1. Tuesday, 26 January

1.1. **Density of the rational and irrational numbers.** We say that a subset D of the real line is *dense* if for any real numbers a, b with $a < b$, $(a, b) \cap D \neq \varnothing$. The main result of this subsection is the following.

Theorem 1.1. *The rational numbers, $\mathbb{Q}$, and the irrational numbers, $\mathbb{J}$, are dense.*

We need the following facts.

Facts. (i) *For any natural number n, $\sup\{\frac{k}{n} : k \in \mathbb{N}\} = \infty$ and $\inf\{\frac{k}{n} : k \in \mathbb{Z}\} = -\infty$.*
(ii) *If p and q are rational, $p - q$ is rational.*
(iii) *If r is irrational and q is rational, then $q + r$ is irrational.*
(iv) *If r is irrational and m is a natural number, $\frac{r}{m}$ is irrational.*
(v) *$\sqrt{2}$ is irrational.*

Proof. (i) Let $A = \{\frac{k}{n} : k \in \mathbb{N}\}$. We know that A is non-empty. We therefore only need to show that it is not bounded above. To obtain a contradiction, suppose that $s = \sup A < \infty$. We claim that $s - \frac{1}{n}$ is also an upper bound for A, which will contradict the fact that s is the *least* upper bound for this set. In order to see that $s - \frac{1}{n}$ is an upper bound for A, fix $\frac{k}{n} \in A$. Since $k \in \mathbb{N}$, $k + 1 \in \mathbb{N}$, and $\frac{k+1}{n} \in A$. Since s is an upper bound for A and since $\frac{k+1}{n} \in A$,

$$\frac{k+1}{n} \leqslant s.$$

Subtracting $\frac{1}{n}$ from both sides yields that $\frac{k}{n} \leqslant s - \frac{1}{n}$. Since $\frac{k}{n}$ was an arbitrary member of A, this shows that $s - \frac{1}{n}$ is an upper bound for A, which gives the contradiction and finishes the proof of the first statement.

The proof that $\inf\{\frac{k}{n} : k \in \mathbb{Z}\} = -\infty$ is similar, and we ask the student to try the proof on their own.

(ii) We know this.

(iii) Suppose that q is rational and $r + q$ is rational. Then by (ii), $r = (r + q) - q$ is rational.

(iv) If $\frac{r}{m} = \frac{i}{j}$ for integers i, j, then $r = \frac{mi}{j}$ must be rational.

(v) We use here without proof the fact that prime factorizations are unique. This means that if p is a prime, then p is a divisor of $n \in \mathbb{N}$ if and only if p occurs in the prime factorization of n. Moreover, if $p_1 \ldots p_s$ is the prime factorization of m and if $q_1 \ldots q_t$ is the prime factorization of n, then $p_1 \ldots p_s q_1 \ldots q_t$ is the prime factorization of mn, and p divides mn if and only if it is one of the primes $p_1, \ldots, p_s, q_1, \ldots, q_t$, which happens if and only if it divides either m or n.

To obtain a contradiction, suppose that $\sqrt{2}$ is rational. Then we may write $\sqrt{2} = \frac{m}{n}$. By cancelling, we may assume that the greatest common divisor of m and n is 1. Then note that $n\sqrt{2} = m$, and squaring both sides gives that $2n^2 = m^2$. Since 2 is a divisor of the left side, 2 must be a divisor of the right side. By the previous paragraph, this means 2 is a

divisor of m. Write $m = 2k$. Then $2n^2 = m^2 = (2k)^2 = 4k^2$. Dividing both sides by 2, we see that $n^2 = 2k^2$. Since 2 divides the right side of this equation, it must also divide the left side, and by the previous paragraph 2 must divide n. Therefore 2 was a divisor of m and n, which contradicts our assumption that the greatest common divisor of m and n is 1. This contradiction finishes the proof.

$\square$

Proof of Theorem 1.1. Fix $a < b$. We need to show that $(a, b) \cap \mathbb{Q} \neq \varnothing$ and $(a, b) \cap \mathbb{J} \neq \varnothing$. Recall that $\mathbb{J}$ is our notation for the irrational numbers. We first prove that $(a, b) \cap \mathbb{Q} \neq \varnothing$. First fix numbers c, d such that $a < c < d < b$. We will actually show that $[c, d] \cap \mathbb{Q} \neq \varnothing$. Fix a natural number m such that $(d - c)^{-1} < m$, which we may do since $\sup \mathbb{N} = \infty$ by our first fact above (applied with $n = 1$). Then $\frac{1}{m} < d - c$. Let $A = \{\frac{k}{n} : k \in \mathbb{N}, \frac{k}{n} \leqslant d\}$ and $B = \{\frac{k}{n} : k \in \mathbb{Z}\}$.

