LECTURE 1.5: SHRINKING AND BOUNDEDLY-COMPLETE BASES

1. A description of the dual

Recall that a Schauder basis (e_i) for a Banach space is a sequence such that for every $x \in X$, there exists a unique scalar sequence (a_n) such that $\sum a_n e_n$ converges in norm to x. In this case, it is known that there exists a constant K, called the *basis constant* of (e_i) in X, such that for all $n \in \mathbb{N}$, the projection $P_n : X \to X$ given by $P_n \sum_m a_m e_m = \sum_{m=1}^n a_m e_m$ has norm at most K. We refer to the projection P_n as the n^{th}-*basis projection*. We say $(e_i) \subset X$ is a *basic sequence* if it is a Schauder basis for its closed span.

Proposition 1.1. *If (e_i) is a basis for the Banach space X, then X^* is the collection of all w^*-converging series $\sum_{i=1}^\infty a_i e_i^*$, which is the collection of all formal series $\sum_{i=1}^\infty a_i e_i^*$ such that $\sup_n \| \sum_{i=1}^n a_i e_i^* \|$ is bounded.*

Proof. First, we note that if $\sup_n \| \sum_{i=1}^n a_i e_i^* \| = C < \infty$, then $\sum b_i e_i \overset{x^*}{\mapsto} \sum a_i b_i$ is a well-defined, continuous, linear functional on X and $\sum_{i=1}^n a_i e_i^* \underset{w^*}{\to} x^*$ as $n \to \infty$. Indeed, for fixed $x = \sum b_i e_i \in X$ and $\varepsilon > 0$, there exists p such that for $p \leqslant m < n$, $\|(P_n - P_m)x\| < \varepsilon$. Then

$$| \sum_{i=m+1}^n a_i b_i | = |(\sum_{i=1}^n a_i e_i^*)((P_n - P_m)(x))| \leqslant C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $\sum_{i=1}^\infty a_i b_i$ is convergent, and x^* is well-defined. Moreover,

$$\|x\| = \lim_n \|P_n x\| \geqslant \lim_n C^{-1} |(\sum_{i=1}^n a_i e_i^*)(P_n x)| = C^{-1}|x^*(x)|,$$

whence $\|x^*\| \leqslant C$ is continuous at 0. Of course, x^* is linear, and therefore it is continuous. Since $x_n^* := \sum_{i=1}^n a_i e_i^*$ is bounded, in order to check that $x_n^* \underset{w^*}{\to} x^*$, we need only check that $x_n^*(x) \to x^*(x)$ for all x in a subset of X which has dense span in X. But obviously this is true for the basis of X. Thus the formal series $\sum_{i=1}^\infty a_i e_i^*$ is w^*-convergent and can be identified with a member of X^*.

If $x^* \in X^*$, then let $a_i = x^*(e_i)$ for each $i \in \mathbb{N}$. Then $\sup_n \| \sum_{i=1}^n a_i e_i^* \| = \sup_n \|P_n^* x^*\| \leqslant K\|x^*\|$. Thus every functional in X^* arises as the w^*-limit of partial sums of $\sum_{i=1}^n a_i e_i^*$ with $\sup_n \| \sum_{i=1}^n a_i e_i^* \| < \infty$. Of course, the identification $x^* \mapsto \sum_{i=1}^\infty a_i e_i^*$ is a bijection onto the set of formal series with bounded partial sums with inverse $\sum_{i=1}^\infty a_i e_i^* \mapsto w^* - \lim_n \sum_{i=1}^n a_i e_i^*$.

Finally, if $\sum_{i=1}^n a_i e_i^*$ is w^*-convergent to x^*, then $\sup_n \| \sum_{i=1}^n a_i e_i^* \| \leqslant \sup_n \|P_n^* x^*\| < \infty$. □

2. Shrinking bases

Let us say that a Schauder basis (e_i) for X is *shrinking* if (e_i^*) is a basis for X^*. Of course, we know that (e_i^*) is always a Schauder basis for its closed span, so (e_i) is shrinking if and only if the span of (e_i^*) is norm dense in X^*. Moreover, one easily checks that the restriction of P_n^* to $[e_i^*]$ is the n^{th} basis projection of (e_i^*), and therefore maps into $[e_i^*]$.

