1D Quadrature Rules for Sparse Grids

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Abstract

Sparse grids are constructed as logical sums of product grids. Each product grid is formed by selecting, for each dimension, one element of an indexed family of quadrature rules. The identical family may be employed in each dimension, or the same rule may be used, but with certain parameters varying, or the family itself may change from one dimension to another. While many quadrature families are known, sparse grids are typically employed for functions with a stochastic component, in which case there are a small set of preferred families. Another issue with sparse grids involves the *growth rule* employed in a family, that is, the rate at which the order of the rules grows with the index. This article considers some typical 1D quadrature rules, and some of the issues that arise when using them to build a sparse grid.

1 Introduction

The sparse grid construction of Smolyak [8] is a method for producing a multidimensional quadrature rule as a weighted sum of product rules, which are in turn typically formed by the product of 1D rules selected from an indexed family.

The Clenshaw Curtis families "CC_*" and the Newton Cotes Closed families "NCC_*" are examples of *closed rules*; that is, the endpoints of the interval are included as abscissas in every rule (except for the very first one-point rule). The other families are *open rules*, and never include the endpoints. There exist *half-open rules* which include one endpoint but not the other; they will not occur in this discussion.

Given any set of n distinct points x, and an interval $[a, b]$, it is possible to write down a linear system of Vandermonde type which determines a set of corresponding weights w, so that the pair x and w form a quadrature rule of exactness $n-1$ for integrals over $[a, b]$. Thus, as a first principle, we may conclude n abscissas will "buy" you exactness for the first n monomials, $1, x, x^2, ..., x^{n-1}$.

A quadrature rule is said to be *Gaussian* if the abscissas x and weights w have been derived from the usual Gauss procedure, so that a rule that uses n points will have exactness $2n-1$. The doubled accuracy of a Gauss rule comes at the price of losing the freedom to choose the locations of the abscissas.

Gauss-Patterson rules are formed by a sort of half-way version of the Gaussian procedure. We are given a set of n abscissas that are fixed. We wish to compute a rule of order $2n+1$, by computing $n+1$ new points and $2n+1$ weights. The result is typically a rule of exactness $3n+2$, which is more or less halfway between the exactness $2n+1$ we could get from a Vandermonde procedure after prescribing all the points, and the exactness $4n+1$ we could get by leaving all the points free and using a Gauss process. The Gauss-Patterson procedure allows us to seek a nested family of rules.

A quadrature rule is said to be *symmetric* if the following are all true:

- the integral to be approximated is over a symmetric interval $[-a, +a]$ or $(-\infty, +\infty)$;

- if there is a weight function $w(x)$ in the integral, it is a symmetric function of x;
- the set of abscissas x is symmetric about 0.0;
- for each abscissa x_i, the weight w_i is the same as for the abscissa $-x_i$.

One attraction of symmetric quadrature rules is that, while a typical (non-Gaussian) quadrature rule of order n will have exactness $n-1$, a symmetric quadrature rule of odd order n will have exactness n. In particular, the midpoint rule will have exactness 1.

The families display several kinds of nesting of the abscissas:

- *fully nested rules* have the property that all points in the rule of level l will occur again in the rule of level $l+1$;
- *weakly nested rules* have the property that one point in the rule of level l will occur again in the rule of level $l+1$; generally, this point is 0.0;
- *very weakly nested rules* have the property that there is one value that will occur in many, but not all, of the rules; generally, this point is 0.0, and it occurs in the rules of even order.
- *non-nested rules* have the property that no points in the rule of level l occur again in the rule of level $l+1$; in fact, for the non-nested rules in this list, abscissas used in one rule never occur again.

1.1 Growth Rules

Each family is an indexed set of quadrature rules. We call the index the *level*, denote it by l, and assume its lowest value is 0, usually corresponding to a 1-point rule. Now it is often the case that a given quadrature rule is available for every possible order, or number of points. However, for a particular quadrature family, we wish to make a particular selection of these rules, and it is the purpose of the level index l to retrieve the particular rules we are interested in.

Note that we could select all the rules, arranged by size, or we could select only the even ones. Our indexing could be done in such a way that some rules are selected more than once. It is theoretically possible that the indexing is done somewhat chaotically. However, the natural purpose of the indexing is to select a sequence of rules of increasing exactness, which may also satisfy some side condition, such as being nested, or having an exactness that grows at a satisfactory rate with the level index.

