LECTURE 1: THE MAUREY-ROSENTHAL SPACE

Recall that a sequence \((x_i)\) in a Banach space is called \(K\)-basic if for any \(m, n \in \mathbb{N}\) with \(m \leq n\) and any scalars \((a_i)_{i=1}^n\), \(\|\sum_{i=1}^m a_i x_i\| \leq K \|\sum_{i=1}^n a_i x_i\|\). The sequence is called \(K\)-unconditional if for any \(n \in \mathbb{N}\), any scalars \((a_i)_{i=1}^n\), and any \((\varepsilon_i)_{i=1}^n \in \{\pm 1\}^n\), \(\|\sum_{i=1}^n \varepsilon_i a_i x_i\| \leq K \|\sum_{i=1}^n a_i x_i\|\). The sequence \((x_i)\) is called conditional if it is not \(K\)-unconditional for any \(K\).

Recall that \((y_i)\) is a block of the basic sequence \((x_i)\) if \(y_i \neq 0\) and if there exist scalars \((a_i)_{i=1}^\infty\) and \(0 \leq k_0 < k_1 < \ldots\) such that for all \(i \in \mathbb{N}\), \(y_i = \sum_{j=k_i-1}^{k_i-1} a_j x_j\). Note that a block of a \(K\)-basic (resp. \(K\)-unconditional) sequence is also \(K\)-basic (\(K\)-unconditional), which is easily checked.

Recall that \(c_{00}\) denotes all finitely non-zero scalar sequences.

The summing basis Let \((s_i)\) denote the canonical \(c_{00}\) basis endowed with the norm

\[
\|\sum_{i=1}^n a_i s_i\| = \max_{1 \leq n} \|\sum_{i=1}^n a_i\|
\]

where \((a_i)_{i=1}^\infty \in c_{00}\). Note that \((s_i)\) is a normalized, monotone basis for the completion of \(c_{00}\) with respect to this norm.

Note that \(\max_{1 \leq i \leq n} |a_i| \leq 2\|\sum_{i=1}^n a_i s_i\|\) for any scalar sequence \((a_i)_{i=1}^n\), and this is sharp. Indeed, \(\frac{1}{2}s_1 - s_2\) demonstrates that this is sharp.

Note that \(\|\sum_{i=1}^n s_i\| = n\) while \(\|\sum_{i=1}^n (-1)^i s_i\| = 1\), so that \((s_i)\) is conditional.

1. The Maurey-Rosenthal space

Fix \(\varepsilon \in (0, 1/2)\). Fix positive numbers \((\varepsilon_i)_{i=1}^\infty\) such that \(\sum_{i=1}^\infty \sum_{j=1}^\infty \varepsilon_{\max\{|i,j\}} < \varepsilon\). Fix natural numbers \(1 < m_1 < m_2 < \ldots\) such that for all \(i < j\), \(m_i / \sqrt{m_j} < \varepsilon_j\). Let \(M = \{1, m_1, m_2, \ldots\}\). Let \(M\) denote the set of all sequences of finite sets of the form \((E_i)_{i=1}^n\), \(E_i \subset \mathbb{N}\), \(n \in \mathbb{N}\), \(E_1 < \ldots < E_n\), as well as the empty sequence. Here, \(E < F\) means \(\max E < \min F\). Fix a bijection \(\phi : M \rightarrow M\) such that if \(s, t \in M\) and \(s\) is a proper initial segment of \(t\), \(\phi(s) < \phi(t)\), and note that such a bijection necessarily satisfies \(\phi(\emptyset) = 1\). Let us say that a \((finite\ or\ infinite)\ sequence \((E_i)\) of successive subsets of \(\mathbb{N}\) is special if for any initial segment \(t = (E_i)_{i=1}^n\) of \((E_i)\), \(|E_n| = \phi(t_{n-1})\). Let us agree that the empty sequence is also special, and let \(\sigma\) denote all special sequences. Given \(t = (E_i) \in \sigma\), let \(g_t = \sum_{i=1}^{1_{E_i}}\). Let

\[
L = \{g_t : t \in \sigma, |t| < \infty\} \subset [0, 1]^\mathbb{N}.
\]