Lemma 2.1. *Let (e_i) be a Schauder basis for X. The following are equivalent.*

(i) The basis (e_i) is a shrinking basis for X.
(ii) For each $x^* \in X^*$, $\lim_n \|x^* - P_n^* x^*\| = 0$.
(iii) Every bounded block sequence in X is weakly null.

Proof. $(i) \Rightarrow (ii)$ As we have already mentioned, $P_n^* : [e_i^*] \to [e_i^*]$ are the basis projections. If (e_i) is shrinking, then (e_i^*) is a basis for X^*, whence $\lim_n \|x^* - P_n^* x^*\| = 0$ for any $x^* \in X^*$.

$(ii) \Rightarrow (iii)$ Fix a bounded block sequence (x_n) in X and let $C = \sup_n \|x_n\|$. Fix $x^* \in X^*$ and $\varepsilon > 0$. Fix $n_0 \in \mathbb{N}$ such that $\|x^* - P_n^* x^*\| < \varepsilon$ for all $n \geqslant n_0$. Then for all $n > n_0$, $\min \operatorname{supp}(x_n) > n_0$, and $(I - P_{n_0})x_n = x_n$. Then

$$|x^*(x_n)| = |x^*(I - P_{n_0} x_n)| = |(x^* - P_{n_0}^* x^*)x_n| \leqslant C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $x^*(x_n) \to 0$. Since $x^* \in X^*$ was arbitrary, (x_n) is weakly null.

$(iii) \Rightarrow (i)$ Suppose there exists $x^* \in X^*$ such that $\|x^*\|_{X^*/[e_n^*]} > \varepsilon > 0$. We may fix $x_1 \in B_X$ with finite support such that $x^*(x_1) > \varepsilon$. Let $n_1 = \max \operatorname{supp}(x_1)$. Next, suppose we have chosen $x_1, \dots, x_{k-1} \subset B_X$, $n_1, \dots, n_{k-1}$ such that $(x^* - P_{n_{i-1}}^* x^*)(x_i) > \varepsilon$ and $\operatorname{supp}(x_i) \subset [1, n_i]$ for each $1 \leqslant i < k$. Then since $\|x^* - P_{n_{k-1}}^* x^*\| > \varepsilon$, there exists $x_k \in B_X$ having finite support such that $(x^* - P_{n_{k-1}}^* x^*)(x_k) > \varepsilon$. Let $n_k = \max \operatorname{supp}(x_k)$. This completes the recursive construction.

For each $i \in \mathbb{N}$, let $y_i = (I - P_{n_{i-1}})x_i$. Note that $\operatorname{supp}(y_i) \subset (n_{i-1}, n_i]$ and $\|y_i\| \leqslant 1 + \|P_{n_{i-1}}\| \leqslant 1 + K$. Therefore (y_i) is a bounded block sequence. But by our choice, $x^*(y_i) = (x^* - P_{n_{i-1}} x^*)(x_i) > \varepsilon$ for all i. Thus (y_i) is a bounded block sequence which fails to be weakly null.

□

Remark 2.2. The ℓ_1 basis is the canonical example of a basis which fails to be shrinking. The bases of ℓ_p, $1 < p < \infty$, and c_0 are shrinking, since every bounded block sequence must be weakly null.

Lemma 2.3. *If (e_i) is an unconditional basis for X, then (e_i) either ℓ_1 embeds into X or (e_i) is shrinking, and exactly one of these alternatives holds.*

Proof. Of course, both alternatives cannot hold simultaneously, since the shrinkingness of (e_i) implies the separability of X^*, while the embeddability of ℓ_1 into X implies the non-separability of X^*.

Suppose (e_i) is non-shrinking. Then there exists a bounded block sequence (x_n) not weakly null. By scaling by unimodular multiples and passing to a subsequence, we may assume there exists $x^* \in B_{X^*}$ and $\varepsilon > 0$ such that $x^*(x_n) \geqslant \varepsilon$ for all $n \in \mathbb{N}$. This means that for any non-negative scalars $(a_i)_{i=1}^n$,

$$\|\sum_{i=1}^n a_i x_i\| \geqslant x^*(\sum_{i=1}^n a_i x_i) \geqslant \varepsilon \sum_{i=1}^n a_i.$$

If (e_i) is K-unconditional, so is (x_i), so that for any scalars,

$$\|\sum_{i=1}^n a_i x_i\| \geqslant \varepsilon/K \sum_{i=1}^n |a_i|,$$

and (x_i) is equivalent to the ℓ_1 basis.