Typically there is a simple relationship between the level l and the order $o(l)$, that is, the number of abscissas in successive rules. For example, many sparse grids use the Clenshaw Curtis family. A Clenshaw Curtis rule can be developed with any number of points. However, if we increase the order in the right way, the successive rules we choose are nested, which can be a useful efficiency. For the Clenshaw Curtis rules, as with most nested rules, the natural growth pattern means that the order essentially doubles when l is incremented. This is termed *exponential growth* in the order. It is a very common growth rule, and has been used even with quadrature rules that are not fully nested. It occurs here with the families CC_E, CH2_E, F2_E, GGH_E, GH_E, GJ_E, GL_E, GLG_E, GP_E, LG_E, NCC_E, and NCO_E.

This rapid growth can be very undesirable, and so alternatives have been investigated.

In particular, if a family does not have nesting, or only weak nesting, there is no reason to advance the order so rapidly. And even for families that admit nesting, it is useful to have an alternative family whose order grows more slowly. A simple *linear growth rule* can be specified, which at level l seeks the rule of order. This growth rule is used with the families GGH_L, GH_L, GJ_L, GL_L, GLG_L, and LG_L.

In some cases, a *linear 2 growth rule* may be specified, in which case we have $o(l) = 2l + 1$. If our underlying rule is Clenshaw Curtis, then we need such a growth rule in order to satisfy the Novak and Ritter exactness requirement, resulting in the CC_L2 rule. For the Gauss-Legendre rule, (and its symmetric relatives), one disadvantage of the linear growth rule is that the point 0.0 recurs in the rules of odd order. One consequence of this is that it is difficult to derive an exact formula for the number of points in a sparse grid derived from such rules. Hence, the linear 2 growth rule can also be applied here, resulting, for instance, int eh GL_L2 family.

Another variant of the linear growth rule, known as the *odd growth rule*, can be applied for Gaussian rules. The growth rate is chosen so that at level l we have $o(l) = 2\lceil l/2 \rceil + 1$, that is, the order sequence is $\{1, 3, 3, 5, 5, 7, 7, \ldots\}$. The repetition of rules in the sequence yields a certain benefit from nesting that can outweigh the cost incurred by using, on every odd level, a rule of slightly greater exactness than is required. This growth is used with the families GGH_O, GH_O, GJ_O, GLG_O, and LG_O.

It is also possible to try to preserve the advantages of nesting while slowing down the associated exponential growth. This can be done, for instance, by using a somewhat irregular growth rule, in which we first assume that the goal of the rule of level l is to achieve an exactness of at least $2l + 1$. If we then select, from an exponential growth family, the rule of minimum order that satisfies this exactness requirement, we have a *slow exponential growth* family. This growth is used with the families CC_SE, F2_SE, GKH_SE, and GP_SE.

A related choice in slowing the exponential growth assumes that the goal of the rule of level l is to achieve a linear exactness of at least $4l+1$. If we then select, from an exponential growth family, the rule of minimum order that satisfies this exactness requirement, we have a *moderate exponential growth* family. This growth is used with the families CC_ME, F2_ME and GP_ME.

A few of the rules, such as CH2_E, GG_E, NCC_E and NCO_E are included for completeness, although they are currently not of active interest for sparse grid usage. Other rules, such as many of the exponential growth families, are described primarily to contrast them with the superior properties of their linear growth or slow exponential growth variations.

1.2 Finite Rules

Two families, GKH_SE and GP_*, have the property that they are limited, that is, the set of rules in the family appears to be finite. For the Gauss-Patterson rules ("GP"), each rule is derived from the previous rule by a process that potentially could fail. Nested rules are known at least up to order 255, but the sequence is not guaranteed to be endless. For the Genz-Keister family of rules for Hermite quadrature, described in [5] and [7], the construction actually could not be continued beyond the rule of order 35, so this family of rules is known, in fact, to be finite.

1.3 Golub Welsch Rules

The Golub Welsch quadrature families described here are really any family derived from the generic procedure described in [6]. This process begins with an integral of the form $\int_a^b w(x)\, f(x)\, dx$. It is assumed that the three-term recurrence relation is known for the orthogonal polynomials generated by the nonnegative weight function $w(x)$, and that the moments of $w(x)$ are known or can be computed.

