimensional irreducible component. □

2.2 General 3-uniform case

We now generalize the above approach to all 3-uniform loose hyperpaths. Define p_k to be $x_{k-2}x_{k-1} + x_{k+1}x_{k+2}$ for some integer k. For integer $n \geq 1$, define $A'_n := \{2k+1 : 1 \leq k \leq n-1\}$, and let $A_n = A'_n \setminus \{3, 2n-1\}$. Define G_n to be the following graph.



Define $\mathcal{F}_n$ be the collection of "Fibonacci subsets" of A'_n, i.e., sets containing at least one of each two consecutive elements:

$$\mathcal{F}_n = \{S \subset A'_n : \forall k \in [n-2], (2k+1 \in S) \vee (2k+3 \in S)\}$$

Let S be any element of $\mathcal{F}_n$. We say that a set of polynomials $B \subset \{x_i : i \in [2n+1]\} \cup \{p_i : i \in A'_n\}$ is S-admissible if it can be obtained in the following manner. Define U_i, $i = 1, 2, 3, 4$, in the following way.

1. $U_1 = \{x_i : i \in S\}$
2. $U_2 = \begin{cases} \{x_1, x_2\} & \text{if } x_3 \notin U_1, x_5 \in U_1 \\ \{x_1\} \text{ or } \{x_2\} & \text{if } \{x_3, x_5\} \subseteq U_1 \\ \{p_3\} & \text{if } x_3 \in U_1, x_5 \notin U_1 \end{cases}$
3. $U_3 = \begin{cases} \{x_{2n}, x_{2n+1}\} & \text{if } x_{2n-1} \notin U_1, x_{2n-3} \in U_1 \\ \{x_{2n}\} \text{ or } \{x_{2n+1}\} & \text{if } \{x_{2n-3}, x_{2n-1}\} \subseteq U_1 \\ \{p_{2n-1}\} & \text{if } x_{2n-1} \in U_1, x_{2n-3} \notin U_1 \end{cases}$
4. $U_4 = \bigcup_{a \in A} \begin{cases} \emptyset & \text{if } \{x_{a-2}, x_{a+2}\} \subseteq U_1 \\ \{x_{a+1}\} & \text{if } x_{a-2} \in U_1, x_{a+2} \notin U_1 \\ \{x_{a-1}\} & \text{if } x_{a-2} \notin U_1, x_{a+2} \in U_1 \\ \{p_a\} & \text{if } \{x_{a-2}, x_{a+2}\} \subseteq \{x_j : j \in A'_n\} \setminus U_1 \end{cases}$

Note that the only choices that do not depend only on S arise from cases of U_2 and U_3. If we let $\mathcal{U}_i$ denote the collection of all allowable U_i, $i = 2, 3$, then $\mathcal{T}_S = \{U_1 \cup U_2 \cup U_3 \cup U_4 : U_2 \in \mathcal{U}_2, U_3 \in \mathcal{U}_3\}$ is the collection of S-admissible sets. We also remark that for each $B \in \mathcal{T}_S$, $B \subseteq \mathbb{C}[x_1, \ldots, x_{2n+1}]$. Define I_B as the ideal in $\mathbb{C}[x_1, \ldots, x_{2n+1}]$ generated by the polynomials in B. Furthermore, let $\mathcal{I}_n$ denote the collection of all such ideals generated by S-admissible sets in $\mathcal{F}_n$, i.e.,

$$\mathcal{I}_n = \{I_B : B \in \mathcal{T}_S \text{ for some } S \in \mathcal{F}_n\}.$$

Before proceeding, we note the following useful fact.

Proposition 2.3 (Prop. 5.20 in [9])**.** *If V and W are irreducible affine varieties over an algebraically closed field, then $V \times W$ is as well.*

In fact, the way we will often use Proposition 2.3 is: if $I \subset \mathbb{C}[x_1, \ldots, x_n]$ and $J \subset \mathbb{C}[y_1, \ldots, y_m]$ are prime ideals and I', J' are the ideals they generate in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$, respectively, then $I' + J'$ is also a prime ideal, and $\mathcal{V}(I' + J') = \mathcal{V}(I) \times \mathcal{V}(J)$. The following lemma establishes that the ideals in $\mathcal{I}_n$ are prime.

Lemma 2.4. *For $n \geq 3$ and each $I_B \in \mathcal{I}_n$, I_B is a prime ideal in $\mathbb{C}[x_1, \ldots, x_{2n+1}]$.*

Proof. First, since polynomial rings over $\mathbb{C}$ are UFDs, primality is equivalent to irreducibility throughout. Note that the generators of I_B are a finite collection of variables and polynomials of the form p_k for some odd integer(s) k. Let $\mathcal{X} = \{x_i : x_i \in I_B\}$ and $\mathcal{X}' = \{x_1, \ldots, x_{2n+1}\} \setminus \mathcal{X}$. Furthermore, let $\mathcal{K} = \{p_k : p_k \in I_B\}$. By Proposition 2.3, it suffices to show the primality of the ideal generated by $\mathcal{K}$ in the ring $\mathbb{C}[\mathcal{X}']$, since the variables appearing in $\mathcal{K}$ are disjoint from those of $\mathcal{X}$. The base case $|\mathcal{K}| = 1$ holds if and only if the polynomial in $\mathcal{K}$ is irreducible. Let $i \in \mathbb{Z}$ so that $p_i \in \mathcal{K}$. It is easy to see that $p_i = x_{i-2}x_{i-1} + x_{i+1}x_{i+2}$ is irreducible. Fix an integer $k \geq 1$ and suppose that the result holds for all $\mathcal{K}'$ with $|\mathcal{K}'| = k$. Let $|\mathcal{K}| = k + 1$ and let p_i be any element of $\mathcal{K}$. By the induction hypothesis, $\mathcal{K} \setminus \{p_i\}$ generates a prime ideal. From here we split into the following two cases.

Case 1: The variables of p_i are disjoint from those of $\mathcal{K} \setminus \{p_i\}$. As noted above, p_i generates a prime ideal in $\mathbb{C}[x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}]$, so it also generates a prime ideal in $\mathbb{C}[\mathcal{X}]$. Moreover, the induction hypothesis gives that $\mathcal{K} \setminus \{p_i\}$ generates a prime ideal in $\mathbb{C}[\mathcal{X}' \setminus \{x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}\}]$, further implying that $\mathcal{K} \setminus \{p_i\}$ generates a prime ideal in $\mathbb{C}[\mathcal{X}]$ by Proposition 2.3.

Case 2: Some variables of p_i also occur as variables of polynomials in $\mathcal{K} \setminus \{p_i\}$. Since i is odd, $i - 1$ and $i + 1$ are even. Moreover, the variables x_{i-1} (and x_{i+1}) appear in no other polynomial of $\mathcal{K}$, since $p_i, p_{i-2} \in \mathcal{K}$ (respectively, p_i and p_{i+2}) implies both i and $i - 2$ (respectively, i and $i + 2$) are outside the set B used to generate the original ideal I_B, contradicting that B is generated by a Fibonacci subset of A'_n. Therefore, the only overlap in variables comes from x_{i-2} and x_{i+2}.

