MATHEMATICAL GAMES

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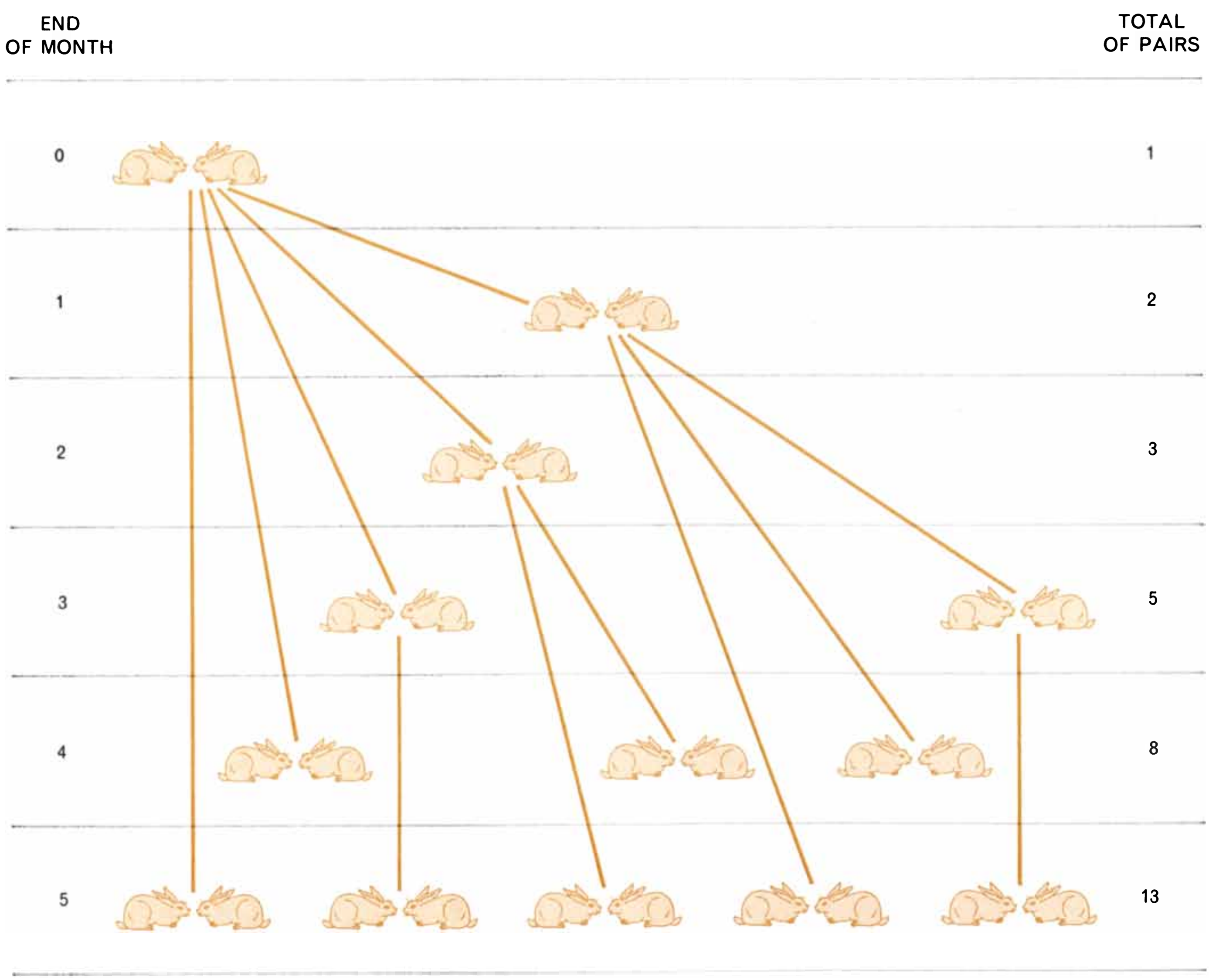
MATHEMATICAL GAMES