For the rule of order n, the abscissas are the roots of the $n - th$ degree orthogonal polynomial generated by $w(x)$. The abscissas and corresponding weights can be determined by solving an eigenvalue problem.

Golub Welsch rules can be generated of any order. A Golub Welsch rule of order o will have exactness $e = 2 * o - 1$. For an arbitrary weight function $w(x)$, we assume that there is no nesting.

To construct a Golub Welsch Linear or Exponential Growth family, we simply compute a sequence of Golub Welsch rules so that the rule of level l has the requested order.

1.4 Discussion of the Tables

In the following tables, under the heading *Type:* we make assertions about whether the rules in a given family are nested or not. In particular, certain families are claimed to be non-nested, by which we understand that no point is shared by any two rules of the family, although it is usually enough to assume that no point is shared by any two *successive* rules. Especially when the family includes parameters, or is derived by the Golub-Welsch procedure, it is possible in certain cases that some points might actually be shared between pairs of rules, or even across all rules. However, we will assume that these are exceptions to the general behavior of the given family, and that even if some nesting occurs, it is not of such a nature that we can or wish to take advantage of it.

Most of the quadrature rules are defined over a standard interval, such as $[-1,+1]$, with a standard weight function $w(x)$ while having a generalization to a general interval and perhaps a weight function with one or more parameters. The tables include the information for the general interval and weight, largely taken from the forms used in IQPACK, Elhay and Kautsky's ACM TOMS Algorithm 655 [3].

2 CC_E: Clenshaw Curtis, Exponential Growth

Standard Interval:	$[-1,+1]$
Weight Function:	$w(x)=1,\ \int_{-1}^{+1} w(x)dx=2$
General Interval:	$[A,B]$
General Weight Function:	$w(x)=1$
Order:	$o(l)=\begin{cases}1, \text{if } l=0;\\ 2^l+1, \text{if } l>0.\end{cases}$
Order vector:	$\{1,3,5,9,17,33,65,129,257,513,1025,\ldots\}$
Exactness:	$e(l)=o(l)$
Exactness vector:	$\{1,3,5,9,17,33,65,129,257,513,1025,\ldots\}$
New vector:	$\{1,2,2,4,8,16,32,64,128,256,512,\ldots\}$
Type:	closed, fully nested, symmetric
Comment:	Same "logical" properties as NCC_E, but a stable rule.

3 CC_ME: Clenshaw Curtis, Moderate Exponential Growth

Standard Interval:	$[-1,+1]$
Weight Function:	$w(x)=1,\ \int_{-1}^{+1} w(x)dx=2$
General Interval:	$[A,B]$
General Weight Function:	$w(x)=1$
Order:	$o(l)=\begin{cases}1 \text{ if } l=0;\\ 2^k+1 \text{ where } k=\lceil\log_2(l)\rceil+2, \text{ otherwise.}\end{cases}$
Order vector:	$\{1,5,9,17,17,33,33,33,33,65,65,\ldots\}$
Exactness:	$e(l)=o(l)$
Exactness vector:	$\{1,5,9,17,17,33,33,33,33,65,65,\ldots\}$
New vector:	$\{1,4,4,8,0,16,0,0,0,32,0,\ldots\}$
Type:	closed, fully nested, symmetric

4 CC_SE: Clenshaw Curtis, Slow Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1,\ \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = \begin{cases} 1 \text{ if } l = 0; \\ 2^k + 1 \text{ where } k = \lceil \log_2(l) \rceil + 1, \text{ otherwise.} \end{cases}$
Order vector:	$\{1, 3, 5, 9, 9, 17, 17, 17, 17, 33, 33, \ldots\}$
Exactness:	$e(l) = o(l)$
Exactness vector:	$\{1, 3, 5, 9, 9, 17, 17, 17, 17, 33, 33, \ldots\}$
New vector:	$\{1, 2, 2, 4, 0, 8, 0, 0, 0, 16, 0, \ldots\}$
Type:	closed, fully nested, symmetric
Comment:	Should have the same exactness as CC_E sparse grids, while using fewer abscissas.

5 CC_L2: Clenshaw Curtis, Linear 2 Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1,\ \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2l + 1$
Order vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
Exactness:	$e(l) = 2l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots\}$
Type:	closed, fully nested, symmetric
Comment:	Repeatedly add two symmetric points from the "next" nested CC rule. Orders 1, 3, 5, 9, 17, ..., match the usual nested CC rule.