Let X be the collection of variables in p_i that also appear in polynomials of $\mathcal{K} \setminus \{p_i\}$. Define $Y := \{x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}\} \setminus X$, and let Z be the collection of variables in polynomials of $\mathcal{K}$ except the variables contained in X. Define a collection of new variables $X' := \{x'_m : x_m \in X\}$. Let the polynomial p'_i be p_i evaluated at the variables of X' and Y, where each input variable matches the index of the existing variable. Let I be the ideal generated by $\mathcal{K} \setminus \{p_i\}$. The induction hypothesis gives that I is prime. The ideal $\langle p'_i \rangle$ is prime because p'_i is irreducible. Proposition 2.3 gives the primality of the ideal generated by $I + \langle p'_i \rangle$. Let $\sigma : \mathbb{C}[X \cup Z] \times \mathbb{C}[X' \cup Y] \to \mathbb{C}[X \cup Y \cup Z]$ be the quotient homomorphism $\sigma : f \mapsto f + \langle \{x_i - x'_i : x_i \in X\} \rangle$. Clearly, σ is surjective, so Proposition 3.34b in [9] (that surjective homomorphisms preserve primality) completes the proof. □

If we let $\mathcal{W}_n$ denote the collection of affine varieties generated by ideals of $\mathcal{I}_n$, i.e., $\mathcal{W}_n := \bigcup_{I \in \mathcal{I}_n} \mathcal{V}(I)$, then the previous lemma implies that all varieties in $\mathcal{W}_n$ are irreducible. However, some of these varieties may not be inclusion-maximal, so they are not irreducible components, a matter we address presently.

Lemma 2.5. *Let I_V denote the ideal in $\mathcal{I}_n$ which generates the variety V. Furthermore, let Θ_V denote the collection of all maximal sets of consecutive odd-indexed $x_i \in I_V$ whose indices are contained in A'_n. Then, the variety $V \in \mathcal{W}_n$ is inclusion-maximal if and only if Θ_V does not contain a set with odd cardinality $m \geq 3$.*

Proof. Suppose first that there exists an odd $m \geq 3$ so that $X := \{x_a, x_{a+2}, \ldots, x_{a+2(m-1)}\} \in \Theta_V$. Let B be the generating set for I_V which corresponds to an S-admissible set for some $S \in \mathcal{F}_n$. If $a > 3$, then the maximality of X implies $x_{a-2} \notin I_V$, giving that $x_{a-1} \in I_V$ by condition (4) in the definition of S-admissible. On the other hand, if $a = 3$, then condition (2) gives the presence of either x_1 or x_2 in I_V. In either case, there exists $q_1 \in \mathbb{N}$ so that $x_{q_1} \in I_V \cap \{x_{a-2}, x_{a-1}\}$. Similarly, there exists q_2 so that $x_{q_2} \in I_V \cap \{x_{a+2m-1}, x_{a+2m}\}$. Now, define $X' := \{x_{q_1}, x_{q_2}\} \cup \{x_{a+2}, x_{a+6}, \ldots, x_{a+2(m-2)}\}$, which is well-defined since $|X|$ is odd, and, let $P' = \{p_a, p_{a+4}, \ldots, p_{a+2(m-1)}\}$. Note that $|X'| = |P'|+1$, so that $|B| > |(B \setminus X') \cup P'|$. Inspection shows that $(B \setminus X') \cup P'$ is an S-admissible set for some $S \in \mathcal{F}_n$. Moreover, if every polynomial in B yields 0 when evaluated at a tuple $(c_1, c_2, \ldots, c_{2n+1}) \in \mathbb{C}^{2n+1}$, then $(c_1, c_2, \ldots, c_{2n+1})$ is also a common zero of all polynomials in $(B \setminus X') \cup P'$, since all polynomials of P' evaluate to zero if those of $X \cup \{x_{q_1}, x_{q_2}\} \subseteq B$ do as well. Then V is not maximal.

It remains to establish the converse. If $n \leq 3$, it is straightforward to check that the varieties in $\mathcal{W}_n$ are maximal. Suppose now that $n \geq 4$ and that $V \in \mathcal{W}_n$ is not maximal, so there exists $V' \in \mathcal{W}_n$ with $V \subsetneq V'$. Let B and B' be the admissible sets which generate I_V and $I_{V'}$ respectively, meaning B and B' also generate V and V'. Since $V \subset V'$, if values for $x_1, \ldots, x_{2n+1}$ are chosen so that all polynomials in B are zero, then all the polynomials in B' are also zero for the same choice of values for $x_1, \ldots, x_{2n+1}$. By the definition of admissible sets, B and B' are each minimal generating sets of their respective ideals, and additionally $B \cap B' \notin \{B, B'\}$, i.e., neither is a subset of the other.

Next, we establish the following claim regarding the inclusion of single-variable monomials between B and B'. Let $i \in [2n-1]$.

Claim 2.6.

If $x_i \notin B$, then $x_i \notin B'$.

Proof of claim. Suppose $i \in [2n-1]$ and $x_i \notin B$. Let $\mathbf{c} = (c_1, \ldots, c_{2n+1}) \in V$. If $c_i \neq 0$, then $x_i \notin B'$, as otherwise $\mathbf{c} \notin V'$, contradicting that $V \subset V'$. Suppose now that $c_i = 0$. The following cases construct another point $\mathbf{c}'$ so that $\mathbf{c}' \in V$ with $c'_i \neq 0$, again obtaining a contradiction to $V \subset V'$.

Case 1: $i \in \{1, 2, 2n, 2n+1\}$. Without loss of generality, suppose $i = 1$, and note that the only polynomials of any admissible set in which x_1 occurs are x_1 and p_3, and, in this case, $x_1 \notin B$. If $p_3 \in B$, define $\mathbf{c}'$ so that $c'_i = c_i$ for $i \notin \{1, 2\}$, but $c'_1 = 1$ and

$c'_2 = -x_4x_3$. The choice of c'_2 gives $p_3(\mathbf{c}') = 0$. Since $p_3 \in B$ implies $x_2 \notin B$, $\mathbf{c}' \in V$. If $p_3 \notin B$, define $\mathbf{c}'$ so that $c'_j = c_j$ for $j \neq 1$, but $c'_1 = 1$. All polynomials of B are satisfied by $\mathbf{c}'$.

Case 2: $3 \leq i \leq 2n-1$ and i odd. Note that the only possible polynomials containing x_i are p_{i-2}, p_{i+2}, and x_i. By assumption, $x_i \notin B$, leaving only p_{i-2} and p_{i+2}. Define $\mathbf{c}'$ so that $c'_j = c_j$ for $j \notin \{i-1, i, i+1\}$ and $c'_i = 1$. If $p_{i-2} \in B$ (resp. $p_{i+2} \in B$), define $c'_{i-1} = -x_{i-4}x_{i-3}$ (resp. $c'_{i+1} = -x_{i+4}x_{i+3}$), so that $p_{i-2}(\mathbf{c}') = 0$ (resp. $p_{i+2}(\mathbf{c}') = 0$). The existence of $p_{i-2} \in B$ (resp. p_{i+2}) implies $x_{i-1} \notin B$ (resp. $x_{i+1} \notin B$). Clearly, p_i is the only other polynomial containing either x_{i-1} or x_{i+1}, but $x_i \notin B$ implies $x_{i-2}, x_{i+2} \in B$, further giving that $p_i \notin B$. Therefore, all polynomials of B are satisfied by $\mathbf{c}'$.

