# 1. TUESDAY, 15 MARCH

## 1.1. **Left and right limits and derivatives.**

1.1.1. *Limits.* Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geqslant 1$. Then for $x$ which are near 0 and to the right of 0, $f(x) = 1$. But for $x$ near 0 and to the left of 0, $f(x) = 0$. Therefore the limit $\lim_{y\to 0} f(y)$ does not exist. However, if we only wish to approach 0 from the right, the function values will approach (and in this example be equal to) 1. If we only wish to approach 0 from the left, the function values will approach (and in this example be equal to) 0. Functions like this are the reason we define *limits from the left* and *limits from the right.*

Given a function $f : A \to \mathbb{R}$, $x \in A$, and $L \in \mathbb{R}$, we write $\lim_{y\to x^-} f(y) = L$ (and say "the limit of $f$ as $y$ approaches $x$ from the left equals $L$") if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in A$ with $y < x$ and $|y-x| < \delta$, $|f(y)-L| < \varepsilon$. The difference between this and the usual definition of the limit is the restriction that $y < x$, which means we only consider $y$ to the left of $x$.

Similarly, we define $\lim_{y\to x^+} f(y) = L$ (and say "the limit of $f$ as $y$ approaches $x$ from the right") if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in A$ with $y > x$ and $|y-x| < \delta$, $|f(y) - L| < \varepsilon$.

**Note that the limit of $f$ at $x$ exists if and only if the left and right limits exist and are equal.** We will use this later in the proof of the Interior Extremum Theorem.

1.1.2. *Derivatives.* Recall that for $f : A \to \mathbb{R}$ and $x \in A$, our definition of what it meant for $f$ to be differentiable at $x$ required that $x \in \text{int}(A)$. We wish to consider derivatives at endpoints of intervals, which are not in the interior of the interval. We do this as follows.

Suppose $f : [a,b] \to \mathbb{R}$ is a function for some $a < b$. We say $f$ is differentiable at $a$ if

$$\lim_{y\to a^+} \frac{f(y)-f(a)}{y-a}$$

exists. Note that this is only a limit from the right. We say $f$ is differentiable at $b$ if

$$\lim_{y\to b^-} \frac{f(y)-f(b)}{y-b}$$

exists. Note that this is only a limit from the left.

## 1.2. **Interior Extremum Theorem.**

**Theorem 1.1** (Interior Extremum Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable on $(a,b)$. If $c \in (a,b)$ is a point at which $f$ attains is maximum (or minimum) on $[a,b]$, then $f'(c) = 0$.*

The idea behind the theorem is clear. If $f(c)$ is a maximum, then in the graph of $f$, we get a hilltop at $c$. The slope of the tangent line at the hilltop must be zero. If $f(c)$ is a minimum, then we have the bottom of a valley at $c$, and the slope of the tangent line is derivative there. This is the geometric idea behind the IET.

*Proof.* Assume $f(c)$ is the maximum of $f$ (the proof in the case that $f(c)$ is a minimum is similar). The idea is that, since $f(c)$ is at a hilltop, if we move to the right, the function will decrease, so the slope of secant lines to the right will be negative. But as we move to the left, the slope of the secant lines will be positive. Since $f'(c)$ is the limit from the left of these slopes, it must be non-negative, and since it is the limit from the right of these slopes, it must be non-positive.

For any $y \in [a, b]$, since $f(c)$ is the maximum of $f$, $f(y) - f(c) \leqslant 0$. If $y < c$, then $y - c$ is negative. Therefore $\frac{f(y)-f(c)}{y-x} \geqslant 0$, since the numerator is non-positive and the denominator is negative. We will use our remark that is in bold at the end of the first section. Then

$$f'(c) = \lim_{y \to c} \frac{f(y) - f(c)}{y - c} = \lim_{y \to c^-} \frac{f(y) - f(c)}{y - c} \geqslant 0.$$

If $y > c$, $y - c$ is positive, and $\frac{f(y)-f(c)}{y-x} \leqslant 0$. Then

$$f'(c) = \lim_{y \to c} \frac{f(y) - f(c)}{y - c} = \lim_{y \to c^+} \frac{f(y) - f(c)}{y - c} \leqslant 0.$$

Then $0 \leqslant f'(c) \leqslant 0$, which means $f'(c) = 0$.

□

### 1.3. **Rolle's theorem and the Mean Value Theorem.**

**Theorem 1.2** (Rolle's Theorem)**.** *Suppose $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then if $f(a) = f(b)$, there exists $c \in (a, b)$ such that $f'(c) = 0$.*

The idea behind this theorem is that either $f$ is constant, in which case the derivative is *always* zero, or $f$ must have either a hilltop or valley between $a$ and $b$. In the second case, the hilltop or the valley is the location of a point with the zero derivative. Alternatively, if this function $f$ is describing one-dimensional motion as a function of time, then if our starting point is the same as our ending point, then either we say still over the entire interval (in which case the derivative, which is the velocity), is always zero, or we must move. If we move, then since our starting point is the same as our ending point, there has to be an instant when we are changing direction (at which point our velocity is zero).

*Proof.* Case 1: $f(a) = f(b)$ is both the maximum and minimum of $f$ on the interval $[a, b]$. Then $f$ is constant and $f'(c) = 0$ for all $c \in (a, b)$.