6 CH2_E: Chebyshev Kind 2, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = \sqrt{1 - x^2},\ \int_{-1}^{+1} w(x)dx = \frac{\pi}{2}$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = \sqrt{(x - A)(B - x)}$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = o(l)$
Exactness vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
New vector:	$\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots\}$
Type:	open, fully nested, symmetric
Comment:	Same "logical" properties as F2_E.

7 F2_E: Fejer Type 2, Exponential Growth

Standard Interval:	$[-1,+1]$
Weight Function:	$w(x)=1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A,B]$
General Weight Function:	$w(x)=1$
Order:	$o(l)=2^{l+1}-1$
Order vector:	$\{1,3,7,15,31,63,127,255,511,1023,2047,\ldots\}$
Exactness:	$e(l)=o(l)$
Exactness vector:	$\{1,3,7,15,31,63,127,255,511,1023,2047,\ldots\}$
New vector:	$\{1,2,4,8,16,32,64,128,256,512,1024,\ldots\}$
Type:	open, fully nested, symmetric
Comment:	Same "logical" properties as NCO_E, but a stable rule.

8 F2_ME: Fejer Type 2, Moderate Exponential Growth

Standard Interval:	$[-1,+1]$
Weight Function:	$w(x)=1$, $\int_{-1}^{+1} w(x)dx = 2$
Standard Interval:	$[A,B]$
Weight Function:	$w(x)=1$
Order:	$o(l)=2^k-1$ where $k=\lfloor\log_2(l)\rfloor+3$
Order vector:	$\{1,7,15,15,31,31,31,31,63,63,63,\ldots\}$
Exactness:	$e(l)=o(l)$
Exactness vector:	$\{1,7,15,15,31,31,31,31,63,63,63\ldots\}$
New vector:	$\{1,6,8,0,16,0,0,0,32,0,0,\ldots\}$
Type:	open, fully nested, symmetric

9 F2_SE: Fejer Type 2, Slow Exponential Growth

Standard Interval:	$[-1,+1]$
Weight Function:	$w(x)=1$, $\int_{-1}^{+1} w(x)dx = 2$
Standard Interval:	$[A,B]$
Weight Function:	$w(x)=1$
Order:	$o(l)=2^k-1$ where $k=\lfloor\log_2(l)\rfloor+2$
Order vector:	$\{1,3,7,7,15,15,15,15,31,31,31,\ldots\}$
Exactness:	$e(l)=o(l)$
Exactness vector:	$\{1,3,7,7,15,15,15,15,31,31,31\ldots\}$
New vector:	$\{1,2,4,0,8,0,0,0,16,0,0,\ldots\}$
Type:	open, fully nested, symmetric
Comment:	Should have the same exactness as F2_E sparse grids, while using fewer abscissas.

10 GG_E: Gauss-Gegenbauer, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = (1-x^2)^\alpha$, $\int_{-1}^{+1} w(x)dx = \frac{\sqrt{\pi}\Gamma(1+\alpha)}{\Gamma(1.5+\alpha)}$, $\alpha > -1$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = ((x-A)(B-x))^\alpha, \alpha > -1$
Order:	$o(l) = 2\,l + 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\,l + 1$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Type:	open, non-nested, symmetric

11 GGH_E: Generalized Gauss Hermite, Exponential Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = \|x\|^\alpha e^{-x^2}$, $\int_{-\infty}^{\infty} w(x)dx = \frac{1+(-1)^\alpha}{2}\Gamma(\frac{1+\alpha}{2})$, $\alpha > -1$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = \|x-a\|^\alpha e^{-b(x-a)^2}, \alpha > -1$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2^{l+2} - 3$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 2, 6, 14, 30, 62, 126, 254, 510, 1022, 2046, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

12 GGH_L: Generalized Gauss Hermite, Linear Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = \|x\|^\alpha e^{-x^2}$, $\int_{-\infty}^{\infty} w(x)dx = \frac{1+(-1)^\alpha}{2}\Gamma(\frac{1+\alpha}{2})$, $\alpha > -1$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = \|x-a\|^\alpha e^{-b(x-a)^2}, \alpha > -1$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2\,l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, very weakly nested (0.0 recurs), symmetric