Case 3: $3 \leq i \leq 2n-1$ and i even. Note that the only possible polynomials containing x_i are p_{i-1}, p_{i+1}, and x_i. By assumption, $x_i \notin B$, leaving only p_{i-1} and p_{i+1}. If $x_{i-1}, x_{i+1} \in B$, then $p_{i-1}, p_{i+1} \notin B$, so defining $c'_j = c_j$ for $j \neq i$ and $c'_i = 1$ yields a $\mathbf{c}'$ satisfying all polynomials of B. Suppose now that not both of x_{i-1} and x_{i+1} are in B. Condition (1) gives that at least one of x_{i-1} and x_{i+1} are in B, so B cannot contain both of p_{i-1} and p_{i+1}. Without loss of generality, suppose $p_{i-1} \in B$, giving that $x_{i+1} \notin B$. In this case, define $\mathbf{c}'$ so that $c'_j = c_j$ for $j \notin \{i, i+1, i+2\}$, $c'_i = 1$, and $c'_{i+1} = -c_{i-3}c_{i-2}$. Then $p_{i-1}(\mathbf{c}') = 0$, so the only other polynomial containing x_{i+1} is p_{i+3}. If $c'_{i+1} = 0$, then we already have $\mathbf{c}' \in V$. Suppose now that $c'_{i+1} \neq 0$. If $p_{i+3} \notin B$, then take $c'_{i+2} = c_{i+2}$, and $\mathbf{c}' \in V$. Otherwise, take $c'_{i+2} = -c_{i+4}c_{i+5}/c'_{i+1}$. In this case, $x_{i+3}, p_{i+3} \in B$ gives that $x_{i+2} \notin B$. Furthermore, $x_{i+3} \in B$ also implies $p_{i+1} \notin B$, meaning p_{i+3} is the only polynomial of B containing x_{i+2}. Therefore, in this case, $\mathbf{c}' \in V$. $\diamondsuit$

We will often use the above claim in contrapositive form, i.e., if $x_i \in B'$, then $x_i \in B$.

Suppose the polynomial p_a is an element of $B' \setminus B$. Thus, $x_{a-2} \in B \setminus B'$ or $x_{a-1} \in B \setminus B'$ by condition (4). The same conclusion can be drawn of x_{a+1} or x_{a+2}. Without loss of generality, there are three cases: $x_{a-2}, x_{a+2} \in B \setminus B'$ and $x_{a-1}, x_{a+1} \notin B \setminus B'$, $x_{a-1}, x_{a+1} \in B \setminus B'$, and $x_{a-1}, x_{a+2} \in B \setminus B'$. Suppose, by way of contradiction, that Θ_V does not contain a set of odd cardinality greater than 1.

Case 1: $\{x_{a-2}, x_{a+2}\} \subset B \setminus B'$ and $x_{a-1}, x_{a+1} \notin B \setminus B'$. Since B' is an admissible set, then $x_a \in B'$, further implying $x_a \in B$ by the above claim. Let M_a be the element of Θ_V containing x_a. By assumption, $|M_a|$ is even. If M_a^L denotes the subset of variables in M_a with indices less than a and M_a^R denotes the subset of variables in M_a with indices greater than a, then exactly one of $|M_a^L|$ and $|M_a^R|$ is odd. Without loss of generality, suppose $|M_a^R|$ is odd, and let x_{a+2q} be the variable of largest index in M_a. Clearly $q \geq 1$.

Since $x_{a+2m} \in B$ for all $0 \leq m \leq q$, we have that $x_{a+2m+1} \notin B$ for each $0 \leq m \leq q-1$ by condition (4). The above claim gives that $x_{a+2m+1} \notin B'$ for each $0 \leq m \leq q-1$. Therefore, $x_{a+2}, x_{a+3} \notin B'$, so $p_{a+4} \in B'$ by condition (4), so $x_{a+6} \notin B'$. Repeating this argument, $B' \setminus B$ contains polynomials p_i for $i \in \{a, a+4, a+8, \ldots, a+2(q-1)\}$, since $|M_a^R|$ odd implies q odd. Furthermore, $x_{a-2}, x_{a+2}, x_{a+6}, \ldots, x_{a+2q} \notin B'$. If $a + 2(q+1) \leq 2n-1$, then

x_{a+2q} being the variable with maximum index in M_a implies $x_{a+2(q+1)} \notin B$. The above claim gives $x_{a+2(q+1)} \notin B'$, and this together with $x_{a+2q} \notin B'$ contradicts condition (1). Therefore, $a + 2q = 2n + 1$. Since $x_{a+2(q-1)}, x_{a+2q} \in B$, then exactly one of x_{2n} and x_{2n+1} are not in B. Without loss of generality, suppose $x_{2n} \notin B$. By the above claim, we have that $x_{2n} \notin B'$. This together with $x_{a+2q} \notin B'$ contradicts condition (3), completing the case.

Case 2: $\{x_{a-1}, x_{a+1}\} \subseteq B \setminus B'$. By the definition of an admissible set, we have $x_a \notin B$ (as otherwise implies $x_{a-2} \notin B$ and $x_{a+2} \notin B$, giving that $p_a \in B$, a contradiction). The absence of x_a in B further implies that $x_a \notin B'$ by the above claim. If $3 < a < 2n - 1$, then $\{x_{a-2}, x_{a+2}\} \subseteq B$. If $a = 3$ or $a = 2n - 1$, suppose without loss of generality that $a = 3$, in which case $x_{a+2} \in B$. For any a, there exists x_j with $j \in \{a + 2, a - 2\}$ so that $3 \leq j \leq 2n - 1$ and $x_j \in B$. The presence of $p_a \in B'$ requires $x_j \notin B'$. This together with $x_a \notin B'$ contradicts the definition of an admissible set.

Case 3: Without loss of generality, $\{x_{a-1}, x_{a+2}\} \subseteq B \setminus B'$. Suppose that $3 < a < 2n - 1$. Since B' is an admissible set, $x_{a+2} \notin B'$ implies $x_a \in B'$, so $x_a \in B$ by the above claim. Furthermore, $\{x_a, x_{a-1}\} \subset B$ implies $x_{a-2} \notin B$, giving that $x_{a-2} \notin B'$, again by the above claim. Let M_a be the element of Θ_V containing x_a. We have that x_a is the variable with smallest index in M_a, since $x_{a-2} \notin B$. Let x_{a+2q} be the variable with largest index in M_a. Since $|M_a|$ is even, we have that $q \geq 1$ is odd. Therefore, applying the argument from case 1 completes this case as well.

Since this considers all cases, this completes the proof that, if V is not maximal, then B contains a maximal odd order collection of monomials with consecutive indices in A'_n. □

Let $\mathcal{J}_n$ denote the collection of all ideals in $\mathcal{I}_n$ which generate inclusion-maximal irreducible varieties. Furthermore, define $\mathcal{T}_n$ to be the subcollection of $\bigcup_{S \in \mathcal{F}_n} \mathcal{T}_S$ containing all admissible sets which generate ideals in $\mathcal{J}_n$. Lastly, define $\hat{\mathcal{F}}_n$ to be the subcollection of $\mathcal{F}_n$ containing all Fibonacci subsets of A'_n which give rise to at least one admissible set in $\mathcal{T}_n$, i.e., subsets S of $A'_n = \{3, 5, \ldots, 2n - 1\}$ so that at least one of every two consecutive elements of A'_n belong to S, and so that maximal intervals of A'_n contained in S are either a single element or have even length.

Theorem 2.7. *If $\mathcal{H} = P_n^3$ for some $n \geq 3$, then the null variety V_0 of $\mathcal{H}$ can be written $\cup_{J \in \mathcal{J}_n} \mathcal{V}(J)$, where $\mathcal{J}_n$ is as defined above and each $J \in \mathcal{J}_n$ is an irreducible component of V_0.*

Proof. Recall that the hyperpath $\mathcal{H}$ has exactly $2n + 1$ vertices, and we label them with $\{v_1, \ldots, v_{2n+1}\}$ so that the j-th edge is $e_j = \{v_{2(j-1)+1}, \ldots, v_{2j+1}\}$ for $j = 1, \ldots, n$.

In constructing the equations that define V_0, there are $n - 1$ vertices giving rise to equations of the form $p_k = 0$, while the other $n + 2$ vertices give equations of the form $x_i x_j = 0$. We begin by considering the variety defined by all polynomials of the second form.