The multiple fascinations of the Fibonacci sequence

by Martin Gardner

The greatest European mathematician of the Middle Ages was Leonardo of Pisa, better known as Fibonacci, meaning "son of Bonaccio." Although Leonardo was born in Pisa, his father was an official of an Italian mercantile factory in Bougie in Algeria, and it was there that young Leonardo received his early mathematical training from Moslem tutors. He quickly recognized the enormous superiority of the Hindu-Arabic decimal system, with its positional notation and zero symbol, over the clumsy Roman system still used in his own country. His best-known work, *Liber abaci* (literally "Book of the Abacus" but actually a comprehensive merchant's handbook on arithmetic and algebra), defended the merits of the Hindu-Arabic notation. The arguments made little impression on the Italian merchants of the time but the book eventually became the most influential single work in introducing the Hindu-Arabic system to the West. Although *Liber abaci* was completed in Pisa in 1202, it survives only in a revised 1228 edition dedicated to a famous astrologer of the period. There has never been an English translation.

It is ironic that Leonardo, who made valuable contributions to mathematics, is remembered today mainly because a 19th-century French number theorist, Édouard Lucas (who edited a classic four-volume work on recreational mathematics), attached the name Fibonacci to a number sequence that appears in a trivial problem in *Liber abaci*. Suppose, Leonardo wrote, a male-female pair of adult rabbits is placed inside an enclosure to breed. Assume that rabbits start to bear young two months after



Tree graph for Fibonacci's rabbits

their own birth, producing only a single male-female pair, and that they have one such pair at the end of each subsequent month. If none of the rabbits die, how many pairs of rabbits will there be inside the enclosure at the end of one year?

The tree graph [*see illustration on opposite page*] shows what happens during the first five months. It is easy to see that the numbers of pairs at the close of each month form the sequence 1, 2, 3, 5, 8, . . . , in which each number (as Fibonacci pointed out) is the sum of the two numbers preceding it. At the end of 12 months there will be 377 pairs of rabbits.

Fibonacci did not investigate the series and no serious study of it was undertaken until the beginning of the 19th century, when, as a mathematician once put it, papers on the sequence began to multiply almost as fast as Fibonacci's rabbits. Lucas made a deep study of sequences (now called "generalized Fibonacci sequences") that begin with *any* two positive integers, each number thereafter being the sum of the preceding two. He called the simplest such series, 1, 1, 2, 3, 5, 8, 13, 21, . . . , the Fibonacci sequence. The position of each number in this sequence is traditionally indicated by a subscript, so that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so on. (The first 50 Fibonacci numbers are listed in the illustration on this page). F_n refers to any Fibonacci number. F_{n+1} is the number following F_n; F_{n-1} is the number preceding F_n; F_{2n} is the F-number with a subscript twice that of F_n, and so on.

The Fibonacci sequence has intrigued mathematicians for centuries, partly because it has a way of turning up in unexpected places but mainly because the veriest amateur in number theory, with no knowledge beyond simple arithmetic, can explore the sequence and discover a seemingly endless variety of curious theorems. Recent developments in computer programming have reawakened interest in the series because it turns out to have useful applications in the sorting of data, information retrieval, the generation of random numbers and even in rapid methods of approximating maxima and minima values of complicated functions for which derivatives are not known.

Early results are summarized in Chapter 17 of the first volume of Leonard Eugene Dickson's *History of the Theory of Numbers.* For the most recent discoveries interested readers can consult *The Fibonacci Quarterly,* published since 1963 by the Fibonacci Association. (The annual price is now $6 and subscriptions are handled by the managing editor, Brother Alfred Brousseau, at St. Mary's College in St. Mary's College, Calif.) Edited by Verner E. Hoggatt, Jr., of San Jose State College in San Jose, Calif., the quarterly is concerned primarily with generalized Fibonacci numbers and similar numbers (such as "Tribonacci numbers," which are sums of the preceding *three* numbers), but the journal is also devoted "to the study of integers with special properties."

Surely the most remarkable property of the Fibonacci series (which holds for the generalized series too) is that the ratio between two consecutive numbers is alternately greater or smaller than the golden ratio and that, as the series continues, the differences become less and less; the ratios approach the golden ratio as a limit. The golden ratio is a famous irrational number, 1.61803 . . . , that is obtained by halving the sum of 1 and the square root of 5. There is a considerable literature, some of it dubious, about the appearance of the golden ratio and the closely related Fibonacci sequence in organic growth and about their applications to art, architecture and even poetry. George Eckel Duckworth, professor of classics at Princeton University, maintains in his book *Structural Patterns and Proportions in Vergil's Aeneid* (University of Michigan Press, 1962) that the Fibonacci series was consciously used by Vergil and other Roman poets of the time. I dealt with such matters in an earlier column on the golden ratio, which is reprinted in *The 2nd Scientific American Book of Mathematical Puzzles & Diversions.*

