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MATHEMATICAL GAMES

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MATHEMATICAL GAMES

The multiple charms of Pascal's triangle

by Martin Gardner

Harry Lorayne, a professional magician and memory expert who lives in New York, has recently been puzzling his friends with an unusual mathematical card trick of his own invention. A spectator is given a deck from which the face cards and tens have

been removed. He is asked to place any five cards face up in a row. Lorayne immediately finds a card in the deck that he puts face down at a spot above the row, as shown in the illustration below. The spectator now builds a pyramid of cards as follows:

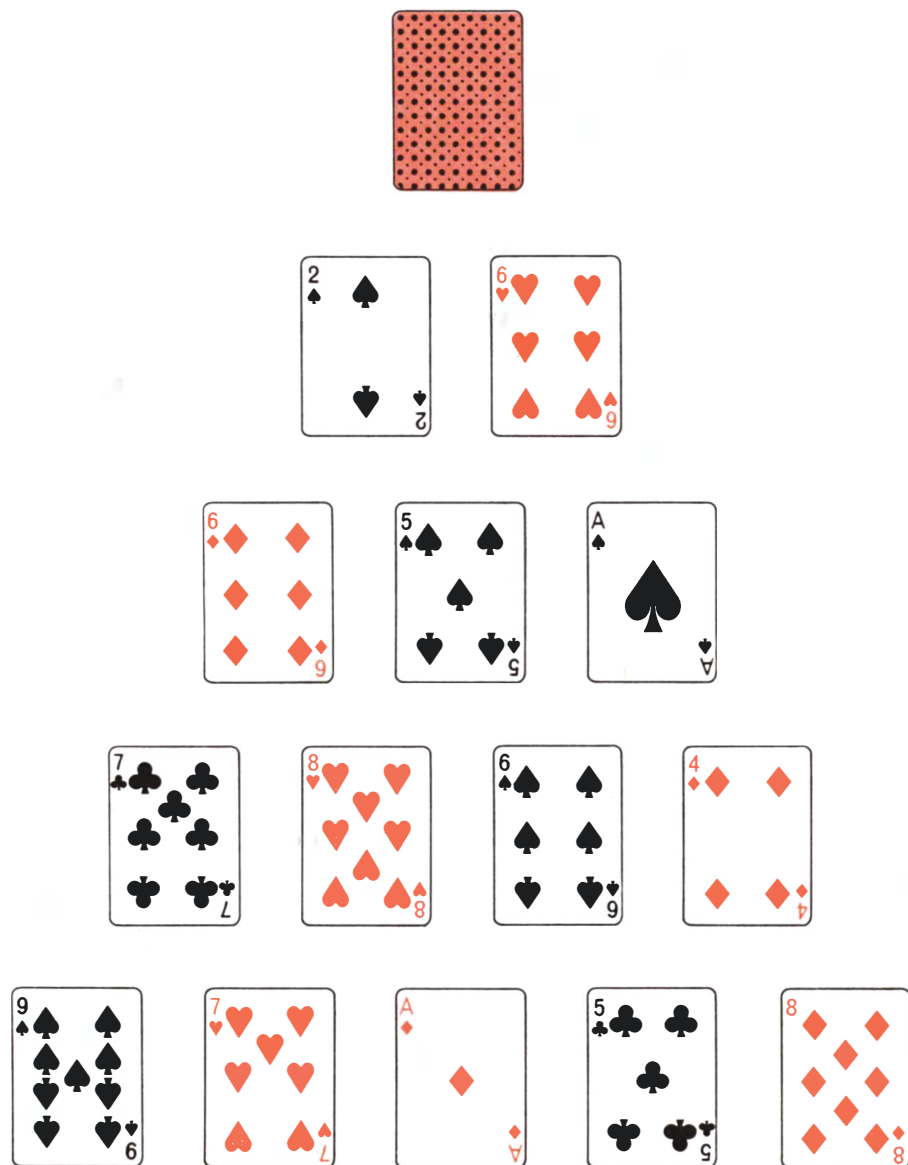
Each pair of cards in the row is added by the process of "casting out nines." If the sum is above 9, 9 is subtracted. This can be done rapidly by adding the two digits in the sum. For example, the first

two cards in the bottom row of the illustration add to 16. Instead of subtracting 9 from 16, the same result is obtained by adding 1 and 6. The sum is 7; therefore the spectator puts a seven above the first pair of cards. The second and third cards add to 8, so an eight goes above them. This is continued until a new row of four cards is obtained, and the procedure is repeated until the pyramid reaches the face-down apex card. When this card is turned over, it proves to be the correct value for the final sum.

The trick can be done with any number of cards in the initial row, although if there are too many there may not be enough cards to supply all the needed values for the pyramid. The computations can, of course, always be done on paper. A good version of the trick is to ask someone to jot down a row of 10 random digits. You can calculate the pyramid's apex digit quickly in your head if you know the secret, and it will always turn out to be correct. How is the apex digit determined? One's first thought is that perhaps it is the "digital root" of the first row—the sum of the digits reduced to a single digit by casting out nines—but this is not the case.

