# Numerical Integration: Trapezoidal Approximation, Simpson's Rule, and Newton-Cotes Formulas

Douglas Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

## *Overview*

As we have learned in Calculus I, there are two ways to evaluate a definite integral: using the Fundamental Theorem of calculus or numerical approximations. While FTC is nice in theory, it cannot be applied in many cases, as antiderivatives are often difficult or even impossible to find. Using computers, numerical integration is the most general and practical way to evaluate definite integrals. In Calculus I, we explored Riemann Sum approximations. In this lab, we will use Maple to experience some new approximations that provide more accuracy with less computation.

## *Maple Essentials*

The *Approximate Integrals* tutor can be started from maple's Tools menu:

**Tools → Tutors → Calculus - Single Variable → Approximate Integrals ...**

Note: *The* Approximate Integrals *tutor is identical to the* Riemann Sums *tutor.*

## *Related course material/Preparation*

§8.7.

The trapezoidal approximation is often introduced as the average of the Riemann left endpoint approximation and the Riemann right endpoint approximation and, similarly, Simpson's rule as a weighted average of the Riemann midpoint approximation and the trapezoidal approximation. Geometrically, as shown in §8.7 of the text, the trapezoidal approximation and Simpson's rule use a line and a parabola (polynomials of degree 1 and 2), respectively, to approximate $y = f(x)$ in each subinterval. In the trapezoidal approximation, the line is determined by the left and right endpoints of the curve in each subinterval. To determine a parabola, you need three points. Therefore, the left endpoint, the right endpoint, and the midpoint of the curve in each subinterval are used in Simpson's rule. In this point of view, the above two approximations are special cases (order 1 and order 2) of the general Newton-Cotes formula of order k, where a polynomial of degree k determined by $k+1$ points on $y = f(x)$ is used to approximate the function $y = f(x)$ in each subinterval. Using Maple, we can experience and compare all those approximations with plots, animations, formulas, and more.

## *Activities*

Use the `Approximate Integrals` tutor to approximate the following integrals by

a) the Riemann Sums midpoint approximation;

b) the trapezoidal approximation;

c) Simpson's rule;

d) another method of your choice.

Start with a small number of subintervals, say $n = 3$, so you can see the geometry. Gradually increase $n$ and describe what happens to your approximation. In each case, what is the smallest $n$ needed such that the absolute error is less than 0.0001? How do different methods compare with each other?

$$1)\ \int_0^{\pi} \sin x\ dx \qquad 2)\ \int_0^1 e^{-x^2} dx \qquad 3)\ \int_1^2 \sqrt{x^3-1}\ dx \qquad 4)\ \int_0^2 \sin(x^2)\ dx \qquad 5)\ \int_1^3 \sqrt{\ln x}\ dx$$

## *Example: Integral 1*

1. Launch the *Approximate Integrals* tutor from maple's Tools menu.
2. Choose `Subintervals` in the `Partition type` box
3. Plug in $f(x) = \sin(x)$, $a = 0$, $b = Pi$, and $n = 3$.
4. Choose your method, say Simpson's rule, and click `Display`. How close is the approximation (the blue graph) to the original curve (the red one)?
5. You should see that the `Approximate Integral` = 2.000863190 and the `Actual Integral` = 2. Therefore, the absolute error is 0.000863190.
6. Gradually increase $n$ and click on `Display` each time. You should see that $n = 6$ is the smallest number of subintervals needed to have the required accuracy (the absolute error=0.000052625). This is very nice as you need to have about $n = 125$ subintervals in order to have the same accuracy using Riemann Sum midpoint approximations.
7. Choose a method and a small value, say $n = 2$, and click on `Animate`.
8. Repeat the above for other methods.
9. Choose a value, say $n = 10$, and click on `Compare` to see all approximations at once and compare. (Don't forget to choose `Subintervals` in the new window, too.)
10. Before you exit from the tutor and start the next activity, copy the command from `Maple Command` box on the bottom (use Ctrl+C) so you do not have to type as much or to remember the long command.

## *Working with maple command:*

- Start by including the `Student[Calculus1]` package.
  `>with(Student[Calculus1]):`
- Paste the command copied from the tutor and click on the plot generated to bring up the animation tool-bar.
  `>ApproximateInt(sin(x), 0 .. Pi, 'partition' = 2, 'method' = trapezoid, 'partitiontype' = subintervals, 'iterations' = 6, 'output' = 'animation', 'showarea' = false, 'boxoptions' = ['filled' = ['transparency'=.5]]);`
- To see the formula for your approximation, change `‘output’` option from `‘animation’` to `sum` and name it as, say `Approx`. You may delete some options such as `‘iteration’=6` since it is only needed for animations.
  `>Approx:=ApproximateInt(sin(x), 0..Pi, ‘partition’ = 2, ‘method’ = trapezoid, ‘output’ = sum);`
- Find the absolute error associated with this approximation.
  `>Err:=evalf(abs(int(sin(x), x=0..Pi) - Approx));`
- To see the formula in general form for $k$ subintervals, change `‘partition’` option from `=2` to `=k`.
  `>Approx:=ApproximateInt(sin(x), 0..Pi, ‘partition’ = k, ‘method’ = trapezoid, ‘output’ = sum);`
- Find the limit of the above formula, which should be equal to the actual value of the integral.
  `>limit(Approx, k=infinity);`
  Note: The limit of some infinite sums can be very hard to find (even for Maple)
- You may want to refer to the help menu and try other fun examples with the command.

## *Assignment*

Finish lab activities and your Lab instructor will give other assignment.