The most striking appearance of Fibonacci numbers in plants is in the spiral arrangement of seeds on the face of certain varieties of sunflower. There are two sets of logarithmic spirals, one set turning clockwise, the other counterclockwise, as indicated by the two colored spirals in the illustration on the next page. The numbers of spirals in the two sets are different and tend to be consecutive Fibonacci numbers. Sunflowers of average size usually have 34 and 55 spirals, but giant sunflowers have been developed that go as high as 89 and 144. In the letters department of *The Scientific Monthly* (November, 1951) Daniel T. O'Connell, a geologist at City College of the City of New York, and his wife reported having found on their Vermont farm one mammoth sunflower with 144 and 233 spirals!

The intimate connection between the Fibonacci series and the golden ratio

n	F_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1,597
18	2,584
19	4,181
20	6,765
21	10,946
22	17,711
23	28,657
24	46,368
25	75,025
26	121,393
27	196,418
28	317,811
29	514,229
30	832,040
31	1,346,269
32	2,178,309
33	3,524,578
34	5,702,887
35	9,227,465
36	14,930,352
37	24,157,817
38	39,088,169
39	63,245,986
40	102,334,155
41	165,580,141
42	267,914,296
43	433,494,437
44	701,408,733
45	1,134,903,170
46	1,836,311,903
47	2,971,215,073
48	4,807,526,976
49	7,778,742,049
50	12,586,269,025

The first 50 Fibonacci numbers

can be seen in the following formula for the nth Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

This equation gives the nth Fibonacci number exactly but it is cumbersome to use for high F-numbers, although good approximations can be obtained with logarithms. A much simpler formula for the nth F-number is the golden ratio raised to the power of n and then divided by the square root of 5. When this result is rounded off to the nearest integer, it too provides the exact number sought. Both formulas are nonrecursive because they compute the nth F-number directly from n. A "recursive procedure" is a series of steps each of which is dependent on previous steps. If you compute the nth F-number by summing consecutive F-numbers until you reach the nth, you are computing it recursively; a definition of the nth F-number as the sum of the preceding two numbers is a simple example of a recursive formula. (Two highly efficient computer algorithms for computing large F-numbers exactly are given as the answer to exercise No. 26 on page 552 of *Seminumerical Algorithms,* the second volume of *The Art of Computer Programming* by Donald E. Knuth.)

To find the sum of the first n Fibonacci numbers the best procedure is to determine F_{n+2} and then subtract 1. Example: What is the sum of the first 20 F-numbers? Subtract 1 from 17,711, the 22nd F-number, to get the answer: 17,710.

Here are some more well-known properties of the Fibonacci sequence, most of them not difficult to prove:

1. The square of any F-number differs by 1 from the product of the two F-numbers on each side. The difference is alternately plus or minus as the series continues. Like so many properties of the Fibonacci series, this is a special case of a property that applies to the general sequence starting with any two integers. In the general case too the difference is a constant that is alternately plus or minus. For example, the next-simplest series after the Fibonacci, 1, 3, 4, 7, 11, 18, ... (now called the Lucas series after the French mathematician), has a constant difference of 5.

2. The sum of the squares of any two consecutive F-numbers, F^2_n and F^2_{n+1}, is F^2_{2n+1}. Since the last number must have an odd subscript, it follows from this theorem that if you write in sequence the squares of the F-numbers, sums of consecutive squares will produce in sequence the F-numbers with odd subscripts.

3. For any four consecutive F-numbers, A, B, C, D, the following formula holds: $C^2 - B^2 = A \times D$.

4. The sequence of final digits of the Fibonacci sequence repeats in cycles of 60. The last two digits repeat in cycles of 300. The repeating cycle is 1,500 for three final digits, 15,000 for four digits, 150,000 for five and so on for all larger numbers of digits.

5. For every integer m there is an infinite number of F-numbers that are evenly divisible by m, and at least one can be found among the first m^2 numbers of the Fibonacci sequence.