□

3. Boundedly-complete bases

We say a basis (e_i) for X is *boundedly-complete* provided that if (a_i) is a sequence of scalars such that $\sup_n \|\sum_{i=1}^n a_i e_i\| < \infty$, then $\sum a_i e_i$ converges in norm.

Let $\phi : X \to [e_i^*]^*$ be the map given by $\phi(x)(\sum a_i e_i^*) = \sum a_i e_i^*(x)$. Of course, since $[e_i^*] \subset X^*$, $\|\phi(x)\| \leqslant \|x\|$. For any $x \in X$, for each $n \in \mathbb{N}$, we may fix x_n^* such that $\|P_n x\| = x_n^*(P_n x)$. Let $y_n^* = P_n^* x_n^*/K$, where K is the basis constant of (e_i). Then (y_n^*) lies in the unit ball of $[e_i^*]$ and $y_n^*(x) = x_n^*(P_n x)/K = \|P_n x\|/K \to \|x\|/K$. Thus $\|\phi(x)\| \geqslant \|x\|/K$, and ϕ is an isomorphic embedding.

Note that ϕ is just the restriction of the image of x under the canonical embedding of X into X^{**} to $[e_i^*]$.

Proposition 3.1. *Let (e_i) be a basis and let $H = [e_i^*]$. The following are equivalent.*

(i) The basis (e_i) is boundedly-complete in X.
(ii) If (x_n) is a block sequence bounded away from zero, $\sup_N \|\sum_{n=1}^N x_n\| = \infty$.
(iii) $\phi : X \to H^*$ is onto.

Proof. *(i)* ⇒ *(ii)* Note that if (x_i) is a block sequence, $e_i^*(\sum_{j=1}^i x_j) = e_i^*(\sum_{j=1}^k x_j)$ for each $j \geqslant i$. Let $a_i = e_i^*(\sum_{j=1}^i x_j)$. Note that $\sum_{i=1}^N a_i e_i = P_N \sum_{i=1}^N x_i$, and therefore $\|\sum_{i=1}^N a_i e_i\| \leqslant K\|\sum_{i=1}^N x_i\|$, where K is the basis constant of (e_i). Therefore if $\sup_N \|\sum_{i=1}^N x_i\| < \infty$, $\sup_N \|\sum_{i=1}^N a_i e_i\| < \infty$. Therefore $\sum_{i=1}^\infty a_i e_i$ converges. If $m_0 = 0$ and $m_n = \max \mathrm{supp}(x_n)$ for each $n \in \mathbb{N}$,

$$\|x_n\| = \|\sum_{i=1}^{m_n} a_i e_i - \sum_{i=1}^{m_{n-1}} a_i e_i\| \underset{n}{\to} 0.$$

(ii) ⇒ *(iii)* Fix $f \in H^*$. Let $y_n = \sum_{i=1}^n f(e_i^*) e_i$, so that for any $\sum a_i e_i^* \in H$,

$$|(\sum a_i e_i^*)(y_n)| = |\sum_{i=1}^n a_i f(e_i^*)| = |f(P_n^* \sum a_i e_i^*)| \leqslant K\|f\|.$$

This means (y_n) must be norm convergent. Indeed, if it were not so, there would exist $n_1 < \dots$ and $\varepsilon > 0$ such that with $x_1 = y_1$ and $x_k = y_{n_k} - y_{n_{k-1}}$ for $k > 1$, (x_k) is a block sequence in (e_i) bounded away from 0 with $\sup_n \|\sum_{i=1}^n x_i\| = \sup_n \|y_n\| \leqslant CK$, contradicting (ii). Then $(\phi(y_n))$ is also norm convergent, and also converging w^* to f in H^*, so that f must be the norm limit of (y_n). To see that $\phi(y_n) \underset{w^*}{\to} f$, since $(\phi(y_n))$ is bounded, it is sufficient to check that $\phi(y_n)(x^*) \to f(x^*)$ for all x^* in a subset of H with dense span. But by our choice of y_n, $\phi(y_n)(e_i^*) = f(e_i^*)$ for all $n \geqslant i$.