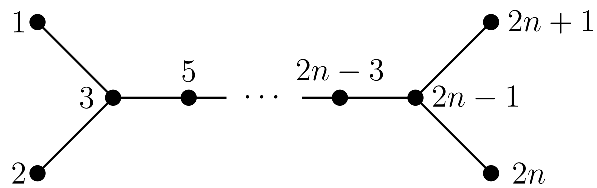
Let $x_{i_k}x_{j_k}$ for $1 \le k \le n+2$ be the $n+2$ polynomials of this form. Then

$$\mathcal{V}\left(\{x_{i_k}x_{j_k}\}_{k=1}^{n+2}\right) = \bigcap_{k=1}^{n+2} \mathcal{V}(x_{i_k}x_{j_k}) = \bigcap_{k=1}^{n+2} \left(\mathcal{V}(x_{i_k}) \cup \mathcal{V}(x_{j_k})\right).$$

Let $\ell_k \in \{i_k, j_k\}$ for each $1 \le k \le n+2$, so that

$$\bigcap_{k=1}^{n+2} \left(\mathcal{V}(x_{i_k}) \cup \mathcal{V}(x_{j_k})\right) = \bigcup_{\{\ell_k\}_{k=1}^{n+2}} \mathcal{V}(\{x_{\ell_k}\}_{k=1}^{n+2}).$$

Let L be the collection of all choices of $\{\ell_k\}$. To facilitate analysis of the sets in L, we construct a graph G, where the vertices of G are labeled with the distinct ℓ_k, and edges connect ℓ_k and $\ell_{k'}$ if and only if $x_{\ell_k}x_{\ell_{k'}} \in \{x_{i_k}x_{j_k}\}_{k=1}^{n+2}$. Based on the structure of $\mathcal{H}$ and the vertex labeling given originally, G has the following form.



An element of L corresponds to a set of vertices in G covering $E(G)$, since the vertices of G are labeled by variable indices, edges are given by pairs of indices in a term of $\bigcap_{k=1}^{2n+1} (\mathcal{V}(x_{i_k}) \cup \mathcal{V}(x_{j_k}))$, and $\bigcap_{k=1}^{2n+1} (\mathcal{V}(x_{i_k}) \cup \mathcal{V}(x_{j_k}))$ is the union of intersections over one term from each element of L.

A subset S of vertices in G which is an edge cover must, in particular, cover the edges $\{3,5\}, \{5,7\}, \ldots, \{2n-3, 2n-1\}$, so no two consecutive elements of A'_n are absent from any such set. In particular, $S \cap A'_n \in \mathcal{F}_n$. Let $\mathcal{X}_S = \{x_i : i \in S\}$. Since $3 \notin S$ implies $1, 2 \in S$ so that S covers the edges $\{1,3\}$ and $\{2,3\}$, if $x_3 \notin \mathcal{X}_S$, then $x_1, x_2 \in \mathcal{X}_S$. Similarly, if $x_{2n-1} \notin \mathcal{X}_S$, then $x_{2n}, x_{2n+1} \in \mathcal{X}_S$. Note that, for any odd a, if

$$[(x_{a-2} = 0) \vee (x_{a-1} = 0)] \wedge [(x_{a+1} = 0) \vee (x_{a+2} = 0)] \qquad (1)$$

then $p_a = 0$. Then let P be the set of p_a so that (1) is *not* satisfied, and define $B = \mathcal{X}_S \cup P$. Then, for each $i \in A'_n$:

1. If $i \notin S$ and $i-4 \notin S$, then $x_i \notin B$, $x_{i-4} \notin B$, and $p_{i-2} \in B$.

2. If $i \notin S$ and $i+4 \notin S$, then $x_i \notin B$, $x_{i+4} \notin B$, and $p_{i+2} \in B$.

3. If $i \notin S$ and $i-4 \in S$, then $x_i \notin B$, $x_{i-4} \in B$, and $x_{i-1} \in B$.

4. If $i \notin S$ and $i+4 \in S$, then $x_i \notin B$, $x_{i+4} \in B$, and $x_{i+3} \in B$.

5. If $5 \notin S$ and $x_1 \notin B$, then $p_3 \in B$.

6. If $2n-3 \notin S$ and $x_{2n+1} \notin B$, then $p_{2n-1} \in B$.
7. If $5 \in S$ and $x_1 \notin B$, then $x_2 \in B$.
8. If $5 \in S$ and $x_2 \notin B$, then $x_1 \in B$.
9. If $2n-3 \in S$ and $x_{2n} \notin B$, then $x_{2n+1} \in B$.
10. If $2n-3 \in S$ and $x_{2n+1} \notin B$, then $x_{2n} \in B$.

Let $\mathcal{B}$ be the set of all such B generated by the above conditions. Then, we have that the null variety of $\mathcal{H}$ is $\cup_{B\in\mathcal{B}}\mathcal{V}(B)$, and it is easy to see that this is exactly the same as the construction given by $\bigcup_{I\in\mathcal{I}_n}\mathcal{V}(I)$. Since Lemma 2.4 gives that each of these ideals are prime, the corresponding varieties are irreducible, giving that $\bigcup_{I\in\mathcal{I}_n}\mathcal{V}(I)$ is a decomposition of V_0 into irreducible varieties. Furthermore, Lemma 2.5 determines the inclusion-maximal varieties under the inclusion relation, implying that $\cup_{J\in\mathcal{J}_n}\mathcal{V}(J)$ is a decomposition of V_0 into its irreducible components. □

Corollary 2.8. *For $n \geq 3$, the null variety V_0 of P_n^3 has dimension $2\lfloor n/2\rfloor + 1$.*

As an illustration of Theorem 2.7, we list all the ideals that generate irreducible components of V_0 for P_5^3:

$$\begin{array}{ccc}
\langle x_1, x_2, x_5, x_9, p_5, p_9\rangle & \langle x_1, x_2, x_4, x_5, x_7, x_8, x_{10}, x_{11}\rangle & \langle x_3, x_7, x_{10}, x_{11}, p_3, p_7\rangle \\
\langle x_3, x_6, x_7, x_9, x_{10}, p_3\rangle & \langle x_3, x_6, x_7, x_9, x_{11}, p_3\rangle & \langle x_1, x_3, x_5x_6, x_9, p_9\rangle \\
\langle x_2, x_3, x_5x_6, x_9, p_9\rangle & \langle x_1, x_3, x_5, x_7, x_9, x_{10}\rangle & \langle x_1, x_3, x_5, x_7, x_9, x_{11}\rangle \\
\langle x_2, x_3, x_5, x_7, x_9, x_{10}\rangle & \langle x_2, x_3, x_5, x_7, x_9, x_{11}\rangle &
\end{array}$$

2.3 Enumeration of Components by Dimension

From here we work to determine the quantity of irreducible components of V_0 of different dimensions for each P_n^3. Fix an n. Let $B \in \mathcal{T}_n$, and let S be such that $S \in \hat{\mathcal{F}}_n$ with B an S-admissible set. Let U_1, U_2, U_3, U_4 be given so that $B = U_1 \cup U_2 \cup U_3 \cup U_4$ as in the definition above. Noting that $|U_1| = |S|$, $|U_2| = \begin{cases} 2 & \text{if } x_3 \notin U_1 \\ 1 & \text{otherwise}\end{cases}$, $|U_3| = \begin{cases} 2 & \text{if } x_{2n-1} \notin U_1 \\ 1 & \text{otherwise}\end{cases}$, and $|U_4| = |\{a \in A_n : a-2 \notin S \text{ or } a+2 \notin S\}|$, the following computation gives an expression for $|B|$.