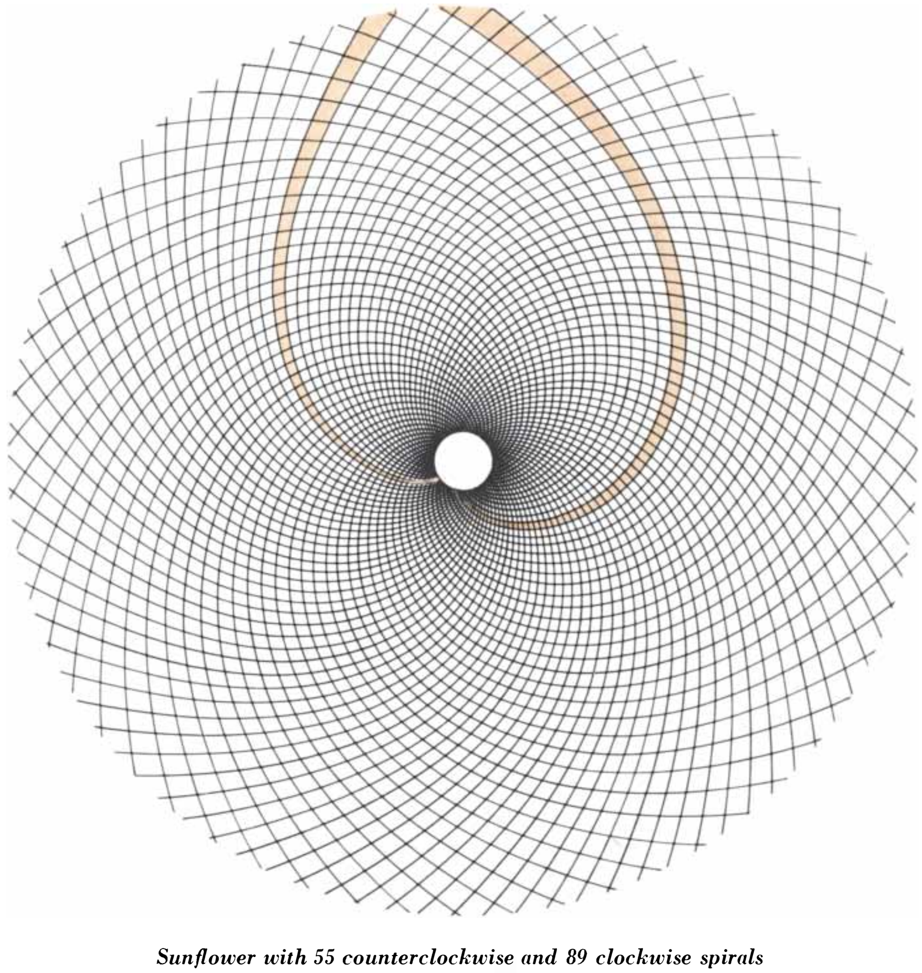
6. Every third F-number is divisible by 2, every fourth number by 3, every fifth number by 5, every sixth number by 8 and so on, the divisors being F-numbers in sequence. Consecutive Fibonacci numbers (as well as consecutive Lucas numbers) cannot have a common divisor other than 1.

7. With the exception of 3, every F-number that is prime has a prime subscript (for example, 233 is prime and its subscript, 13, is also prime). Put another way, if a subscript is composite (not prime), so is the number. Unfortunately the converse is not always true: a prime subscript does not necessarily mean that the number is prime. The first counterexample is F_{19}, 4,181. The subscript is prime but 4,181 is 37 times 113.