Case 2: $f(a) = f(b)$ is the minimum of $f$ but not the maximum. Then there must be some point $c \in [a, b]$ where $f$ attains its maximum, and this point cannot be $a$ or $b$. Thus $c \in (a, b)$, and the Interior Extremum Theorem gives that $f'(c) = 0$.

Case 3: $f(a) = f(b)$ is not the minimum. Then there must be some point $c \in (a, b)$ where $f$ attains its minimum. By the Interior Extremum Theorem, $f'(c) = 0$.

□

**Theorem 1.3** (Mean Value Theorem)**.** *Suppose $a < b$. If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exists a point $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The idea behind the proof is to shift the graph of $f$ (by subtracting a particular line from it), apply Rolle's Theorem, and then shift back. If the line has slope $\frac{f(b)-f(a)}{b-a}$, the shifted point will be the one we want.

*Proof.* Let

$$g(x) = f(x) - \Big[\Big(\frac{f(b) - f(a)}{b - a}\Big)(x - a)\Big].$$

Then

$$g(a) = f(a) - \Big[\Big(\frac{f(b) - f(a)}{b - a}\Big)(a - a)\Big] = f(a)$$

and

$$g(b) = f(b) - \Big[\Big(\frac{f(b) - f(a)}{b - a}\Big)(b - a)\Big] = f(b) - [f(b) - f(a)] = f(a).$$

Then $g(a) = g(b) = f(a)$. Since $g$ is $f$ plus a line, and since the line is differentiable everywhere, $g$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Therefore we may apply Rolle's Theorem to $g$ to find some $c \in (a,b)$ such that $g'(c) = 0$. Note that, using the sum rule, the fact that the derivative is a line is just the slope of that line, and the fact that $-\Big[\Big(\frac{f(b)-f(a)}{b-a}\Big)(x - a)\Big]$ is a line with slope $-\frac{f(b)-f(a)}{b-a}$, we deduce that

$$0 = g'(c) = f'(c) - \Big[\frac{f(b) - f(a)}{b - a}\Big].$$

Adding $\frac{f(b)-f(a)}{b-a}$ to both sides of this equation yields

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

## 1.4. **Intermediate value theorem for derivatives.**

**Theorem 1.4.** *Fix $a < b$. Let $f : [a,b] \to \mathbb{R}$ be a function which is differentiable on $[a,b]$.*

- *(i) If $f'(a) < 0 < f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*
- *(ii) For any real number $\lambda$, if $f'(a) < \lambda < f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = \lambda$.*
- *(iii) For any real number $\lambda$, if $f'(a) > \lambda > f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = \lambda$.*

This seems to be a simple application of the regular intermediate value theorem applied to $f'$. However, just because $f'$ exists does not mean it is continuous, which is required to apply the IVT.

The idea behind this proof is as follows: For $(i)$, since $f'(a) < 0$, the value of $f$ must dip below $f(a)$ immediately to the right of $a$ (draw a picture with the tangent line at $a$ having a negative slope to see this). Similarly, since $f'(b) > 0$, the value of $f$ must dip below $f(b)$ immediately to the left of $b$ (again, draw a picture). These facts mean that the minimum of $f$ must occur at some point in $(a, b)$, and by the Interior Extremum Theorem, the derivative at this point must be zero.

*Proof.* $(i)$ Since $f$ is differentiable on $[a, b]$, it is continuous on $[a, b]$, and has a minimum. Fix $c \in [a, b]$ such that $f(c)$ is the minimum of $f$ on $[a, b]$. We have three possibilities: Case 1: $c = a$. Case 2: $c = b$. Case 3: $c \in (a, b)$. We will show that cases 1 and 2 actually cannot occur.

First, suppose $c = a$. Then since $f(c) = f(a)$ is the minimum of $f$ on $[a, b]$, $f(y) - f(a) \geqslant 0$ for any $y \in [a, b]$. Then since $\frac{f(y)-f(a)}{y-a}$ is a non-negative number divided by a positive number for every $y \in (a, b]$,

$$f'(a) = \lim_{y \to a^+} \frac{f(y) - f(a)}{y - a} \geqslant 0.$$

But this contradicts the hypothesis that $f'(a) < 0$, and case 1 cannot occur.

Next, suppose $c = b$. Then $f(y) - f(b) \geqslant 0$ as in the previous paragraph. Since $y - b < 0$ for any $y \in [a, b)$, $\frac{f(y)-f(b)}{y-b} \leqslant 0$ for every $y \in [a, b)$. Then

$$f'(b) = \lim_{y \to b^-} \frac{f(y) - f(b)}{y - b} \leqslant 0.$$

But this contradicts the hypothesis that $f'(b) > 0$. Therefore case 2 cannot occur.

This means $c \in (a, b)$. By the Interior Extremum Theorem, $f'(c) = 0$.

$(ii)$ Let $g(x) = f(x) - \lambda x$ for each $x \in [a, b]$. Then apply $(i)$ to the function $g$.

$(iii)$ Let $g(x) = \lambda x - f(x)$ for each $x \in [a, b]$. Then apply $(i)$ to the function $g$.

□