The truth is that Lorayne's trick operates with simple formulas derived from one of the most famous number patterns in the history of mathematics. The pattern is known as Pascal's triangle because Blaise Pascal, the 17th-century French mathematician and philosopher, was the first to write a treatise about it: *Traité du triangle arithmétique* (*Treatise on the Arithmetic Triangle*). The pattern was well known, however, long before 1653, when Pascal first wrote his treatise. It had appeared on the title page of an early 16th-century arithmetic by Petrus Apianus, an astronomer at the university in Ingolstadt. An illustration in a 1303 book by a Chinese mathematician also depicts the triangular pattern, and recent scholarship has traced it back still earlier. Omar Khayyám, who was a mathematician as well as a poet and philosopher, knew of it about 1100, having in turn probably got it from still earlier Chinese or Indian sources.

The pattern is so simple that a 10-year-old can write it down, yet it contains such inexhaustible riches, and links with so many seemingly unrelated aspects of mathematics, that it is surely one of the most elegant of all number arrays. The triangle begins with 1 at the apex [see illustration on opposite page]. All other numbers are the sums of the two numbers directly above them. (Think of each 1, along the two borders, as the sum of the 1 above it on one side and 0,



The apex card trick

State College who edits *The Fibonacci Quarterly*, a fascinating journal that has published many articles about Pascal's triangle. If the zero diagonal on the left side is sliced off, the Fibonacci diagonals have sums that are the partial sums of the Fibonacci series ($1 = 1$; $1 + 1 = 2$; $1 + 1 + 2 = 4$; $1 + 1 + 2 + 3 = 7$ and so on). If diagonals 0 and 1 are eliminated from the left side, the Fibonacci diagonals give the partial sums of the partial sums ($1 = 1$; $1 + 2 = 3$; $1 + 2 + 4 = 7$ and so forth). In general, if k diagonals are trimmed, the Fibonacci diagonals give the k -fold partial sums of the Fibonacci series.

Each horizontal row of Pascal's triangle gives the coefficients in the expansion of the binomial $(x + y)^n$. For example, $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. The coefficients of this expansion are 1, 3, 3, 1 (a coefficient of 1 is customarily omitted from a term), which is the third row of the triangle. To find the coefficients of $(x + y)^n$, in proper order, merely look at the triangle's n th row. This basic property of the triangle ties it in with elementary combinatorial and probability theory in ways that make the triangle a useful calculating device. Suppose an Arab chief offers to give you any three of his seven wives. How many different selections can you make? You have only to find the intersection of diagonal 3 and row 7 to get the answer: 35. If (in your eager confusion) you commit

the blunder of looking for the intersection of diagonal 7 and row 3, you will find that they do not intersect, so that the method can never go wrong. In general the number of ways to select a set of n elements from a set of r distinct elements is given by the intersection of diagonal n and row r .

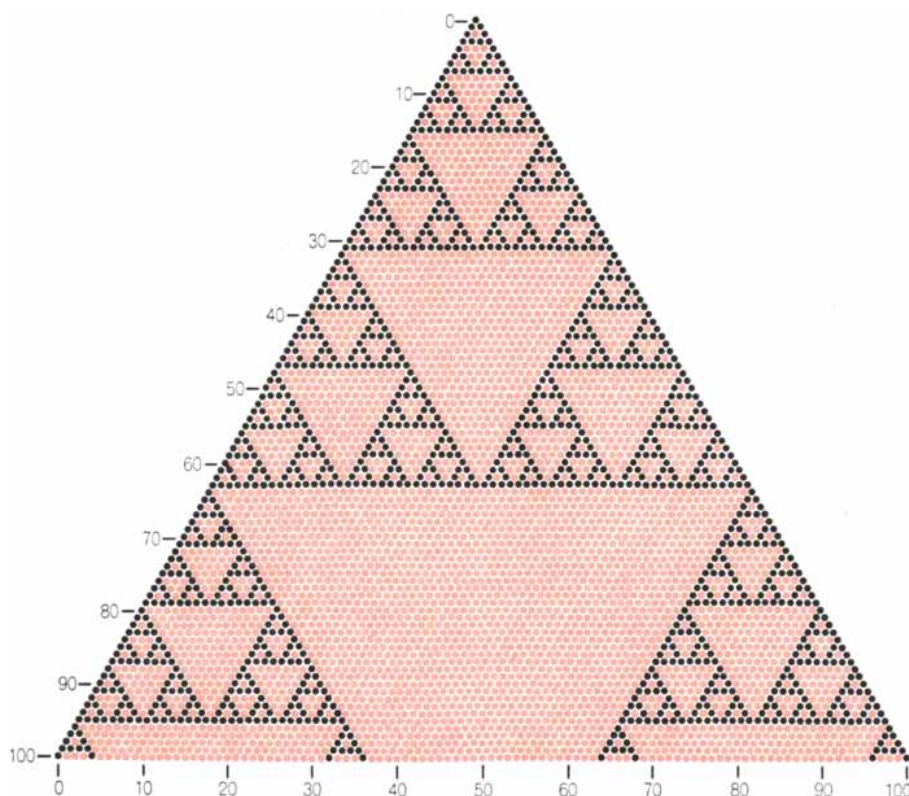
The connection between this and probability is easily seen by considering the eight equally possible outcomes of getting heads or tails when flipping three pennies: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. There is one way to get three heads, three ways to get two heads, three ways to get one head and one way to get no heads. These numbers (1, 3, 3, 1) are, of course, the triangle's third row. Suppose you want to know the probability of exactly five heads showing if you toss 10 pennies in the air. First determine how many different ways five pennies can be selected from 10. The intersection of diagonal 5 and row 10 provides the answer: 252. Now you must add the numbers in the 10th row to obtain the number of equally possible cases. You can short-cut this addition by remembering that the sum of the n th row of Pascal's triangle is always 2^n . (The sum of each row is obviously twice the sum of the preceding row, since every number is carried down twice to enter into the numbers of the row below; therefore the sums of the rows form the doubling series 1, 2, 4,