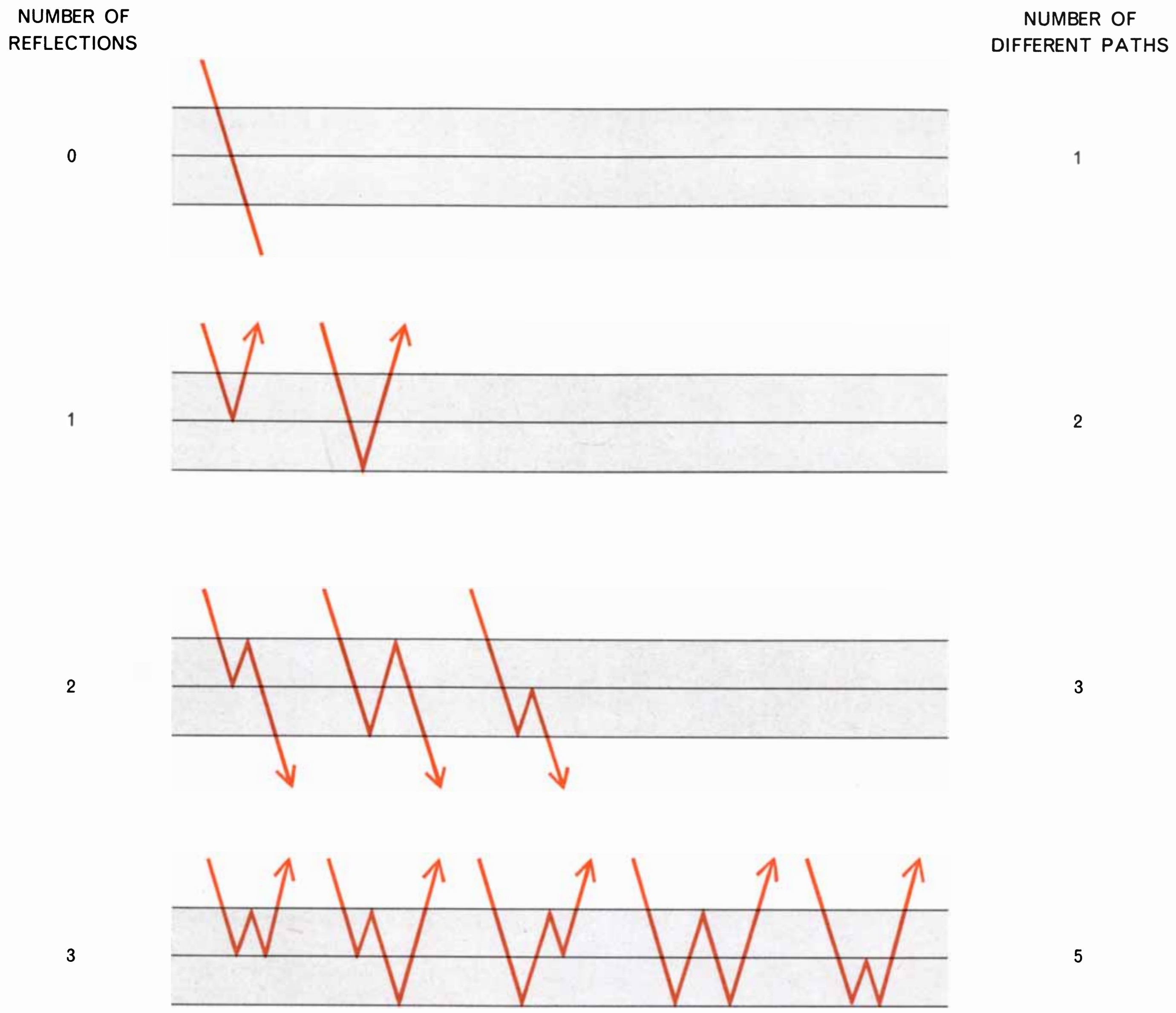
If the converse theorem held in all cases, it would answer the greatest unsolved question about Fibonacci numbers: Is there an infinity of Fibonacci primes? We know that the number of primes is infinite, and therefore if every F-number with a prime subscript were prime, there would be an infinity of prime F-numbers. As it is, no one today knows if there is a largest Fibonacci prime.

8. With the trivial exceptions of 0 and 1 (taking 0 to be F_0), the only square F-number is F_{12}, 144—which, surprisingly, is the square of its subscript. Whether or not there is a square F-number greater than 144 was an open question until the matter was finally settled, as recently as 1963, by John H. E. Cohn of Bedford College in the University of London. He also proved that 1 and 4 are the only squares in the Lucas sequence.

9. The reciprocal of 89, the 11th F-number, can be generated by writing



Sunflower with 55 counterclockwise and 89 clockwise spirals



There are F_{n+2} *paths by which a ray can be reflected* n *times through two panes of glass*

the Fibonacci sequence, starting with 0, and then adding as follows:

.0112358
13
21
34
55
89
144
233
. . .