We first claim that A is non-empty, bounded above, and $a < \sup A \in A$. This will show that $\sup A \in [c, d] \cap \mathbb{Q} \subset (a, b) \cap \mathbb{Q}$, and will give the first part of the theorem. Let us see why the claim is true.

Our first fact above says that $\inf B = -\infty$, so there must exist some $\frac{k}{n} \in B$ less than d, and this is a member of A. Therefore A is non-empty.

Of course, A is bounded above by d.

Since d is an upper bound for A, $\sup A \leqslant d$, since $\sup A$ is the least upper bound. Note that since $\sup A$ is the least upper bound of A, $\sup A - \frac{1}{2n}$ is not an upper bound for A. Therefore there exists $\frac{k}{n} \in A$ such that $\sup A - \frac{1}{2n} < \frac{k}{n}$. We will actually show that $\sup A = \frac{k}{n}$, which shows that $\sup A = \frac{k}{n} \in A$. Since $\sup A$ is an upper bound for A, $\frac{k}{n} \leqslant \sup A$. We will next show that $\frac{k}{n}$ is an upper bound for A, so that $\sup A \leqslant \frac{k}{n}$ since $\sup A$ is the *least* upper bound. In order to see that $\frac{k}{n}$ is an upper bound for A, fix $\frac{j}{n} \in A$. We need to show that $\frac{j}{n} \leqslant \frac{k}{n}$, which happens if and only if $j \leqslant k$. If $j > k$, then since j and k are integers, $j \geqslant k + 1$. This means that

$$\frac{j}{n} \geqslant \frac{k+1}{n} = \frac{k}{n} + \frac{1}{n} > (\sup A - \frac{1}{2n}) + \frac{1}{n} = \sup A + \frac{1}{2n} > \sup A.$$

But this is a contradiction, since $\sup A$ is an upper bound for A, which contains $\frac{j}{n}$. Therefore the assumption that $j > k$ leads to a contradiction, which means that $j \leqslant k$ and $\frac{j}{n} \leqslant \frac{k}{n}$. This shows that $\frac{k}{n}$ is an upper bound for A, so $\sup A \leqslant \frac{k}{n}$. Then

$$\frac{k}{n} \leqslant \sup A \leqslant \frac{k}{n},$$

and these are equal. Therefore $\sup A = \frac{k}{n} \in A$.

Next, still letting $\frac{k}{n} = \sup A$, note that $\frac{k+1}{n} \notin A$, since it exceeds the supremum of A. But since $\frac{k+1}{n} \in B$, in order for $\frac{k+1}{n}$ to fail to be in A, it must be that $\frac{k+1}{n} > d$. Therefore $\frac{k}{n} > d - \frac{1}{n} > d - (d - c) = c$. Therefore we have shown that

$$\frac{k}{n} \in [c, d] \cap \mathbb{Q} \subset (a, b) \cap \mathbb{Q}.$$

This shows that for any $a < b$, $(a, b) \cap \mathbb{Q} \neq \varnothing$.

Next, we will show that $(a, b) \cap \mathbb{J} \neq \varnothing$. Let $\frac{k}{n} \in (a, b) \cap \mathbb{Q}$ be the rational number we found above. Since $\frac{k}{n} < b$, there exists a natural number so large that $\frac{k}{n} + \frac{\sqrt{2}}{m} < b$ (indeed, by our first fact above, $\sup \mathbb{N} = \infty$, and there exists a natural number m such that $\sqrt{2}\left(b - \frac{k}{n}\right) < m$, and this m satisfies $\frac{k}{n} + \frac{\sqrt{2}}{m} < b$). Then by facts (iii), (iv), and (v) above, $\frac{k}{n} + \frac{\sqrt{2}}{m}$ is irrational. Moreover,

$$a < \frac{k}{n} < \frac{k}{n} + \frac{\sqrt{2}}{m} < b,$$

and

$$\frac{k}{n} + \frac{\sqrt{2}}{m} \in (a, b) \cap \mathbb{J}.$$

$\square$

1.2. Closures and interiors: Examples.

Recall that the interior of A is the largest open subset of A, and A is open if and only if $\mathrm{int} A = A$. Recall also that x is in the interior of A if and only if there exists $r > 0$ such that $(x - r, x + r) \subset A$.