Let \(K\) denote the closure of \(L\) in \([0, 1]^\mathbb{N}\). Since \([0, 1]^\mathbb{N}\) is compact (by Tychonoff’s theorem) and metrizable \((d(s, t) = \sum 2^{-|s_i - t_i|}\) is a metric compatible with the product topology), \(K\) is compact metrizable.
Remark 1.1. Let \((E_i)_{i=1}^n\) and \((F_j)_{j=1}^m\) be two special sequences. Then there exists a natural number \(l\) such that

(i) for every \(1 \leq i < l\), \(E_i = F_i\),
(ii) for each \(l \leq i \leq n\) and \(l \leq j \leq m\), \(E_i \neq F_j\),
(iii) for each \(l < i \leq n\) and \(l < j \leq m\), \(|E_i| \notin \{|F_k| : 1 \leq k \leq m\}\) and \(|F_j| \notin \{|E_k| : 1 \leq k \leq n\}\).

Indeed, if one of these sequences is an initial segment of the other, we may take \(l\) to be \(1 + \min\{m, n\}\). Otherwise, we let \(l\) be the first \(i\) such that \(E_i \neq F_i\). Then suppose that for some \(1 \leq i \leq n\) and \(1 \leq j \leq m\), if \(|E_i| = |F_j|\), then

\[
\phi(\langle E_1, \ldots, E_{i-1} \rangle) = |E_i| = |F_j| = \phi(\langle F_1, \ldots, F_{j-1} \rangle).
\]

It follows from the injectivity of \(\phi\) that \((E_k)_{k=1}^{i-1} = (F_k)_{k=1}^{j-1}\), so that \(i = j < l\).

Lemma 1.2. The set \(K\) is contained in \(c_0\).

Proof. Fix \(\delta > 0\) and for each \(k \in \mathbb{N}\), let \(U_k = \{(a_n) \in [0, 1]^\mathbb{N} : |\{n : a_n > \delta\}| \geq k\}\). Note that \(U_k\) is open in \([0, 1]^\mathbb{N}\). We will show that there exists \(k = k(\varepsilon)\) such that \(U_k \cap L = \emptyset\), so that \(U_k \cap K = \emptyset\). Since \(\delta > 0\) was arbitrary, this will give the result.

Fix \(l \in \mathbb{N}\) such that \(m_l^{-1/2} < \delta\) and let \(T = \{\phi^{-1}(m_i) : 1 \leq i \leq l\} \cap \sigma\). Note that any non-empty initial segment of \(T\) is also a member of \(T\) by the properties of \(\phi\). Let

\[
k = \max\left\{\sum_{i=1}^n |E_i| + \phi(\langle E_1, \ldots, E_n \rangle) : (E_i)_{i=1}^n \in T\right\}.
\]

Fix any \((F_j)_{j=1}^m \in \sigma\). In the case that for all \((E_i)_{i=1}^n \in T\), \(E_1 \neq F_1\), let \(r = 0\). Otherwise let \(r\) be the largest value of \(j \in \{1, \ldots, m\}\) such that there exists some \((E_i)_{i=1}^n \in T\) such that for each \(1 \leq i \leq j\), \(E_i = F_i\).

If \(r = 0\), by Remark 1.1, for each \(1 < j \leq m\) and any \((E_i)_{i=1}^n \in T\), \(|F_j| \notin \{|E_i| : 1 \leq i \leq n\}\). This means that for each \(1 < j \leq m\),

\[
|F_j| \notin \bigcup_{(E_i)_{i=1}^n \in T} \{|E_i| : 1 \leq i \leq n\} \supset \{m_1, \ldots, m_l\}.
\]

Therefore \(|F_j|^{-1/2} \leq m_l^{-1/2} < \varepsilon\), and \(g(F_j)_j=1^m\) has at most \(|F_1| = 1 \leq k\) coordinate which is larger than \(\delta\).

If \(r > 0\), fix \((G_i)_{i=1}^p \in T\) such that \(G_i = F_i\) for each \(1 \leq i \leq r\). Again using Remark 1.1, for each \(r + 1 < j \leq m\),

\[
|F_j| \notin \bigcup_{(E_i)_{i=1}^n \in T} \{|E_i| : 1 \leq i \leq n\} \supset \{m_1, \ldots, m_l\}.
\]

Then \(g(F_j)_j=1^m\) has at most

\[
\sum_{j=1}^{r+1} |F_j| = \sum_{j=1}^r |G_j| + \phi(\langle G_1, \ldots, G_r \rangle) \leq k.
\]
coordinates which are larger than \( \delta \).