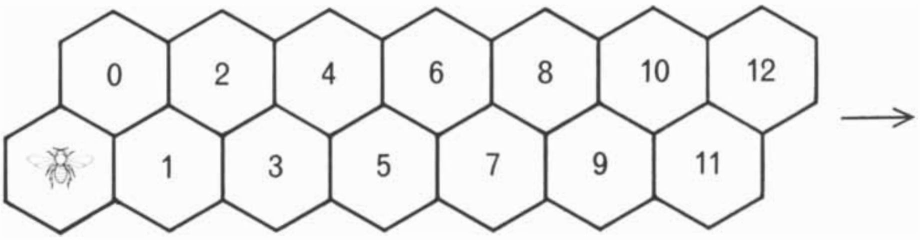
.011235955040673 . . . = 1/89 .

This list of properties could be extended to fill a book. One could do the same with instances of how the series appears in physical and mathematical situations. (For its appearance in the diagonals of Pascal's triangle see this department for December, 1966.) Leo Moser in 1963 studied the paths of slanting light rays through two face-to-face glass plates. An unreflected ray goes through the plates in only one way [*see illustration above*]. If a ray is reflected once, there are two paths; if it is reflected twice, there are three paths, and if three times, there are five paths. As n, the number of reflections, increases, the numbers of possible paths fall into the Fibonacci sequence. For n reflections the number of paths is F_{n+2}.

The sequence can be applied similarly to the different paths that can be taken by a bee crawling over hexagonal cells [*see top illustration on next page*]. The cells extend as far as desired to the right. Assume that the bee always moves to an adjacent cell and always moves toward the right. It is not hard to prove there is one path to cell 0, two paths to cell 1, three to cell 2, five to cell 3 and so on. As before, the number of paths is F_{n+2}, where n is the number of cells involved.

Consider Fibonacci nim, a counter-removal game invented a few years ago by Robert E. Gaskell. The game begins with a pile of n counters. Players take turns removing counters. The first player may not take the entire pile, but thereafter either player may remove all the remaining counters if these rules permit: at least one counter must be taken on each play, but a player may never take more than twice the number of counters his opponent took on his last play. Thus if one player takes three counters, the next player may not take more than six. The person who takes the last counter wins.

It turns out that if n is a Fibonacci number, the second player can always win; otherwise the first player can win. If a game begins with 20 counters (not an F-number), how many must the first



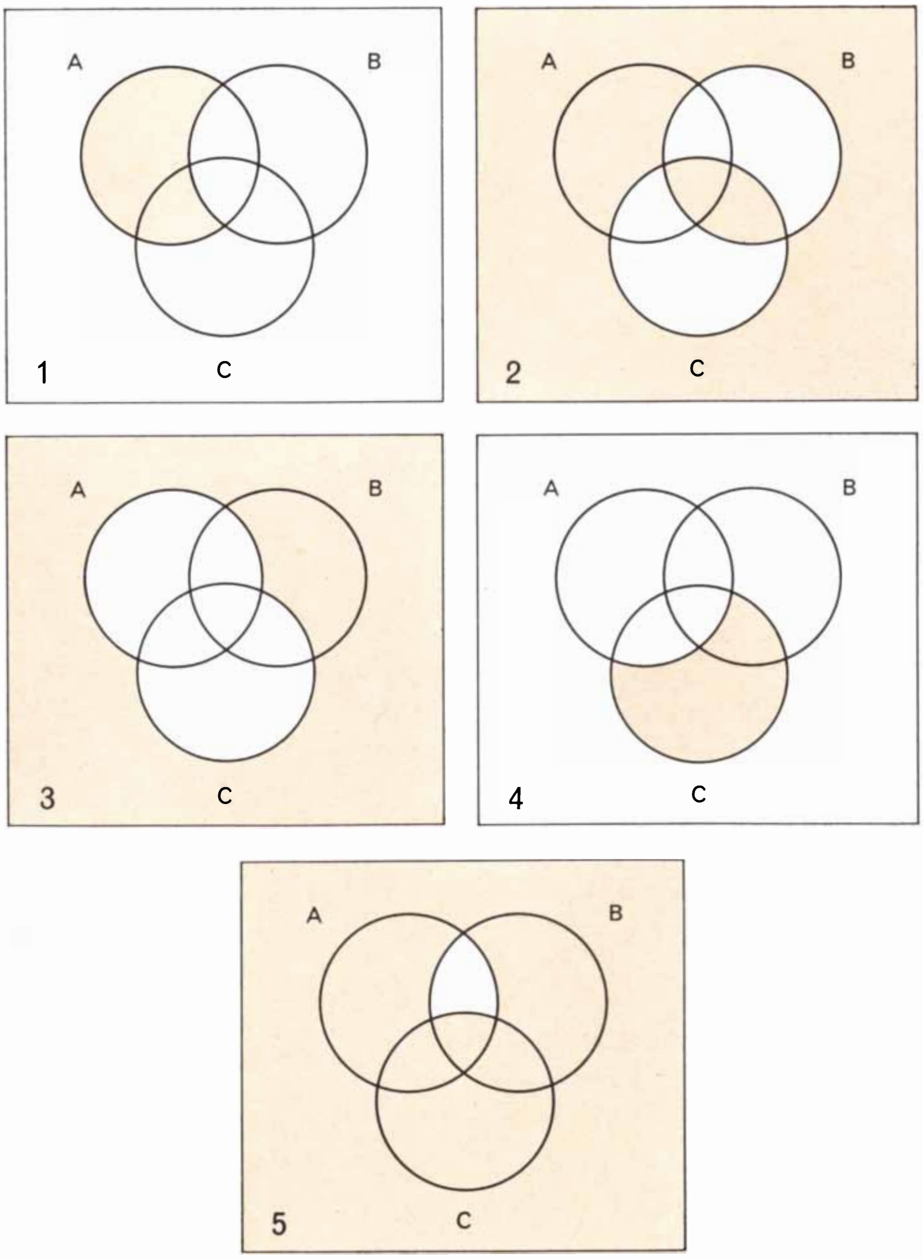
There are F_{n+2} *paths by which the bee can crawl to cell* n