Recall that the closure of A is the smallest closed set which contains A, and that A is closed if and only if $\overline{A} = A$. Recall also that x is in the closure of A if and only if for every $r > 0$, $(x - r, x + r) \cap A \neq \varnothing$.

Since the interior of A is always a subset of A, and since A is always a subset of $\overline{A}$, we know that

$$\mathrm{int} A \subset A \subset \overline{A}$$

for any set A.

We will now compute the interiors and closures of some sets.

Example 1.1.
$$\mathrm{int}(0, 1) = \mathrm{int}[0, 1] = \mathrm{int}[0, 1) = (0, 1)$$

and

$$\overline{(0, 1)} = \overline{[0, 1]} = \overline{[0, 1)} = [0, 1].$$

Since each set $(0, 1)$, $[0, 1)$, and $[0, 1]$ all include $(0, 1)$, which we know to be open, the interior of each set must include $(0, 1)$ (since the interior of a set is the largest open subset of that set). This shows that $(0, 1)$ is its own interior. In order to see that $(0, 1)$ is the interior of $[0, 1)$, we note that there are only two sets which contain $(0, 1)$ and which are contained in $[0, 1)$, namely these sets themselves. Since the interior of $[0, 1)$ contains $(0, 1)$ and is contained in $[0, 1)$, we just need to show that $0 \notin \mathrm{int}[0, 1)$ in order to know that the interior is $(0, 1)$. But for any $r > 0$, $(-r, r) = (0 - r, 0 + r) \not\subset [0, 1)$. By our remark above, 0 is not in the interior of $[0, 1)$.

Similarly, we argue that neither 0 nor 1 can lie in the interior of $[0, 1]$. Since the interior of $[0, 1]$ contains $(0, 1)$, is contained in $[0, 1]$, and (by the previous sentence) does not contain 0 or 1, it follows that the interior of $[0, 1]$ is $(0, 1)$.

Note that for any $r > 0$, $(-r, r) \cap (0, 1) = (0 - r, 0 + r) \cap (0, 1) \neq \varnothing$, $(-r, r) \cap [0, 1) \neq \varnothing$, and $(-r, r) \cap [0, 1] \neq \varnothing$. By the remark above, this means 0 lies in the closures of each of the sets $(0, 1)$, $[0, 1)$, and $[0, 1]$. Similarly, for any $r > 0$, $(1 - r, 1 + r) \cap (0, 1) \neq \varnothing$, $(1 - r, 1 + r) \cap [0, 1) \neq \varnothing$, and $(1 - r, 1 + r) \cap [0, 1] \neq \varnothing$. Thus 1 is in the closure of each of these three sets. Since $(0, 1)$ is contained in each of these sets, it must be contained in their closures (since the closure of a set contains that set and therefore all of its subsets), it follows that $(0, 1)$ is a subset of the closures of each of the sets $(0, 1)$, $[0, 1)$, and $[0, 1]$. We have argued that $[0, 1] \subset \overline{(0, 1)}, \overline{[0, 1)}, \overline{[0, 1]}$.

We need to show that the closures of each of these sets contains nothing else. However, since $[0, 1]$ is a closed set which contains $(0, 1)$, the *smallest* closed set containing $(0, 1)$ must be a subset of $[0, 1]$, and therefore $\overline{(0, 1)} \subset [0, 1]$. Therefore $\overline{(0, 1)} \subset [0, 1] \subset \overline{(0, 1)}$, and these sets are equal. We argue that $\overline{[0, 1)}, \overline{[0, 1]} \subset [0, 1]$ similarly.

Example 1.2. $\text{int}(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R} \setminus \mathbb{Z}$ and $\overline{\mathbb{R} \setminus \mathbb{Z}} = \mathbb{R} \setminus \mathbb{Z}$.

In order to see this, note that

$$\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n + 1),$$

which is open. Therefore this set is equal to its own interior. Note that this implies $\mathbb{Z} = \overline{\mathbb{R} \setminus \mathbb{Z}}$ is closed!