13 GGH_O: Generalized Gauss Hermite, Odd Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = \lvert x\rvert^{\alpha} e^{-x^2}$, $\int_{-\infty}^{\infty} w(x)dx = \frac{1+(-1)^{\alpha}}{2}\Gamma(\frac{1+\alpha}{2})$, $\alpha > -1$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = \lvert x - a\rvert^{\alpha} e^{-b(x-a)^2}, \alpha > -1$
Order:	$o(l) = 2\lceil l/2\rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\lceil l/2\rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 2, 0, 4, 0, 6, 0, 8, 0, 10, 0, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

14 GGP_E: Genz-Gauss-Patterson, Exponential Growth with Intermediates

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l)$ is irregular.
Order vector:	$\{1, 3, 7, 13, 15, 25, 27, 29, 31, 49, 51, 53, 55, 57, 59, 61, 63\}$
Exactness:	$e(l)$ is irregular.
Exactness vector:	$\{1, 5, 11, 13, 23, 25, 27, 29, 47, 49, 51, 53, 55, 57, 59, 61, 95\}$
New vector:	$\{1, 2, 4, 6, 2, 10, 2, 2, 2, 18, 2, 2, 2, 2, 2, 2, 2\}$
Type:	open, fully nested, finite, symmetric
Comment:	interleaves cheaper "ad hoc" non-Gaussian but nested rules.

15 GH_E: Gauss-Hermite, Exponential Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = e^{-x^2}$, $\int_{-\infty}^{+\infty} w(x)dx = \sqrt{\pi}$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = e^{-b(x-a)^2}$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2^{l+2} - 3$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 2, 6, 14, 30, 62, 126, 254, 510, 1022, 2046, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

16 GH_L: Gauss-Hermite, Linear Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = e^{-x^2}$, $\int_{-\infty}^{+\infty} w(x)dx = \sqrt{\pi}$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = e^{-b(x-a)^2}$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2\,l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, very weakly nested (0.0 recurs), symmetric

17 GH_O: Gauss-Hermite, Odd Growth

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = e^{-x^2}$, $\int_{-\infty}^{+\infty} w(x)dx = \sqrt{\pi}$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = e^{-b(x-a)^2}$
Order:	$o(l) = 2\lceil l/2 \rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\lceil l/2 \rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 2, 0, 4, 0, 6, 0, 8, 0, 10, 0, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

18 GJ_E: Gauss-Jacobi, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = (1-x)^\alpha (1+x)^\beta$, $\int_{-1}^{+1} w(x)dx = \frac{{}_2F_1(1,-\alpha,2+\beta,-1)}{\beta+1} + \frac{{}_2F_1(1,-\beta,2+\alpha,-1)}{\alpha+1}$, $\alpha, \beta > -1$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = (B-x)^\alpha (x-A)^\beta$, $\alpha, \beta > -1$
Order:	$o(l) = 2\,l + 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\,l + 1$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Type:	open, non-nested, non-symmetric (unless $\alpha = \beta$)

19 GJ_L: Gauss-Jacobi, Linear Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = (1-x)^\alpha (1+x)^\beta$, $\int_{-1}^{+1} w(x)dx = \frac{{}_2F_1(1,-\alpha,2+\beta,-1)}{\beta+1} + \frac{{}_2F_1(1,-\beta,2+\alpha,-1)}{\alpha+1}$, $\alpha, \beta > -1$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = (B-x)^\alpha (x-A)^\beta, \alpha, \beta > -1$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2\,l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, non-nested, non-symmetric (unless $\alpha = \beta$)

20 GJ_O: Gauss-Jacobi, Odd Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = (1-x)^\alpha (1+x)^\beta$, $\int_{-1}^{+1} w(x)dx = \frac{{}_2F_1(1,-\alpha,2+\beta,-1)}{\beta+1} + \frac{{}_2F_1(1,-\beta,2+\alpha,-1)}{\alpha+1}$, $\alpha, \beta > -1$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = (B-x)^\alpha (x-A)^\beta, \alpha, \beta > -1$
Order:	$o(l) = 2\lceil l/2 \rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\lceil l/2 \rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 3, 0, 5, 0, 7, 0, 9, 0, 11, 0, \ldots\}$
Type:	open, non-nested, non-symmetric (unless $\alpha = \beta$)