In the sequel, we will treat the members of \( c_{00} \) as vectors as well as functionals on \( c_{00} \) via the standard inner product action.

**Lemma 1.3.** Define \( \| \cdot \| \) on \( c_{00} \) by \( \| x \| = \sup_{f \in L} |f(x)| \). This defines a norm making the canonical \( c_{00} \) basis normalized, monotone and weakly null in the completion \( X_{MR} \) of \( (c_{00}, \| \cdot \|) \).

**Proof.** We will show that for any \( 1 \leq m \leq n \) and any scalars \( (a_i)_{i=1}^n \), and for any \( f \in L \), there exists \( g \in L \) such that \( f(\sum_{i=1}^m a_i e_i) = g(\sum_{i=1}^n a_i e_i) \). This will yield that \( \| \sum_{i=1}^m a_i e_i \| \leq \| \sum_{i=1}^n a_i e_i \| \), and \( (e_i) \) is monotone. To that end, suppose \( m, n \), \( (a_i)_{i=1}^n \), and \( f \in L \) are fixed. Write \( f = \sum_{i=1}^n \frac{E_i}{|E_i|^{1/2}} \). We may assume \( \min E_i \leq m \) for every \( 1 \leq i \leq m \). To see this, note that if \( q \) is the maximum value of \( i \) such that \( \min E_i \leq m \), \( \langle 1_{E_i}, \sum_{i=1}^m a_i e_i \rangle = 0 \) for every \( q < i \leq p \). Then \( f' = \sum_{i=1}^q \frac{1_{E_i}}{|E_i|^{1/2}} \in L \) and \( f'(\sum_{i=1}^m a_i e_i) = f(\sum_{i=1}^m a_i e_i) \). Thus by replacing \( f' \) with \( f \), we may make the assumption above. Let \( F = E_p \cap [1, m] \) and let \( G \) be any set such that \( \min G > n \) and \( |F| + |G| = |E_p| \). Then

\[
g := \sum_{i=1}^{p-1} \frac{1_{E_i}}{|E_i|^{1/2}} + \frac{1_{E_p \cup G}}{|F \cup G|^{1/2}} \in L
\]

and \( f(\sum_{i=1}^m a_i e_i) = g(\sum_{i=1}^n a_i e_i) \).

We remark that since \( L \subset [0, 1]^N \), \( \| x \| \leq \| x \|_{\ell_1} \) for every \( x \in c_{00} \). Since every length one sequence \( (\{m\}) \) is a special sequence, it follows that the coordinate functionals lie in \( L \), whence \( \| x \| \geq \| x \|_{\ell_1} \) for every \( x \in c_{00} \). From these two inequalities it follows that for every \( i \in \mathbb{N} \), \( 1 = \| e_i \|_{c_0} \leq \| e_i \| \leq \| e_i \|_{\ell_1} = 1 \), so that \( (e_i) \) is normalized in \( X_{MR} \).

Next, recall that \( K \) is the closure of \( L \) in \([0, 1]^N\), which is compact, Hausdorff. Define the linear function \( i : c_{00} \to C(K) \) by \( i(x) = h_x \), where \( f_x \) is given by \( h_x(f) = f(x) \). Note that \( f(x) \) is well-defined, since \( x \) has finite support. Moreover, if \( x = \sum_{i=1}^n a_i e_i \) and if \( f = \sum_{i=1}^\infty b_i e_i \), \( f(x) = \sum_{i=1}^n a_i b_i \), which is continuous on \([0, 1]^N\) with its product topology. Thus \( f_x \) is indeed continuous. Last,

\[
\| f_x \|_{C(K)} = \sup_{f \in K} |f(x)| = \sup_{f \in L} |f(x)| = \| x \|,
\]

where the middle equality comes from the density of \( L \) in \( K \). Thus \( i : c_{00} \to C(K) \) is an isometric map of \( c_{00} \) into \( C(K) \). Thus \( i \) admits a unique extension to an isometric embedding \( j : X_{MR} \to C(K) \). Moreover, \( f_{e_i} \) is pointwise null on \( K \), since \( K \subset c_0 \), and normalized, since \( (e_i) \) is normalized and \( j \) is an isometry. It follows that \( (f_{e_i}) \) is normalized and pointwise null on \( K \), whence weakly null in \( C(K) \) by the Riesz representation theorem. Since \( (e_i) \) is isometrically equivalent to \( (f_{e_i}) \), \( (e_i) \) is weakly null.