$$\begin{aligned}
|B| &= |S| + \begin{cases} 2 & \text{if } x_3 \notin U_1 \\ 1 & \text{otherwise}\end{cases} + \begin{cases} 2 & \text{if } x_{2n-1} \notin U_1 \\ 1 & \text{otherwise}\end{cases} + |\{a \in A_n : a-2 \notin S \text{ or } a+2 \notin S\}| \\
&= |S| + 1_{3\notin S} + 1_{2n-1\notin S} + |(A'_n - 2) \cap \overline{S}| + |(A'_n + 2) \cap \overline{S}| - |(A'_n - 2) \cap (A'_n + 2) \cap \overline{S}| \\
&= |S| + 1_{3\notin S} + 1_{2n-1\notin S} + |(A'_n - 2) \cap \overline{S}| + |(A'_n + 2) \cap \overline{S}| - |A_n \cap \overline{S}| \\
&= |S| + 1_{3\notin S} + 1_{2n-1\notin S} + |A'_n| - |(S-2) \cap (S+2)| \\
&= |S \cap A| + n + 1 - |S \cap (S+4)|
\end{aligned}$$

Additionally, let $\mu_n(S)$ denote $|\mathcal{T}_S \cap \mathcal{J}_n|$, i.e., the number of irreducible components of V_0 generated by sets in $\mathcal{T}_S$. It is clear that $\mu_n(S) \in \{1, 2, 4\}$. All irreducible components generated by sets in $\mathcal{T}_S$ have dimension $2n + 1 - |B|$ for some $B \in \mathcal{T}_S$, since the irreducible components all reside in $\mathbb{C}[x_1, \ldots, x_{2n+1}]$ and $|B_1| = |B_2|$ for all $B_1, B_2 \in \mathcal{T}_S$. Consider the generating function

$$g(y, z) = \sum_{n \geq 0} \sum_{S \in \hat{\mathcal{F}}_n} y^{|B|} z^n.$$

Note that $g(y, z)$ does *not* incorporate the multiplicity $\mu_n(S)$. We first consider the expression given by the inner sum, namely

$$g_n(y) := \sum_{S \in \hat{\mathcal{F}}_n} y^{|B|}$$

for a given $n \in \mathbb{N}$. Computation gives the following results for small values of n.

$$\begin{aligned} g_0(y) &= y & g_3(y) &= 3y^4 \\ g_1(y) &= y^2 & g_4(y) &= 3y^6 + y^4 \\ g_2(y) &= 2y^3 & g_5(y) &= y^8 + 5y^6 \end{aligned}$$

We develop a recurrence for $g_n(y)$ aided by two new sequences of functions, $b_n(y)$ and $c_n(y)$, defined in the following way:

$$\begin{aligned} b_n(y) &= \sum_{S \in \hat{\mathcal{F}}_n, \{2n-3, 2n-1\} \subseteq S} y^{|B|} \\ c_n(y) &= \sum_{S \in \hat{\mathcal{F}}_n, 2n-3 \notin S, 2n-1 \in S} y^{|B|} \end{aligned}$$

For clarity, we define $b_0 = b_1 = b_2 = c_0 = c_1 = c_2 = 0$. Otherwise, we have the following small values of the two new sequences.

$$\begin{aligned} b_3(y) &= y^4 & c_3(y) &= y^4 \\ b_4(y) &= y^6 & c_4(y) &= y^4 \\ b_5(y) &= 2y^6 & c_5(y) &= 2y^6 \end{aligned}$$

Note that, for each S a Fibonacci subset of A'_n, at least one of $2n - 3$ and $2n - 1$ are included in S, so there are three options for $\{2n-1, 2n-3\} \cap S$. All three can be expressed in terms of b_n, c_n, and g_n. A straightforward (if laborious) case analysis provides the following recurrences for the three sequences of functions. Note that these recurrences are valid only for $n \geq 5$.

$$\begin{aligned} g_n(y) &= 2y^2 g_{n-2}(y) + y^4 b_{n-3}(y) + y^2(y^2-1)c_{n-2}(y) \\ b_n(y) &= y^2 g_{n-2}(y) - y^2 c_{n-2}(y) \\ c_n(y) &= y^2 b_{n-2}(y) + y^2 c_{n-2}(y) \end{aligned} \tag{2}$$

Recall that $g(y,z)$ is the generating function for $g_n(y)$. Analogously, let $b(y,z) = \sum_{n\geq 0} b_n(y)z^n$ and $c(y,z) = \sum_{n\geq 0} c_n(y)z^n$. The following computations work towards closed forms for b, c, and g.

$$\begin{aligned} g &= \sum_{n=0}^{4} g_n z^n + 2y^2 \sum_{n\geq 5} g_{n-2} z^n + y^4 \sum_{n\geq 5} b_{n-3} z^n + y^2(y^2-1)\sum_{n\geq 5} c_{n-2} z^n \\ &= \sum_{n=0}^{4} g_n z^n + 2y^2 z^2 (g - \sum_{n=0}^{2} g_n z^n) + y^4 z^3 b + y^2(y^2-1) z^2 c \\ g &= \frac{3y^6z^4 - 4y^5z^4 + y^4z^4 + y^4z^3 + y^2 z + y + y^4z^3 b + y^2(y^2-1)z^2 c}{1 - 2y^2z^2} \\ b &= \sum_{n=0}^{4} b_n z^n + y^2 \sum_{n\geq 5} g_{n-2} z^n - y^2 \sum_{n\geq 5} c_{n-2} z^n = \sum_{n=0}^{4} b_n z^n + y^2 z^2 (g - \sum_{n=0}^{2} g_n z^n) - y^2 z^2 c \\ &= y^6 z^4 - 2y^5 z^4 - y^3 z^2 + y^2 z^2 g - y^2 z^2 c \\ c &= \sum_{n=0}^{4} c_n z^n + y^2 \sum_{n\geq 5} b_{n-2} z^n + y^2 \sum_{n\geq 5} c_{n-2} z^n = \sum_{n=0}^{4} c_n z^n + y^2 z^2 b + y^2 z^2 c \\ c &= \frac{y^4 z^4 + y^4 z^3 + y^2 z^2 b}{1 - y^2 z^2} \end{aligned}$$

Solving the system for g gives the following.

$$g = -\frac{(y^8 - 2\,y^7 + y^6)z^6 - (y^8 - 2\,y^7)z^5 - (2\,y^6 - 3\,y^5 + y^4)z^4 + (y^5 - y^4)z^3 - y^2 z - y}{y^4 z^4 - y^4 z^3 - 2\,y^2 z^2 + 1}$$

Recall that the exponent on y in $g(y,z)$ is the co-dimension of the irreducible component of V_0 for P_n^3. Since we are interested in the dimension of these components, we make the following transformation. The dimension of each component is $2n+1$ minus its co-dimension. Thus, the function we want is given by $h(y,z) = y \cdot g(1/y, y^2 z)$, expressible as follows (computations throughout performed by SageMath [11]).

$$h = \frac{-y^7z^6 + 2y^6z^6 - y^5z^6 + y^5z^4 - 2y^4z^5 - 3y^4z^4 + y^3z^5 + 2y^3z^4 + y^3z^3 - y^2z^3 + yz + 1}{y^4z^4 - y^2z^3 - 2y^2z^2 + 1}$$

To help later with verifying Conjecture 1.1, differentiating with respect to y gives the following expression and then plugging in $y = 2$, because

$$H(z) := \frac{\partial}{\partial y} h(y,z)\bigg|_{y=2} = \sum_{n\geq 0} \sum_{S\in\hat{\mathcal{F}}_n} (\dim \mathcal{V}(B)) 2^{\dim \mathcal{V}(B)-1} z^n.$$

The generating function obtained in this way encodes a lower bound on gm(0) of the conjecture, but four times this function is an upper bound. We get the following expression when substituting $y = 2$:

$$\frac{-1280z^{10} + 384z^9 + 1136z^8 + 192z^7 - 224z^6 - 132z^5 - 20z^4 + 20z^3 + 8z^2 + z}{256z^8 - 128z^7 - 240z^6 + 64z^5 + 96z^4 - 8z^3 - 16z^2 + 1}$$
$$= z + 8z^2 + 36z^3 + 116z^4 + 412z^5 + 1088z^6 + \cdots$$