Of course, $\mathbb{R} \setminus \mathbb{Z} \subset \overline{\mathbb{R} \setminus \mathbb{Z}}$. If k is an integer, then for any $r > 0$, $(k - r, k + r) \cap (\mathbb{R} \setminus \mathbb{Z}) \neq \varnothing$. Indeed, if $r < 1$, $k + \frac{r}{2} \in (k - r, k + r) \cap (\mathbb{R} \setminus \mathbb{Z})$. If $r \geqslant 1$, then $k + \frac{1}{2} \in (k - r, k + r) \cap (\mathbb{R} \setminus \mathbb{Z})$. This means that $k \in \overline{\mathbb{R} \setminus \mathbb{Z}}$. Therefore $\overline{\mathbb{R} \setminus \mathbb{Z}}$ contains all non-integers as well as all integers, and therefore must be all real numbers.

Example 1.3. If D is a dense set, then $\overline{D} = \mathbb{R}$ and $\text{int}(\mathbb{R} \setminus D) = \varnothing$.

Of course, $\overline{D} \subset \mathbb{R}$. We need to show that $\mathbb{R} \subset \overline{D}$. For this, we need to fix $x \in \mathbb{R}$ and show that it is in $\overline{D}$. As we remarked above, we need to show that for this fixed x, and for any $r > 0$, $(x - r, x + r) \cap D \neq \varnothing$. But this is true by our definition of dense.

We next need to show that $\text{int}(\mathbb{R} \setminus D) = \varnothing$. Note that, by our remark above, $\text{int}(\mathbb{R} \setminus D)$ is the set of all x such that there exists an $r > 0$ such that $(x - r, x + r) \subset \mathbb{R} \setminus D$, which happens if and only if there exists $r > 0$ such that $(x - r, x + r) \cap D = \varnothing$. But by the definition of dense, there can be no such x! Therefore no x can be in the interior of $\mathbb{R} \setminus D$, and $\text{int}(\mathbb{R} \setminus D) = \varnothing$.

Exercise 1.1. *Show that the converses of the statements above are true. By this we mean that if $\overline{D} = \mathbb{R}$, then D is a dense set, and if $int(\mathbb{R} \setminus D) = \varnothing$, then D is a dense set.*

2. Thursday, 28 January

2.1. **Business about sequences.** We say that a sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for any natural numbers $p, q \geqslant m$, $|x_p - x_q| < \varepsilon$.

Note that the definition of Cauchy looks strikingly similar to the definition of convergent, but it is not the same definition. In particular, there is no limit anywhere in the definition of a Cauchy sequence. It will turn out that convergent and Cauchy are equivalent properties for sequences of real numbers (and our next goal will be to prove this, although it will take several steps and will not be completed in today's lecture). One may wonder why we give a special name to Cauchy sequences when they are really just convergent sequences. We attempt to illustrate the reason for the difference with the following example. The following example is **not** one which you will be required to remember or know at any point in this class. It is only for fun and to satisfy the curiosity of the student.

Example 2.1. Let P denote the collection of all polynomials (with real coefficients, of course). For any $p, q \in P$, let

$$d(p, q) = \int_0^1 |p(x) - q(x)| dx.$$

Note that this function d behaves perfectly well as a notion of distance. That is, we think of $d(p, q)$ as being the distance from the polynomial p to the polynomial q. Then we call the pair (P, d) a *metric space* (a set of points together with some notion of distance between the points in that set).

It makes sense to define convergent and Cauchy sequences in this metric space: The sequence $(p_n) \subset P$ *converges* if there exists $p \in P$ (note that p must be in P, which is the key to this example!) such that for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $n \geqslant m$, $d(p, p_n) < \varepsilon$. We say the sequence (p_n) is *Cauchy* if for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $k, \ell \geqslant m$, $d(p_k, p_\ell) < \varepsilon$.