$(iii) \Rightarrow (i)$ Suppose (a_i) is a scalar sequence such that $\sup_n \|\sum_{i=1}^n a_i e_i\| = C < \infty$. Then $\sup_n \|\sum_{i=1}^n a_i e_i^{**}\| < \infty$, since $\phi(e_i) = e_i^{**}$. We have already shown that $\sum b_i e_i^* \mapsto \sum a_i b_i$ defines a member of $[e_i^*]^*$. Then there exists $x \in X$ such that $\phi(x) = f$. Of course, $a_i = f(e_i^*) = \phi(x_i)(e_i^*) = e_i^*(x)$ for each $i \in \mathbb{N}$. Thus $x = \sum a_i e_i$, from which it follows that $\sum a_i e_i$ is norm convergent.

□

Remark 3.2. The c_0 basis is the canonical example of a non-boundedly-complete basis. The bases of ℓ_p, $1 \leqslant p < \infty$, are boundedly-complete.

Lemma 3.3. *Suppose (e_i) is an unconditional basis for X. Then either c_0 embeds into X or (e_i) is boundedly-complete. Exactly one of these two alternatives holds.*

Proof. It follows from Proposition 3.1 that at most one of these two alternatives could hold. Suppose (e_i) is not boundedly-complete. Then there exists a seminormalized block sequence (x_n) such that $\sup_N \|\sum_{n=1}^N x_n\| = C < \infty$. Then for any $N \in \mathbb{N}$ and any $(\varepsilon_n)_{n=1}^N$ with $|\varepsilon_n| = 1$, $\|\sum_{n=1}^N \varepsilon_n x_n\| \leqslant CK$, where K is the unconditionality constant of (e_i). Note that for any $N \in \mathbb{N}$ and scalars $(a_i)_{i=1}^N$ with $a = \max_{1 \leqslant i \leqslant N} |a_i|$,

$$\sum_{i=1}^N a_i x_i \in \text{co}\Big\{a \sum_{i=1}^N \varepsilon_i x_i : |\varepsilon_i| = 1\Big\} \subset aCKB_X,$$

so that $\|\sum_{i=1}^N a_i x_i\| \leqslant CK \max_{1 \leqslant i \leqslant N} |a_i|$. Since (x_i) is seminormalized and basic, it dominates the c_0 basis, and therefore (x_i) is equivalent to c_0.

□

Proposition 3.4. *Let (e_i) be a Schauder basis. Then (e_i) is shrinking (resp. boundedly-complete) if and only if (e_i^*) is boundedly-complete (resp. shrinking).*

Proof. Recall that X^* can be identified with the set of all formal series $\sum_{i=1}^\infty a_i e_i^*$ with bounded partial sums, $[e_i^*]^*$ can be identified with the set of all formal series $\sum_{i=1}^\infty a_i e_i^{**}$ with bounded partial sums, and $\phi(e_i) = e_i^{**}$ for all $i \in \mathbb{N}$. Then (e_i) is shrinking if and only if $[e_i^*]$, which is the set of formal series $\sum a_i e_i^*$ with bounded partial sums, is equal to the set of $\sum a_i e_i^*$ which norm converge, and so bounded partial sums are equivalen to norm convergent in this case. Thus (e_i) is shrinking if and only if (e_i^*) is boundedly-complete.

Note that $\phi : X \to [e_i^*]$ is onto if and only if $\phi(X) = \phi([e_i]) = [\phi(e_i)] = [e_i^{**}] = [e_i^*]^*$, which is precisely what it means for (e_i^*) to be shrinking.

□

4. Reflexivity

Lemma 4.1. *The Banach space X admits a non-shrinking basic sequence if and only if it admits a non-boundedly-complete basic sequence.*

We omit the proof of this lemma, since the final theorem of this lecture will be to prove a stronger result due to Zippin.

Lemma 4.2. *If X is not reflexive, there exists a basic sequence $(x_n) \subset B_X$ which is not weakly null.*

Proof. Recall Helly's theorem, which states that for any $x^{**} \in X^{**}$, any finite subset F of X^*, and any $\varepsilon > 0$, there exists $x \in X$ such that $x^{**}(x^*) = x^*(x)$ for all $x^* \in F$ and $\|x\| \leqslant \|x^{**}\| + \varepsilon$.