player take to be sure of winning? The answer will be given next month along with a simple strategy employing *F*-numbers.

A second problem, also to be answered next month, concerns a little-known lightning calculation trick. Turn your back and ask someone to write down any two positive integers (one below the other), add those two numbers to get a third, put the third number below the second, add the last two numbers to get a fourth, and continue in this way until he has a column of 10 numbers. In other words, he writes 10 numbers of a generalized Fibonacci sequence, each the sum of the preceding two numbers except for the first two, which are picked at random. You turn around, draw a line below the last number and immediately write the sum of all 10 numbers.

The secret is to multiply the seventh number by 11. This can easily be done in your head. Suppose the seventh number is 928. Put down the last digit, 8, as the last digit of the sum. Add 8 and 2 to get 10. Put 0 to the left of 8 in the sum, carrying the 1. The sum of the next pair of digits, 9 and 2, is 11. Add the carried 1 to get 12. Put 2 to the left of 0 in the sum, again carrying 1. Add the carried 1 to 9 and put down 10 to the left of 2 in the sum. The completed sum is 10,208. In brief, you sum the digits in pairs, moving to the left, carrying 1 when necessary, and ending with the last digit on the left.

Can you prove, before the simple answer is given next month, that the sum of the first 10 numbers in a generalized Fibonacci sequence is always 11 times the seventh number?



Venn-diagram solution to last month's martini problem

Three Venn circles are shaded as in the illustration at the left to solve last month's problem about the three men who lunch together. Each of the first four diagrams is shaded to represent one of the four premises of the problem. Superimposing the four to form the last diagram shows that if the four premises are true, the only possible combination of truth values is *a, b,* $\sim c$, or true *a*, true *b* and false *c*. Since we are identifying truth with ordering a martini, this means that Abner and Bill always order martinis, whereas Charley never does.

The method of generating 2^n integers to form Boolean algebras, as explained last month, was given by Francis D. Parker in *The American Mathematical Monthly* for March, 1960, page 268. Consider a set of any number of distinct primes, say 2, 3, 5. Write down the multiples of all the subsets of these three primes, which include 0 (the null set) and the original set of three primes. Change 0 to 1. This produces the set 1, 2, 3, 5, 6, 10, 15, 30, the first of the examples given last month. In a similar way the four primes 2, 3, 5, 7 will generate the second example, the $2^4 = 16$ factors of 210. A proof that all such sets provide Boolean algebras, when the algebra is interpreted as explained last month, can be found in *Boolean Algebra*, by R. L. Goodstein (Pergamon Press, 1963), page 126, as the answer to problem No. 10.

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