21 GKH_SE: Genz-Keister-Hermite, Exponential Growth with Intermediates

Standard Interval:	$(-\infty, +\infty)$
Weight Function:	$w(x) = e^{-x^2}$, $\int_{-\infty}^{+\infty} w(x)dx = \sqrt{\pi}$
General Interval:	$(-\infty, +\infty)$, symmetric around $x = a$
General Weight Function:	$w(x) = e^{-b(x-a)^2}$
Order:	$o(l)$ is irregular.
Order vector:	$\{1, 3, 7, 9, 17, 19, 31, 33, 35, \text{(end)}\}$
Exactness:	$e(l)$ is irregular.
Exactness vector:	$\{1, 5, 7, 15, 17, 29, 31, 33, 51, \text{(end)}\}$
New vector:	$\{1, 2, 4, 2, 8, 2, 12, 2, 2, \text{(end)}\}$
Type:	open, fully nested, symmetric.
Comment:	interleaves cheaper "ad hoc" non-Gaussian but nested rules. The rules of order 1, 3, 9, 19 and 35 form a nested family of rules created by a procedure for the Hermite weight similar to Patterson's extension of the Gauss rule for Legendre weight. The rules of order 7, 17, 31 and 33 are formed as efficiency measures. They maintain the nestedness of the original family, but provide a desired level of polynomial Exactness using slightly fewer points than the "nearest" appropriate rule from the original nested family.

22 GL_E: Gauss-Legendre, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2^{l+2} - 3$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 2, 6, 14, 30, 62, 126, 254, 510, 1022, 2046, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

23 GL_L2: Gauss-Legendre, Linear2 Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2l + 1$
Order vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\,l + 1$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric
Comment:	because 0.0 is repeated each time, it is difficult, but not extremely difficult, to count the points in a sparse grid made from this rule.

24 GL_L: Gauss-Legendre, Linear Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2\,l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, very weakly nested (0.0 recurs), symmetric
Comment:	because 0.0 is only repeated every other time, it is extremely difficult to count the points in a sparse grid made from this rule.

25 GL_L: Gauss-Legendre, Odd Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1$, $\int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2\lceil l/2 \rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\lceil l/2 \rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 2, 0, 4, 0, 6, 0, 8, 0, 10, 0, \ldots\}$
Type:	open, weakly nested (0.0 repeated), symmetric

26 GLG_E: Generalized Gauss-Laguerre, Exponential Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = x^\alpha \, e^{-x}$, $\int_0^{+\infty} w(x)dx = \Gamma(\alpha+1)$, $\alpha > -1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = (x-a)^\alpha \, e^{-b(x-a)}$, $\alpha > -1$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\, o(l) - 1 = 4\, l + 1$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Type:	open, non-nested, non-symmetric

27 GLG_L: Generalized Gauss-Laguerre, Linear Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = x^\alpha \, e^{-x}$, $\int_0^{+\infty} w(x)dx = \Gamma(\alpha+1)$, $\alpha > -1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = (x-a)^\alpha \, e^{-b(x-a)}$, $\alpha > -1$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\, o(l) - 1 = 2\, l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, non-nested, non-symmetric

28 GLG_O: Generalized Gauss-Laguerre, Odd Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = x^\alpha \, e^{-x}$, $\int_0^{+\infty} w(x)dx = \Gamma(\alpha+1)$, $\alpha > -1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = (x-a)^\alpha \, e^{-b(x-a)}$, $\alpha > -1$
Order:	$o(l) = 2\lceil l/2 \rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\, o(l) - 1 = 4\lceil l/2 \rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 3, 0, 5, 0, 7, 0, 9, 0, 11, 0, \ldots\}$
Type:	open, non-nested, non-symmetric

29 GP_E: Gauss-Patterson, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1, \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots, \text{(end?)}\}$
Exactness:	$e(l) = \begin{cases} 1, \text{if } l = 0; \\ 1.5 * o(l) + 0.5, \text{if } l > 0. \end{cases}$
Exactness vector:	$\{1, 5, 11, 23, 47, 95, 191, 383, 767, 1535, 3071, \ldots, \text{(end?)}\}$
New vector:	$\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots, \text{(end?)}\}$
Type:	open, fully nested, symmetric
Comment:	combines F2_E style nesting and some of GL_L higher Exactness.