Suppose X is not reflexive. We may fix $x^{**} \in X^{**}$ such that

$$1/2 < \|x^{**}\|_{X^{**}/X} \leqslant \|x^{**}\| < 1.$$

By the Hahn-Banach theorem, there exists $x^{***} \in X^{***}$ such that $\|x^{***}\| < 2$, $x^{***}(x^{**}) =$, and $x^{***}|_X \equiv 0$. Fix a sequence of positive numbers (ε_n) such that $\prod(1 - \varepsilon_n)^{-1} < 2$. Recursively apply Helly's theorem to obtain $(x_n) \subset B_X$, $(x_n^*) \subset 2B_{X^*}$, and finite sets $\varnothing = F_0 \subset F_1 \subset F_2 \subset \dots$ of B_{X^*} such that for all $n \in \mathbb{N}$,

(i) for all $y^{**} \in [x_i - x^{**} : 1 \leqslant i \leqslant n]$, $\max_{x^* \in F_n} |y^{**}(x^*)| \geqslant (1 - \varepsilon_n)\|y^{**}\|$,
(ii) for all $x^* \in F_{n-1} \cup \{x_1^*, \dots, x_{n-1}^*\}$, $x^{**}(x^*) = x^*(x_n)$,
(iii) for all $y^{**} \in \{x_1, \dots, x_{n-1}, x^{**}\}$, $x_n^*(y^*) = x^{***}(y^*)$.

Conditions (i) and (ii) guarantee that $(x_n - x^{**})$ is 2-basic. Then $I - x^{**} \otimes x^{***} : [x_n - x^{**}] \to [x_n]$ is an isomorphism between these spaces with inverse $I + x^{**} \otimes x^{***}$, therefore $(x_n - x^{**})$ and (x_n) are $\|I + x^{**} \otimes x^{***}\| \|I - x^{**} \otimes x^{***}\| \leqslant 9$-equivalent, and (x_n) is bounded and 18-basic. Moreover, for any $n \in \mathbb{N}$,

$$1 = x^{***}(x^{**}) = x^{**}(x_1^*) = x_1^*(x_n),$$

whence (x_n) is not weakly null.

□

Lemma 4.3 (James)**.** *If (e_i) is a basis for X, then X is reflexive if and only if (e_i) is both shrinking and boundedly-complete.*

Proof. (e_i) is both shrinking and boundedly-complete if and only if (e_i) and (e_i^*) are both shrinking. Then $X^* = [e_i^*]$ and $X^{**} = [e_i^*]^*$. Moreover, $\phi : X \to [e_i^*]^* = X^{**}$ defined before

Proposition 3.1 is simply the canonical embedding into the second dual, and is onto if (e_i) is boundedly-complete. Therefore X is reflexive in this case.

Next, suppose X is reflexive. Suppose (x_n) is a bounded block sequence in (e_i) not weakly null. Then there exists $\varepsilon > 0$ and $x^* \in X^*$ such that, by passing to a subsequence, we may assume $|x^*(x_n)| > \varepsilon$ for all $n \in \mathbb{N}$. We may pass to a further subsequence which is weakly converging to some $x \in X$ and note that $|x^*(x)| \geqslant \varepsilon$. But for all $i \in \mathbb{N}$, $e_i^*(x) = \lim_n e_i^*(x_n) = 0$, so $x = 0$. This contradiction implies that (e_i) is shrinking if X is reflexive. Then $X^* = [e_i^*]$. Since X^* is reflexive, (e_i^*) is shrinking, and (e_i) is boundedly-complete.

□

Corollary 4.4 (Singer). *Let X be a Banach space. The following are equivalent.*

(i) X is reflexive.
(ii) Every basic sequence in X is shrinking.
(iii) Every basic sequence in X is boundedly-complete.

Proof. If X is reflexive, so is any subspace spanned by a basic sequence, whence all basic sequences are both shrinking and boundedly-complete. Will show in the final section that X admits a non-shrinking basic sequence if and only if it admits a non-boundedly-complete basic sequence, and the former happens when X is not reflexive.