□
2. THE MAIN INEQUALITY

Since any block of an unconditional sequence is unconditional, it follows that if every subsequence of \((e_i)\) has a conditional block, every subsequence of \((e_i)\) must be conditional. Given an infinite subset \(N\) of \(\mathbb{N}\), let \(\{E_1, E_2, \ldots\}\) be a partition of \(N\) into successive subsets such that \(|E_1| = 1\) and \(|E_{i+1}| = \phi((E_j)_{j=1}^{i})\) for every \(i \in \mathbb{N}\). Let \(x_i = 1_{E_i}/|E_i|^{1/2}\).

**Theorem 2.1.** The sequence \((x_i)\) is equivalent to the summing basis.

**Proof.** Note that by the definition, \(\sum_{i=1}^{m} x_i \in L\) for every \(n \in \mathbb{N}\). Thus for any \(n \in \mathbb{N}\), scalars \((a_i)_{i=1}^{m}, n, \sum_{i=1}^{m} \sum a_i = \max_{1 \leq m \leq n} \left( \left\| \sum_{i=1}^{m} x_i, \sum_{i=1}^{n} a_i x_i \right\| \right) \leq \sum_{i=1}^{n} a_i x_i \|

Fix \(f = \sum_{j=1}^{m} \frac{1}{|F_j|^{1/2}} \in L\) and scalars \((a_i)_{i=1}^{n}\). Let \(x = \sum_{i=1}^{n} a_i x_i\). By Remark 1.1, we may fix some \(l\) such that for each \(1 \leq i < l\), \(E_i = F_i\), and for each \(l < i \leq n\) and \(l < j \leq m\), \(|E_i| \notin \{|F_k| : 1 \leq k \leq m\}\) and \(|F_j| \notin \{|E_k| : 1 \leq k \leq n\}\). Note that for each \(1 \leq i < l\) and \(1 \leq j < i \leq m\), \(F_j = E_j < E_i = F_i < F_k\), so that \(\langle 1_{F_j}, 1_{E_i} \rangle = \langle 1_{F_k}, 1_{E_i} \rangle = 0\). Similarly, for each \(1 \leq j < l\) and \(1 \leq i < j \leq n\), \(\langle 1_{F_j}, 1_{E_i} \rangle = \langle 1_{F_j}, 1_{E_k} \rangle = 0\). Fix \(p_1, \ldots, p_n\) and \(q_1, \ldots, q_m\) such that for each \(1 \leq i \leq n\) and \(1 \leq j \leq m\), \(|E_i| = m_{p_i}\) and \(|F_j| = m_{q_j}\). Note \(p_1 < \ldots < p_n\) and \(q_1 < \ldots < q_m\). Let \(D = \{(i, j) : l \leq i \leq n, l \leq j \leq m, (i, j) \neq (l, l)\}\) and note that \((i, j) \mapsto (p_i, q_j)\) is an injection of \(D\) into \(\mathbb{N}^2\) by the properties of \(l\).

Note that for \((i, j) \in D\), if \(m_{p_i} > m_{q_j}\), which happens if and only if \(p_i > q_j\),

\[
\frac{1_{F_j}}{|F_j|^{1/2}}, \frac{1_{E_i}}{|E_i|^{1/2}} = \frac{|E_i \cap F_j|}{|E_i|^{1/2}|F_j|^{1/2}} \leq \frac{m_{q_j}}{m_{p_i}^{1/2}} < \varepsilon_{p_i}.
\]