The smallest-magnitude root of the denominator lies in the interval $(0.37, 0.371)$. This implies that the coefficients of $H(z)$ have growth rate in the interval $(2.69, 2.71)$. We upper-bound the coefficients $\{\eta_n\}_{n\geq 0}$ of $H(z)$. Recall that Corollary 2.8 gives that the maximum dimension of an irreducible component of V_0 for P_n^3 is $2\lfloor n/2\rfloor + 1$. Since we counted at most one component for each Fibonacci subset of A'_n, there are at most F_n (the n-th Fibonacci number) terms which contribute to η_n. Therefore, η_n is bounded above in the following way, given that $\phi = (1 + \sqrt{5})/2$:

$$\eta_n \leq \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \cdot (n + 1) \cdot 2^n$$

2.4 Incorporating Multiplicity

Recall that $\mu_n(S) \in \{1, 2, 4\}$ for $S \in \hat{\mathcal{F}}_n$, but the above sums ignore this factor. Note that $\mu_n(S) > 1$ when either pair $\{3, 5\}$ or $\{2n-3, 2n-1\}$ are subsets of S. The sequence b_n given above accounts for the subcollection of $\hat{\mathcal{F}}_n$ containing both $2n - 3$ and $2n - 1$, so b is the generating function where the $y^m z^n$ coefficient counts the number of irreducible components of codimension m from a hyperpath of length n generated from a given S containing both $2n - 3$ and $2n - 1$. By the symmetry of these Fibonacci subsets, the coefficients of b also count the same quantity, where now the Fibonacci set S contains both 3 and 5. So, $2b$ counts the $\{3, 5\} \subseteq S$ and $\{2n - 3, 2n - 1\} \not\subseteq S$ components once, the $\{2n - 3, 2n - 1\} \subseteq S$ and $\{3, 5\} \not\subseteq S$ components once, and the $\{3, 5, 2n - 3, 2n - 1\} \subseteq S$ components twice. It only remains to count the $\{3, 5, 2n - 3, 2n - 1\} \subseteq S$ components one additional time.

We now define g'_n, b'_n, and c'_n to have the same conditions on the presence of $2n - 3$ and $2n - 1$ in S as was given for g_n, b_n, and c_n above, but now we require that 3 and 5 be in S, i.e.,

$$g'_n(y) := \sum_{\substack{S\in\hat{\mathcal{F}}_n \\ 3,5\in S}} y^{|B|}$$

and analogously for b'_n and c'_n. We define all three sequences for $n \geq 0$, although some initial values are zero. These modified sequences satisfy the exact same recurrences as displayed in (2) for $n \geq 5$.

Let g', b', and c' be the generating functions with respect to the variable z for the three sequences defined. Then, the generating function b' counts exactly the $\{3, 5, 2n-3, 2n-1\} \subseteq$

S components once. Computation gives the following rational expression for g' and b':

$$g' = \frac{y^6z^4 + y^4z^3}{y^4z^4 - y^4z^3 - 2\,y^2z^2 + 1}$$
$$b' = -\frac{y^6z^5 - y^4z^3}{y^4z^4 - y^4z^3 - 2\,y^2z^2 + 1}$$

Note that the generating function for g' counts the same irreducible components as b from above. Therefore, the generating function of gm(0), which incorporates multiplicity (aside from some initial terms), is given by $G = g + 2g' + b'$, and is given by the following rational function.

$$G = \left(-y^8z^6 + y^8z^5 + 2y^7z^6 - 2y^7z^5 - y^6z^6 - y^6z^5 + 4y^6z^4 - 3y^5z^4 - y^5z^3 + y^4z^4 + 4y^4z^3 + y^2z + y\right)/(y^4z^4 - y^4z^3 - 2y^2z^2 + 1)$$

Similarly to the previous subsection, we compute $h' = y \cdot G(1/y, y^2z)$, which is the generating function for the number of irreducible components of dimension given by the exponent on y in V_0 for P_n^3, if n is the exponent on z.

$$h' = \left(-y^7z^6 + 2y^6z^6 - y^5z^6 - y^5z^5 + y^5z^4 - 2y^4z^5 - 3y^4z^4 + y^3z^5 + 4y^3z^4 + 4y^3z^3 - y^2z^3 + yz + 1\right)/(y^4z^4 - y^2z^3 - 2y^2z^2 + 1).$$

Computing $\frac{\partial}{\partial y}h'(y,z)\Big|_{y=2}$ yields the following generating function:

$$\frac{-1280z^{10} + 128z^9 + 1200z^8 + 352z^7 - 336z^6 - 308z^5 + 4z^4 + 56z^3 + 8z^2 + z}{256z^8 - 128z^7 - 240z^6 + 64z^5 + 96z^4 - 8z^3 - 16z^2 + 1}$$
$$= z + 8z^2 + 72z^3 + 140z^4 + 812z^5 + 1648z^6 + 7280z^7 + 18064z^8 + 60928z^9 + 176576z^{10} + \cdots$$

Here the linear and quadratic coefficients are incorrect, however, because incorporation of multiplicity only adjusts for $n \geq 3$. Modifying this expression via Propositions 2.1 and 2.2, we obtain

$$H'(z) = \frac{-256z^8 + 192z^7 + 272z^6 - 156z^5 - 92z^4 + 24z^3 + 13z^2 + 3z}{256z^8 - 128z^7 - 240z^6 + 64z^5 + 96z^4 - 8z^3 - 16z^2 + 1}$$
$$= 3z + 13z^2 + 72z^3 + 140z^4 + 812z^5 + 1648z^6 + 7280z^7 + 18064z^8 + 60928z^9 + 176576z^{10} + \cdots$$

3 Algebraic Multiplicity of Zero

Let $D_{n,k}$ be the algebraic multiplicity of zero in the characteristic polynomial of $\phi_{P_n^k}(\lambda)$ (the k-uniform linear hyperpath with n edges). We are given the following by the paper of Bao, Fan, Wang, and Zhu.

Theorem 3.1 ([1]). *For $n \geq 2$,*

$$\phi_{P_n^k}(\lambda) = \lambda^{(k-2)(k-1)^{n(k-1)}} \prod_{s=0}^{n} \left(\lambda - \frac{f^{s-1}(1)}{\lambda^{k-1}}\right)^{\nu_{n,k}(s)} \phi_{P_{n-1}^k}(\lambda)^{(k-1)^{k-1}},$$

where

$$\nu_{n,k}(s) = \begin{cases} k^{s(k-2)}((k-1)^{k-1} - k^{k-2})(k-1)^{(n-s-1)(k-1)} & \text{if } s \in [0, n-1], \\ k^{s(k-2)} & \text{if } s = n, \end{cases}$$

and

$$f^i(x) = \begin{cases} 0 & \text{if } i = -1, \\ 1 & \text{if } i = 0, \\ f(x) = \frac{1}{1 - x\lambda^{-k}} = \frac{\lambda^k}{\lambda^k - x} & \text{if } i = 1, \\ f^{i-1}(f(x)) & \text{if } i > 1. \end{cases}$$

We use these facts to prove the following. We start by proving the following lemma concerning the degree of the zero root in $f(x)$.

Lemma 3.2. *Let $k \geq 2$ be given. Let d_s be the degree of the zero root in the rational function $f^s(1)$. If $s \geq 1$, then $d_s = 0$ if s is even and $d_s = k$ if s is odd.*

Proof. We proceed by induction on s, with the base cases given by $s = 1$ and $s = 2$. The definition of $f^s(x)$ includes that $f(1) = \frac{\lambda^k}{\lambda^k - 1}$, giving that $d_1 = k$. For $s = 2$, then,

$$f^2(1) = f(f(1)) = f\left(\frac{\lambda^k}{\lambda^k - 1}\right) = \frac{\lambda^k - 1}{\lambda^k - 2}.$$

Now suppose that the result holds for some $s \geq 1$. Consider the value of d_{s+1}. Since composition of functions is associative, $f^{s+1}(1) = f^s(f(1)) = f(f^s(1))$. Let $q^s(x)$ denote the denominator of $f^s(x)$. Since $f(x) = \frac{\lambda^k}{\lambda^k - x}$, we can think of $f^{s+1}(1)$ as $\lambda^k q^s(1)$ divided by $\lambda^k q^s(1)$ minus the numerator of $f^s(1)$.