□

5. Example: James space

Define the norm $\|\cdot\|$ on c_{00} by letting

$$\|x\|^2 = \sup\Big\{\sum_{i=1}^{k} |(e^*_{m_i} - e^*_{m_{i+1}})(x)|^2 : k \in \mathbb{N}, 1 \leqslant m_1 < \ldots < m_{k+1}\Big\}.$$

One can check that this norm turns (e_i) into a seminormalized, monotone basis for the completion J of c_{00} with this norm. One can also check that every normalized block of (e_i) is dominated by the ℓ_2 basis, and so (e_i) is shrinking. However, $\|\sum_{i=1}^n e_i\| = 1$ for every n, which shows that (e_i) is not boundedly-complete. If we let $s_n = \sum_{i=1}^n e_i$, we obtain a boundedly-complete basis for J, which is necessarily non-shrinking (since J cannot be reflexive). It is also easy to see that the sequence (s_n) itself is not shrinking, since $e_1^*(s_n) = n$ for all n, and this sequence (s_n) is normalized and not weakly null. However, it is not too difficult to see that if (x_n) is a normalized block sequence in J, then (x_{2n}) is equivalent to the ℓ_2 basis, so that neither c_0 nor ℓ_1 can embed into J. This shows that the hypothesis of unconditionality cannot be dropped from Lemmas 2.3 and 3.3.

6. A stronger result

Theorem 6.1 (Zippin). *If (e_i) is a non-shrinking (resp. non-boundedly-complete) basis for X, then X admits a non-boundedly-complete (resp. non-shrinking) basis.*

Lemma 6.2. *(i) If E is a Banach space and F, G are subspaces with $\dim E/F, \dim E/G = 1$, then there exists an automorphism $D : E \to E$ such that $\|D\|, \|D^{-1}\| \leqslant 5$ and $D(F) = G$.*

(ii) Suppose E is a Banach space and $x, y \in X$ and $x^*, y^* \in X^*$ are such that $x^*(x) = y^*(y) = 1$. Then there exists an automorphism $A : E \to E$ taking x to y and $\ker(x^*)$ to $\ker(y^*)$ such that $\|A\|, \|A\|^{-1} \leqslant K$ for some function K which depends only on $\max\{\|x\|, \|y\|, \|x^*\|, \|y^*\|\}$.

Proof. (i) If $F = G$, we take D to be the identity. Otherwise let $M = G \cap F$. Note that $\dim F/M, \dim G/M = 1$. We may fix $x \in F$ such that $\|x\| = 1$ and $\|x\|_{F/M} > 1/2$. By the Hahn-Banach theorem, we may fix $f \in E^*$ such that $f(x) = 1$, $f|_M \equiv 0$, and $\|f\| \leqslant 2$. Moreover, $F \cap \ker(f) = M$. Similarly, we may fix $y \in G$ and $g \in E^*$ such that $g(y) = 1$, $f|_M \equiv 0$, $\|g\| \leqslant 2$, and $G \cap \ker(g) = M$. Define $A : F \to \mathbb{F} \oplus M$ by $Az = (f(z), z - f(z)x)$ and $B : G \to \mathbb{F} \oplus M$ by $Bz = (g(z), z - g(z)y)$. Note that $\|A\|, \|B\| \leqslant 5$, and A, B are inverses. Indeed, $A^{-1}(a, z) = ax + z$ and $B^{-1}(a, z) = ay + z$. Then $D = B^{-1}A$.

(ii) Define $B : E \to \mathbb{F} \oplus_1 \ker(x^*)$, $C : E \to \mathbb{F} \oplus_1 \ker(y^*)$ by

$$Bz = (x^*(z), z - x^*(z)x), \qquad Cz = (y^*(z), z - y^*(z)y).$$

One easily checks that B, C are isomorphisms with inverses given by $B^{-1}(a, z) = ax + z$, $C^{-1}(a, z) = ay + z$. Fix any isomorphism $D : \ker(x^*) \to \ker(y^*)$ such that $\|D\|, \|D^{-1}\| \leqslant 5$. Then we may take $A = C^{-1}D'B$, where $D' : \mathbb{F} \oplus_1 \ker(x^*) \to \mathbb{F} \oplus_1 \ker(y^*)$ is given by $D'(a, z) = (a, Dz)$. Of course, $Ax = y$. For $z \in \ker(x^*)$, $CD'Bz = C(0, z) = C(0, Dz) = Dz \in \ker(y^*)$. Similarly, one may check that $A^{-1}y = x$ and $A^{-1}(\ker(y^*)) = x^*$.