Similarly, if \(m_{q_j} > m_{p_i}\), \(\langle 1_{F_j}, 1_{E_i} \rangle \leq \varepsilon_{q_j}\). Therefore \(\langle 1_{F_j}, 1_{E_i} \rangle \leq \varepsilon_{\max\{p_i, q_j\}}\). Using the fact that \(|E_i| = |F_i|\), we deduce that

\[
\left| \sum_{j=1}^{m} \frac{1_{F_j}}{|F_j|^{1/2}}, \sum_{i=1}^{n} \frac{1_{E_i}}{|E_i|^{1/2}} \right| \leq \left| \sum_{i=1}^{l-1} a_i \right| + \|(a_i)\|_{\infty} \frac{|E_i \cap F_j|}{|E_i|} + \|(a_i)\|_{\infty} \sum_{(i, j) \in D} \left( \frac{1_{F_j}}{|F_j|^{1/2}}, 1_{E_i} \right) \leq \left(1 + 2 + 2\varepsilon\right) \left\| \sum_{i=1}^{n} a_i x_i \right\| < 3 \left\| \sum_{i=1}^{n} a_i x_i \right\|.
\]

Here we have used that \(\|(a_i)\|_{\infty} \leq 2\| \sum_{i=1}^{n} a_i x_i \|\) and that since \((i, j) \mapsto (p_i, q_j)\) is an injection of \(D\) into \(\mathbb{N}^2\),

\[
\sum_{(i, j) \in D} \left( \frac{1_{F_j}}{|F_j|^{1/2}}, \frac{1_{E_i}}{|E_i|^{1/2}} \right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\max\{i, j\}} < \varepsilon < 1/2.
\]

□
3. A pathological norm on $\ell_2$

The following theorem was given in the work of Odell and Zheng. It is a modification of the Maurey-Rosenthal space which is attributed to Johnson.

**Theorem 3.1.** For every $K \geq 1$, there exists an equivalent norm on $\ell_2$ such that no subsequence of the canonical basis is $K$-unconditional.

We have already done most of the hard work. We fix a natural number $r$ such that $\sqrt{r} > 3K$. We will use all of the preparatory work done in the definition of the Maurey-Rosenthal space. Let

$$M = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{|E_i|^{1/2}} : (E_i)_{i=1}^{n} \in \sigma \right\} \cup \frac{1}{\sqrt{r}} B_{\ell_2}. $$

Note that $\frac{1}{\sqrt{r}} B_{\ell_2} \subset M \subset B_{\ell_2}$, so that the norm $|x|_J := \sup_{f \in M} |\langle f, x \rangle|$ defines a norm on $c_00$ making the canonical basis equivalent to the canonical $\ell_2$ basis. Fix any subsequence $N$ of $\mathbb{N}$ and let $E_1, E_2, \ldots, x_i = \frac{1}{|E_i|^{1/2}}$ as with the Maurey-Rosenthal space.

We note that

$$\left\| \sum_{i=1}^{r} (-1)^i x_i \right\|_J \geq \left\langle \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \frac{1}{|E_i|^{1/2}}, \sum_{i=1}^{r} \frac{1}{|E_i|^{1/2}} \right\rangle = \sqrt{r}. $$

We will show that $\left\| \sum_{i=1}^{r} (-1)^i x_i \right\|_J < 3$, showing that no subsequence of $(e_i)$ is $\sqrt{r}/3$-unconditional.

Fix $f \in M$. If $f \in \frac{1}{\sqrt{r}} B_{\ell_2}$, then

$$\left| \left\langle f, \sum_{i=1}^{r} (-1)^i x_i \right\rangle \right| \leq \frac{1}{\sqrt{r}} \left\| \sum_{i=1}^{r} x_i \right\|_{\ell_2} = 1. $$

If $f = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \frac{1}{|F_j|^{1/2}}$ for some $(F_j)_{j=1}^{m} \in \sigma$, let $l$ be as in the proof of Theorem 2.1. Then estimating as we did there,

$$\left| \left\langle f, \sum_{i=1}^{r} (-1)^i x_i \right\rangle \right| \leq \frac{1}{\sqrt{m}} \left| \sum_{i=1}^{l-1} (-1)^i \right| + \frac{1}{\sqrt{m}} \frac{|E_l \cap F_l|}{|E_l|} + \varepsilon < 3.$$