If $f^s(1)$ is rational in λ with $d_s = 0$, then the denominator of $f^{s+1}(1)$ will not be divisible by λ, but the degree of λ in the numerator is k. Thus $d_{s+1} = k$. On the other hand, if $d_s = k$, then $f^s(1)$ is rational in λ with the power of λ in the numerator equal to k. Then, $f^{s+1}(1)$ will have k factors of λ in the numerator after multiplying through by $q^s(1)$, but the denominator is the difference of two polynomials both of which have λ occurring k times as a factor. Factor out the term λ^k from the denominator and cancel it within $f^{s+1}(1)$. This leaves zero factors of λ in the numerator. In the denominator, we have zero factors of λ if and only if the constant term in $q^s(1)$ differs from the coefficient of λ^k in the numerator of $f^s(1)$. This inequality of coefficients is established by the following inductive argument, which need only handle the case of s odd. In fact, we include in the inductive hypothesis as well that the numerator and denominator have no nonzero coefficients of terms of the form λ^j with $0 < j < k$.

By definition, $f(1) = \frac{\lambda^k}{\lambda^k - 1}$, so the constant term in the denominator (namely, -1) and the coefficient of λ^k (namely, 1) in the numerator differ, giving the base case. Suppose now that the result holds for some odd $i \geq 1$. Let $f^s(1)$ have numerator $\alpha(\lambda) + \alpha_1\lambda^k$ and denominator $\beta(\lambda) + \beta_1\lambda^k + \beta_2$, where α and β are both polynomials of degree greater than k, and $\alpha_1 \neq \beta_2$. Then, we have the following.

$$\begin{aligned}
f^{s+2}(1) &= f \circ f\left(\frac{\alpha(\lambda) + \alpha_1\lambda^k}{\beta(\lambda) + \beta_1\lambda^k + \beta_2}\right) \\
&= f\left(\frac{\lambda^k}{\lambda^k - \frac{\alpha(\lambda)+\alpha_1\lambda^k}{\beta(\lambda)+\beta_1\lambda^k+\beta_2}}\right) \\
&= f\left(\frac{\beta(\lambda) + \beta_1\lambda^k + \beta_2}{\beta(\lambda) + \beta_1\lambda^k + \beta_2 - \alpha(\lambda)\lambda^{-k} - \alpha_1}\right) \\
&= \frac{\lambda^k}{\lambda^k - \left(\frac{\beta(\lambda)+\beta_1\lambda^k+\beta_2}{\beta(\lambda)+\beta_1\lambda^k+\beta_2-\alpha(\lambda)\lambda^{-k}-\alpha_1}\right)} \\
&= \frac{\lambda^k(\beta(\lambda) + \beta_1\lambda^k + \beta_2 - \alpha(\lambda)\lambda^{-k} - \alpha_1)}{\lambda^k(\beta(\lambda) + \beta_1\lambda^k + \beta_2 - \alpha(\lambda)\lambda^{-k} + \alpha_1) - \beta(\lambda) - \beta_1\lambda^k - \beta_2}
\end{aligned}$$

From this, we see that the coefficient of λ^k in the numerator is $\beta_2 - \alpha_1$, and the constant term in the denominator is $-\beta_2$. Since $\alpha_1 = 1$ and $\beta_2 = -1$ in $f(1)$, we have that the constant term in the denominator flips back and forth between -1 and 1 as the powers of f increase by two. On the other hand, $\beta_2 - \alpha_1$ takes values of the form $(-1)^{(s-1)/2} \cdot (s-1)/2$ for odd $s \geq 1$. Then the two desired coefficients are never equal, completing the proof. □

Corollary 3.3. *The multiplicity of the zero root of $\lambda^k - f^s(1)$ is the same as the multiplicity of zero in $f^s(1)$.*

Proof. The even case is trivial, because both multiplicities are zero. In the odd case, the ratio of the coefficient of λ^k in the numerator of $f^s(1)$ divided by the constant coefficient in the denominator has absolute value less than 1 except when $s = 1$. However, in that case $\lambda^k - f(1) = \lambda^k - \frac{\lambda^k}{\lambda^k - 1} = \frac{\lambda^{2k} - 2\lambda^k}{\lambda^k - 1}$. □

We now use the preceding lemma and corollary to fully describe the nullity of P_n^k.

Theorem 3.4. *Let $k \geq 1$ and $n \geq 1$. Additionally, let $u = (k-1)^{k-1}$ and $v = k^{k-2}$. If $D_{n,k}$ denotes the multiplicity of λ in the k-uniform hyperpath characteristic polynomial $\phi_{P_n^k}(\lambda)$, then*

$$D_{n,k} = \frac{u^n\left([nk - n + 1]u^2 + [nk - 2n + 2]uv - [k + n - 1]v^2\right) + k(-v)^{n+2}}{(u + v)^2}.$$

Proof. We first separate the $n = 1$ case. Cooper and Dutle [3] showed that $D_{1,k} = k(k-1)^{k-1} - k^{k-1} = k(u-v)$. Plugging $n = 1$ into the suggested formula gives the same expression, verifying the result for the base case. Suppose now that $n \geq 2$. From Theorem 3.1, we have

$$\phi_{P_n^k}(\lambda) = \lambda^{(k-2)(k-1)^{n(k-1)}} \prod_{s=0}^{n} \left(\lambda - \frac{f^{s-1}(1)}{\lambda^{k-1}}\right)^{\nu_{n,k}(s)} \phi_{P_{n-1}^k}(\lambda)^{(k-1)^{k-1}}, \qquad (3)$$

so we develop a recurrence that gives $D_{n,k}$ knowing $D_{n-1,k}$. From the preceding formula, we see

$$D_{n,k} = (k-2)u^n + u \cdot D_{n-1,k} + F_{n,k},$$

where we define $F_{n,k}$ to be the multiplicity of the zero root in the simplified rational function $\prod_{s=0}^{n} \left(\lambda - \frac{f^{s-1}(1)}{\lambda^{k-1}}\right)^{\nu_{n,k}(s)}$ (taking the parameter to be negative if there are excess powers of λ in the denominator). As above, let d_s be the multiplicity of the zero root in $f^s(1)$. By Lemma 3.2, we have that d_{s-1} is zero when $s-1$ is even, and $d_{s-1} = k$ when $s-1$ is odd. Since the s-th term of the product in (3) is $[\lambda^{-(k-1)}(\lambda^k - f^{s-1}(1))]^{\nu_{n,k}(s)}$, and Corollary 3.3 gives that the degree of the zero root in $f^{s-1}(1)$ and $\lambda^k - f^{s-1}(1)$ are the same, we have

$$F_{n,k} = -(k-1)\sum_{s=0}^{n} \nu_{n,k}(s) + \sum_{s=0}^{n} \nu_{n,k}(s) \cdot d_{s-1}.$$

We start by considering the value of the first term above. We have the following.

$$\begin{aligned}
\sum_{s=0}^{n} \nu_{n,k}(s) &= \nu_{n,k}(n) + \sum_{s=0}^{n-1} \nu_{n,k}(s) \\
&= v^n + \sum_{s=0}^{n-1} v^s(u-v)u^{n-s-1} \\
&= v^n + (u-v)u^{n-1}\frac{1-\left(\frac{v}{u}\right)^n}{1-\frac{v}{u}} \\
&= u^n
\end{aligned}$$