□

Proposition 6.3. *If (x_n) is any block of the basis (e_i), there exists a basis (f_i) for $[e_i]$ having (x_n) as a subsequence. Moreover, if x^* is such that $x^*(x_n) = 1$ for all $n \in \mathbb{N}$, then there exists a basis (f_i) for $[e_i]$ and an infinite subset P of $\mathbb{N}$ such that $(f_i)_{i \in P} = (x_i)_{i \in \mathbb{N}}$ and such that $x^*(f_i) = 0$ for all $i \notin P$.*

Proof. We may first assume that (x_n) and (e_n) are normalized. Fix $0 = k_0 < k_1 < \ldots$ such that with $E_i = [e_j : k_{i-1} < j \leqslant k_i]$, $x_i \in E_i$ for all i. Note that (E_i) is an FDD for X. We may fix a seminormalized block sequence (x_n^*) of (e_i^*) biorthogonal to (x_i) and such that $x_n^* \in [e_i^* : k_{n-1} < i \leqslant k_n]$. Then for each i, there exists an automorphism A_n of E_n taking e_{k_n} to x_n. For $k_{n-1} < i \leqslant k_n$, let $f_i = A_n e_i$. Then $(f_i)_{i=k_{n-1}+1}^{k_n}$ is a basis for E_n with basis constant not exceeding K, which depends on $\sup \|x_n^*\|, \|e_n^*\|$, and $f_{k_n} = x_n$. It follows that (f_i) is a basis for X having (x_n) as a subsequence.

The second statement is similar, except we define the automorphism A_i on E_i by using the functionals $e_{k_i}^*$ and x^* rather than $e_{k_i}^*$ and x_i^*. Then in this case, the members $(f_i)_{i=k_{n-1}+1}^{k_n-1}$ of the basis are the image of $(e_i)_{i=k_{n-1}+1}^{k_n-1} \subset \ker(e_{k_n}^*)$ under A_i, which maps $\ker(e_{k_n}^*)$ into $\ker(x^*)$. Therefore with $P = \{k_n : n \in \mathbb{N}\}$, we reach the conclusion.

□

Proof of Theorem 6.1. If (e_i) is not shrinking, there exists a bounded block sequence (x_n) of (e_i) and $x^* \in X^*$ such that $x^*(x) = 1$ for all $n \in \mathbb{N}$. We may fix a basis (f_i) having (x_n) as a subsequence, say $(f_n)_{n\in P}$, such that for $n \notin P$, $x^*(f_n) = 0$. Let $g_i = f_i$ if $i \notin P$, $g_{k_1} = x_1$, and $g_{k_{n+1}} = x_{n+1} - x_n$. Then $\sup_n \|\sum_{i=1}^n g_{k_n}\| = \sup_n \|x_n\| < \infty$, while (x_n) does not converge. Therefore if (g_i) is a basic sequence, it is not boundedly-complete. We will show that (g_i) is a basis for X.

Fix $x = \sum_{i=1}^\infty a_i f_i$. Let $b_i = a_i$ for $i \notin P$ and let $b_{k_n} = \sum_{i=n}^\infty a_{k_i}$. To see that this sum converges, note that for any $m \leqslant n$,

$$|\sum_{i=m}^{n} a_{k_i}| = |x^*(\sum_{i=k_m}^{k_n} a_i f_i)| = |x^*((P_{k_n} - P_{k_m-1})(x))|,$$

and this quantity vanishes as m tends to infinity. Fix any $n \in \mathbb{N}$ with $n \geqslant k_1$. Let p be the maximum natural number i such that $k_i \leqslant n$. Then

$$\begin{aligned}\sum_{i=1}^{n} b_i g_i - \sum_{i=1}^{n} a_i f_i &= \sum_{i=1}^{p} b_{k_i} g_{k_i} - \sum_{i=1}^{p} a_{k_i} x_i \\ &= \sum_{i=1}^{p} b_{k_i} x_i - \sum_{i=1}^{p-1} b_{k_{i+1}} x_i - \sum_{i=1}^{p} (b_{k_i} - b_{k_{i+1}}) x_i \\ &= b_{k_{p+1}} x_{k_p}.\end{aligned}$$

As $n \to \infty$, $p \to \infty$, and this quantity vanishes. Therefore $\sum_{i=1}^\infty b_i g_i = x$.