30 GP_ME: Gauss Patterson, Moderate Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1, \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = \begin{cases} 1 \text{ if } l = 0; \\ 2^{k+1} - 1 \text{ where } k = \lceil \log_2((l+1)/3) \rceil + 2, \text{ otherwise.} \end{cases}$
Order vector:	$\{1, 3, 7, 15, 15, 15, 31, 31, 31, 31, 31, \ldots, \text{(end?)}\}$
Exactness:	$e(l) = \begin{cases} 1 \text{ if } l = 0; \\ 1.5\, o(l) + 0.5, \text{otherwise.} \end{cases}$
Exactness vector:	$\{1, 5, 11, 23, 23, 23, 47, 47, 47, 47, 47, \ldots, \text{(end?)}\}$
New vector:	$\{1, 4, 6, 12, 0, 0, 24, 0, 0, 0, 0, \ldots, \text{(end?)}\}$
Type:	open, fully nested, symmetric

31 GP_SE: Gauss Patterson, Slow Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1, \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = \begin{cases} 1 \text{ if } l = 0; \\ 2^{k+1} - 1 \text{ where } k = \lceil \log_2((l+1)/3) \rceil + 1, \text{ otherwise.} \end{cases}$
Order vector:	$\{1, 3, 3, 7, 7, 7, 15, 15, 15, 15, 15, \ldots, \text{(end?)}\}$
Exactness:	$e(l) = \begin{cases} 1 \text{ if } l = 0; \\ 1.5\, o(l) + 0.5, \text{otherwise.} \end{cases}$
Exactness vector:	$\{1, 5, 5, 11, 11, 11, 23, 23, 23, 23, 23, \ldots, \text{(end?)}\}$
New vector:	$\{1, 2, 0, 4, 0, 0, 8, 0, 0, 0, 0, \ldots, \text{(end?)}\}$
Type:	open, fully nested, symmetric
Comment:	Should have the same exactness as GP_E sparse grids, while using fewer abscissas.

32 LG_E: Gauss-Laguerre, Exponential Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = e^{-x}$, $\int_0^{+\infty} w(x)dx = 1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = e^{-b(x-a)}$
Order:	$2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2^{l+2} - 3$
Exactness vector:	$\{1, 5, 13, 29, 61, 125, 253, 509, 1021, 2045, 4093, \ldots\}$
New vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Type:	open, non-nested, non-symmetric

33 LG_L: Gauss-Laguerre, Linear Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = e^{-x}$, $\int_0^{+\infty} w(x)dx = 1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = e^{-b(x-a)}$
Order:	$o(l) = l + 1$
Order vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 2\,l + 1$
Exactness vector:	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$
New vector:	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$
Type:	open, non-nested, non-symmetric

34 LG_O: Gauss-Laguerre, Odd Growth

Standard Interval:	$[0, \infty)$
Weight Function:	$w(x) = e^{-x}$, $\int_0^{+\infty} w(x)dx = 1$
General Interval:	$[a, \infty)$
General Weight Function:	$w(x) = e^{-b(x-a)}$
Order:	$o(l) = 2\lceil l/2 \rceil + 1$
Order vector:	$\{1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, \ldots\}$
Exactness:	$e(l) = 2\,o(l) - 1 = 4\lceil l/2 \rceil + 1$
Exactness vector:	$\{1, 5, 5, 9, 9, 13, 13, 17, 17, 21, 21, \ldots\}$
New vector:	$\{1, 3, 0, 5, 0, 7, 0, 9, 0, 11, 0, \ldots\}$
Type:	open, non-nested, non-symmetric

35 NCC_E: Newton Cotes Closed, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1,\ \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = \begin{cases} 1, \text{if } l = 0; \\ 2^l + 1, \text{if } l > 0. \end{cases}$
Order vector:	$\{1, 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, \ldots\}$
Exactness:	$e(l) = o(l)$
Exactness vector:	$\{1, 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, \ldots\}$
New vector:	$\{1, 2, 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots\}$
Type:	closed, fully nested, symmetric
Comment:	Equally spaced points; some weights become negative; not a rule of choice.

36 NCO_E: Newton Cotes Open, Exponential Growth

Standard Interval:	$[-1, +1]$
Weight Function:	$w(x) = 1,\ \int_{-1}^{+1} w(x)dx = 2$
General Interval:	$[A, B]$
General Weight Function:	$w(x) = 1$
Order:	$o(l) = 2^{l+1} - 1$
Order vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
Exactness:	$e(l) = o(l)$
Exactness vector:	$\{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \ldots\}$
New vector:	$\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots\}$
Type:	open, fully nested, symmetric
Comment:	Equally spaced points; some weights become negative; not a rule of choice.

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