When considering the second summand in the expression for $F_{n,k}$, we split into cases initially based on the parity of n. Starting with n odd, we have the following simplification of $\sum_{s=0}^{n} \nu_{n,k}(s) \cdot d_{s-1}$:

$$\begin{aligned}
\sum_{s=0}^{n} \nu_{n,k}(s) \cdot d_{s-1} &= \sum_{s=0}^{(n-1)/2} \nu_{n,k}(2s) \cdot k \\
&= k \cdot \sum_{s=0}^{(n-1)/2} v^{2s}(u-v)u^{n-1-2s} \\
&= k(u-v)u^{n-1}\frac{1-\left(\frac{v^2}{u^2}\right)^{(n+1)/2}}{1-\frac{v^2}{u^2}}
\end{aligned}$$

$$= \left(\frac{k}{u+v}\right)(u^{n+1} - v^{n+1})$$

On the other hand, if n is even, we have the following.

$$\begin{aligned}
\sum_{s=0}^{n} \nu_{n,k}(s) \cdot d_{s-1} &= \sum_{s=0}^{n/2} \nu_{n,k}(2s) \cdot k \\
&= k \cdot v^n + k \cdot \sum_{s=0}^{(n-2)/2} v^{2s}(u-v)u^{n-1-2s} \\
&= k \cdot v^n + k(u-v)u^{n-1} \frac{1 - \left(\frac{v^2}{u^2}\right)^{(n)/2}}{1 - \frac{v^2}{u^2}} \\
&= \left(\frac{k}{u+v}\right)(u^{n+1} + v^{n+1})
\end{aligned}$$

Thus, for general n, we have

$$\sum_{s=0}^{n} \nu_{n,k}(s) \cdot d_{s-1} = \left(\frac{k}{u+v}\right)(u^{n+1} - (-v)^{n+1}).$$

This gives the following closed form for $F_{n,k}$.

$$F_{n,k} = -(k-1)u^n + \left(\frac{k}{u+v}\right)(u^{n+1} - (-v)^{n+1})$$

Substituting this back into the original expression for $D_{n,k}$, we have the following simplification.

$$\begin{aligned}
D_{n,k} &= (k-2)u^n + u \cdot D_{n-1,k} + F_{n,k} \\
&= (k-2)u^n + uD_{n-1,k} - (k-1)u^n + \left(\frac{k}{u+v}\right)(u^{n+1} - (-v)^{n+1}) \\
&= uD_{n-1,k} + \frac{u^n[(k-1)u - v] - k(-v)^{n+1}}{u+v}
\end{aligned}$$

For $n = 1$, we noted earlier that $D_{1,k} = k(u-v)$. We continue with the following, completing the proof.

$$\begin{aligned}
D_{n,k} = ku^{n-1}(u-v) + \frac{u^n[(k-1)u - v] - k(-v)^{n+1}}{u+v} + \sum_{i=1}^{n-2} u^i \frac{u^{n-i}[(k-1)u - v] - k(-v)^{n-i+1}}{u+v} \\
= ku^{n-1}(u-v) + \sum_{i=0}^{n-2} u^i \frac{u^{n-i}[(k-1)u - v] - k(-v)^{n-i+1}}{u+v}
\end{aligned}$$

$$= ku^{n-1}(u-v) + \sum_{i=0}^{n-2} \frac{u^n[(k-1)u-v]}{u+v} - \sum_{i=0}^{n-2} \frac{ku^i(-v)^{n+1-i}}{u+v}$$
$$= ku^{n-1}(u-v) + \frac{u^n(n-1)[(k-1)u-v]}{u+v} - \frac{k(-v)^{n+1}}{u+v} \cdot \frac{1-\left(\frac{u}{-v}\right)^{n-1}}{1-\frac{u}{-v}}$$
$$= \frac{u^n([nk-n+1]u^2 + [nk-2n+2]uv - [k+n-1]v^2) + k(-v)^{n+2}}{(u+v)^2}.$$

□

The next result applies the above theorem to obtain an asymptotic expression for $D_{n,k}$.

Corollary 3.5. *Let $k \geq 3$ be fixed and $n \geq 1$. Additionally, let $u = (k-1)^{k-1}$ and $v = k^{k-2}$. Then $\lim_{n\to\infty} D_{n,k} = nu^n$.*

Proof. We are given the following expression for $D_{n,k}$.

$$D_{n,k} = \frac{u^n\left([nk-n+1]u^2 + [nk-2n+2]uv - [k+n-1]v^2\right) + k(-v)^{n+2}}{(u+v)^2}$$

Noting that $k \geq 2$, we first show that $u > v$. We have the following computation.

$$\frac{u}{v} = \frac{(k-1)^{k-1}}{k^{k-2}} = \frac{k^2}{k-1}\left(1-\frac{1}{k}\right)^k \geq \frac{k^2}{k-1} \cdot \frac{1}{4} = \frac{k^2}{4k-4}.$$

Note that for $k \geq 2$, the function $\left(\frac{k-1}{k}\right)^k$ is increasing, so its value for any $k \geq 2$ is bounded below by its value when $k = 2$, namely, $1/4$. Furthermore, the rightmost expression is greater than one if and only if $k^2 \geq 4k-4$, which is true because $(k-2)^2 \geq 0$. Therefore, $u > v$, so u dominates v asymptotically. Then the rational expression is asymptotically the same as a ratio of two polynomials just in the variable u, from which it follows that

$$\lim_{n\to\infty} D_{n,k} = [n(k-1)+1]u^n$$

Since k is constant, this gives the desired result. □

From this, we observe the following lower bound for $D_{n,3}$ when $n \geq 12$.

$$D_{n,3} \geq \frac{4^n}{7}(5n+3)$$

4 Conjecture Verification

Theorem 4.1. *Let $V_0^1, \ldots, V_0^\kappa$ denote the irreducible components of V_0 for P_n^3. For $n \geq 1$, $D_{n,3} \geq \sum_{i=1}^{\kappa} \dim(V_0^i)(2)^{\dim(V_0^i)-1}$.*

Proof. Recall the following bounds on $D_{n,3}$ and η_n, where η_n is the z^n coefficient of the generating function $H(z)$ found in Section 2.

$$D_{n,3} \geq \frac{4^n}{7}(5n+3)$$
$$\eta_n \leq \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \cdot (n+1) \cdot 2^n$$

It is easy to check that $4(\phi^n + 1) \leq 2^n$ for any $n \geq 7$. Furthermore,

$$2^n \leq \frac{4\sqrt{5}}{7} \cdot 2^n \leq \frac{\sqrt{5}}{7} \cdot 2^n \cdot \frac{5n+3}{n+1}$$
$$4(\phi^n + 1) \geq 4(\phi^n - (-\phi)^{-n})$$

Combining the inequalities shows that $D_{n,3} \geq 4\eta_n$ for $n \geq 12$:

$$4(\phi^n - (-\phi)^{-n}) \leq \frac{\sqrt{5}}{7} \cdot 2^n \cdot \frac{5n+3}{n+1}$$
$$4 \cdot \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \cdot (n+1) \leq \frac{2^n}{7} \cdot (5n+3)$$
$$4 \cdot \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \cdot (n+1) \cdot 2^n \leq \frac{4^n}{7} \cdot (5n+3)$$

Therefore, this gives us that the conjecture holds for $n \geq 12$, since $\text{gm}(0) \leq 4\eta_n$. The following table computes values for $n < 12$ exactly, completing the proof.

n	1	2	3	4	5	6	7	8	9	10	11
$D_{n,3}$	3	35	151	891	3983	19795	88071	407531	1792063	7993155	34740791
gm(0)	3	13	72	140	812	1648	7280	18064	60928	176576	509376

□

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