Next, suppose that $\sum_{i=1}^\infty b_i g_i = 0$. Then for $n \notin P$ and $m > n$, $b_n = f_n^*(\sum_{i=1}^m b_i g_i) \underset{m}{\to} 0$. Note that $b_{k_n} \to 0$, since (g_{k_n}) is bounded away from zero. But for any $n \in \mathbb{N}$ and $m > k_{n+1}$,

$$f_{k_n}^*(\sum_{i=1}^{m} b_i g_i) = b_{k_n} - b_{k_{n+1}} \underset{m}{\to} 0.$$

Thus (b_{k_n}) is a constant sequence converging to zero, and is therefore constantly zero. We have shown that $b_n = 0$ for all $n \in \mathbb{N}$.

If (e_i) is not boundedly-complete, there exists a block sequence (x_n) of (e_i) bounded away from 0 such that $\sup_n \|\sum_{i=1}^n x_i\| < \infty$. We may pass to another basis (f_i) which has (x_n) as a subsequence, say (f_{k_n}). Let $g_i = f_i$ if $i \notin \{k_n : n \in \mathbb{N}\}$, $g_{k_n} = \sum_{i=1}^n x_i$. Therefore if (g_i) is a basic sequence, it is not shrinking, since $x^*(g_{k_n}) = 1$ for all $n \in \mathbb{N}$ if x^* is any Hahn-Banach extension of x_1^*. We will show that (g_i) is a basis for X.

Fix $x = \sum a_i f_i \in [f_i]$. Let $b_{k_n} = a_{k_n} - a_{k_{n+1}}$ and $b_i = a_i$ for $i \notin K := \{k_1, k_2, \ldots\}$. Then for any $n \in \mathbb{N}$, if p is the minimum natural number such that $n < k_p$,

$$\begin{aligned}
\sum_{i=1}^{n} a_i f_i &= \sum_{i=1, i\notin K}^{n} a_i f_i + \sum_{i=1}^{p-1} (a_{k_i} - a_{k_p}) x_i + a_{k_p} \sum_{i=1}^{p-1} x_i \\
&= \sum_{i=1, i\notin K}^{n} a_i f_i + \sum_{i=1}^{p-1} \sum_{j=i}^{p-1} b_{k_j} x_i + a_{k_p} \sum_{i=1}^{p-1} x_i \\
&= \sum_{i=1, i\notin K}^{n} a_i f_i + \sum_{j=1}^{p-1} \sum_{i=1}^{j} b_{k_j} x_i + a_{k_p} \sum_{i=1}^{p-1} x_i \\
&= \sum_{i=1, i\notin K}^{n} a_i g_i + \sum_{i=1}^{p-1} b_{k_i} g_{k_i} + a_{k_p} \sum_{i=1}^{p-1} x_i = \sum_{i=1}^{n} b_i g_i + a_{k_p} \sum_{i=1}^{p-1} x_i.
\end{aligned}$$

Note that $a_{k_p} \to 0$ as $p \to \infty$, while $\sum_{i=1}^{p-1} x_i$ stays bounded. Therefore subtracting $\sum_{i=1}^{n} b_i g_i$ from this term leaves a sequence which vanishes as $n \to \infty$. From this it follows that $\sum_{i=1}^{n} b_i g_i \to x$.

We show uniqueness. Suppose $\sum_{i=1}^{\infty} b_i g_i = 0$. For each $i \notin K$, $b_i = f_i^*(\sum_{i=1}^{\infty} b_i g_i) = 0$. Then $\sum_{i=1}^{\infty} b_i g_i = \sum_{i=1}^{\infty} b_{k_i} g_{k_i} = 0$. For any $m, n \in \mathbb{N}$ with $m > n+1$,

$$(x_n^* - x_{n+1}^*)(\sum_{i=1}^{m} \sum_{j=1}^{i} b_{k_i} x_j) = (x_n^* - x_{n+1}^*)(\sum_{j=1}^{m} \sum_{i=j}^{m} b_{k_i}) x_j = b_{k_n}.$$

This vanishes as m tends to infinity, so that $b_{k_n} = 0$ for all $n \in \mathbb{N}$, and $b_n = 0$ for all $n \in \mathbb{N}$.

□