Let $p_n = \sum_{k=0}^n \frac{x^k}{k!}$. Note that p_n is a polynomial for each n. Recall also that $\sum_{k=0}^\infty \frac{x_k}{k!}$ is a power series representation of the function e^x. One can check that the sequence (p_n) is Cauchy (since by the triangle inequality,

$$\int_0^1 |p_n(x) - p_m(x)| dx \leqslant \int_0^1 (|p_n(x) - e^x| + |e^x - p_m(x)|) dx = \int_0^1 |p_n(x) - e^x| dx + \int_0^1 |e^x - p_m(x)| dx,$$

and both of these summands will be small if m and n are large), but it is not convergent. If it were convergent, the definition of convergence would mean that the sequence converges to a polynomial! But if (p_n) were to converge to p, for any $n \in \mathbb{N}$, the triangle inequality again yields that

$$\int_0^1 |p(x) - e^x| dx \leqslant \int_0^1 |p(x) - p_n(x)| dx + \int_0^1 |p_n(x) - e^x| dx.$$

But the right side tends to zero as n tends to infinity, so that $\int_0^1 |p(x) - e^x| dx = 0$. But since p and e^x are continuous, this means that $p(x) = e^x$, which is impossible, since e^x is not a polynomial. Thus in the space (P, d), the notions of Cauchy and convergent to not coincide. We have here a Cauchy, non-convergent sequence. It is somewhat easy to see that the reverse cannot hold. Even in the space (P, d), convergent sequences must be Cauchy. Indeed, if the

points in the sequence are eventually close to the limit, then they must eventually be close to each other. We formally prove this next for the real numbers.

Lemma 2.1. *If (x_n) is convergent, then (x_n) is Cauchy.*

Proof. Let $x = \lim_n x_n$. Fix $\varepsilon > 0$. By the definition of convergent, there exists some number m such that if $n \geqslant m$, $|x - x_n| < \varepsilon/2$. Then for any $p, q \geqslant m$,

$$|x_p - x_q| = |x_p - x + x - x_q| \leqslant |x_p - x| + |x - x_q| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that (x_n) is Cauchy.

$\square$

We say the sequence (x_n) is *increasing* if for any natural numbers m, n with $m < n$, $x_m \leqslant x_n$. We say the sequence is *strictly increasing* if for any natural numbers m, n with $m < n$, $x_m < x_n$. We say (x_n) is decreasing if for any natural numbers m, n with $m < n$, $x_m \geqslant x_n$, and (x_n) is *strictly decreasing* if for any natural numbers m, n with $m < n$, $x_m > x_n$. We say (x_n) is *monotonic* if it is either increasing or decreasing.

We say the sequence (x_n) is *bounded above* if there exists a number M such that for all $n \in \mathbb{N}$, $x_n \leqslant M$. Note that the sequence (x_n) is bounded above if and only if the set $\{x_n : n \in \mathbb{N}\}$ consisting of the numbers which occur in the sequence is a bounded above set. The sequence (x_n) is *bounded below* if there exists a number M such that for all $n \in \mathbb{N}$, $x_n \geqslant M$. We say (x_n) is *bounded* if it is bounded above and bounded below.

Proposition 2.2. *Let (x_n) be a sequence of real numbers.*

  (i) *If (x_n) is increasing and bounded above, (x_n) is convergent.*
  (ii) *If (x_n) is decreasing and bounded below, (x_n) is convergent.*

Proof. We only prove (i). We ask the student to fill in the details of (ii) to test their understanding.

Suppose (x_n) is increasing and bounded above. Let $A = \{x_n : n \in \mathbb{N}\}$. Since (x_n) is bounded above, A is bounded above. Of course, A is non-empty. Therefore $\sup A$ is a real number, call it x. We will show that $x = \lim_n x_n$. Fix $\varepsilon > 0$. Since $x - \varepsilon$ is not an upper bound for A, there must exist a point $y \in A$ such that $x - \varepsilon < y$. But since $y \in A$, there exists $m \in \mathbb{N}$ such that $y = x_m$. Note that since $x - \varepsilon < y = x_m$, $x - x_m < \varepsilon$. Then for any $n \geqslant m$, since $x_m \leqslant x_n$, $x - x_n \leqslant x - x_m < \varepsilon$. But since x is an upper bound for A, $x_n \leqslant x$ and $x - x_n \geqslant 0$. Therefore

$$0 \leqslant x - x_n \leqslant \varepsilon.$$

Therefore $|x - x_n| = x - x_n < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown that $x = \lim x_n$.

$\square$

Corollary 2.3. *If (x_n) is a monotonic, bounded sequence, then it is convergent.*

Proof. Since it is monotonic, it is either increasing or decreasing. Since it is bounded, it is bounded above and bounded below. This means that the sequence is either increasing and bounded above, or decreasing and bounded below. Therefore (x_n) is convergent by Proposition 2.2.

$\square$