8...) The 10th power of 2 is 1,024. The probability of getting five heads is $252/1,024$, or $63/256$. (There is a mechanical device for demonstrating probability, often exhibited at science fairs and museums, in which hundreds of small balls roll down an incline through a hexagonal array of obstacles to enter slots and form an approximation of the bell-shaped normal-distribution curve. For a picture of such a device, and a discussion of how Pascal's triangle underlies it, see "Probability," by Mark Kac; SCIENTIFIC AMERICAN, September, 1964.)

If we represent each number of the triangle by a small dot and then blacken every dot whose number is not exactly divisible by a certain positive integer, the result is always a striking pattern of triangles. Patterns obtained in this way conceal many surprises. Consider the binary pattern that results when the divisor is 2 [see illustration on this page]. Running down the center there are colored triangles of increasing size, each made up entirely of even-numbered dots. At the top is a "triangle" of one dot, then the series continues with triangles of 6, 28, 120, 496... dots. Three of those numbers—6, 28 and 496—are known as perfect numbers because each is the sum of all its divisors, excluding itself (for example, $6 = 1 + 2 + 3$). It is not known if there is an infinity of perfect numbers, or if there is one that is odd. Euclid managed to prove, however, that every number of the form $2^{n-1}(2^n - 1)$, where $(2^n - 1)$ is a prime (primes of this form are called Mersenne primes), is an even perfect number. Leonhard Euler much later showed that all even perfect numbers conform to Euclid's formula. The formula is equivalent to

$$\frac{P(P + 1)}{2},$$

where P is a Mersenne prime. The above expression happens also to be the formula for a triangular number. In other words, if the "side" of a triangular number is a Mersenne prime, the triangular number is also perfect. Going back to the even-odd coloring of Pascal's triangle, it can be shown that the formula for the number of dots in the n th central triangle, moving down from the apex, is $2^{n-1}(2^n - 1)$, the formula for perfect numbers. All even perfect numbers appear in the pattern, therefore, as the number of dots in the n th central triangle whenever $2^n - 1$ is prime. Because $2^4 - 1 = 15$, which is not a prime, the fourth colored triangle is not perfect. The fifth triangle of 496 dots is perfect because $2^5 - 1 = 31$, a prime. (The sixth



Pascal's triangle with numbers represented by dots, the odd numbers by black dots

colored triangle is not perfect, but the seventh, with 8,128 dots, is.)

One final curiosity. If rows 0 through 4 are read as single numbers (1, 11, 121, 1,331 and 14,641), they are the first five powers of 11, starting with $11^0 = 1$. The fifth row *should* be $11^5 = 161,051$, but it is not. Observe, however, that this is the first row with two-digit numbers. If we interpret each number as indicating a multiple of the place value of that spot in decimal notation, the fifth row can be interpreted (reading right to left) as $(1 \times 1) + (5 \times 10) + (10 \times 100) + (10 \times 1,000) + (5 \times 10,000) + (1 \times 100,000)$, which gives the correct value of 11^5 . Interpreted this way, each *n*th row is 11^n .

Almost anyone can study the triangle and discover more properties, but it is unlikely they will be new, for what is said here only scratches the surface of a vast literature. Pascal himself, in his treatise on the triangle, said that he was leaving out more than he was putting in. "It is a strange thing," he exclaimed, "how fertile it is in properties!" There are also endless variants on the triangle, and many ways to generalize it, such as building it in tetrahedral form to give the coefficients of trinomial expansions.

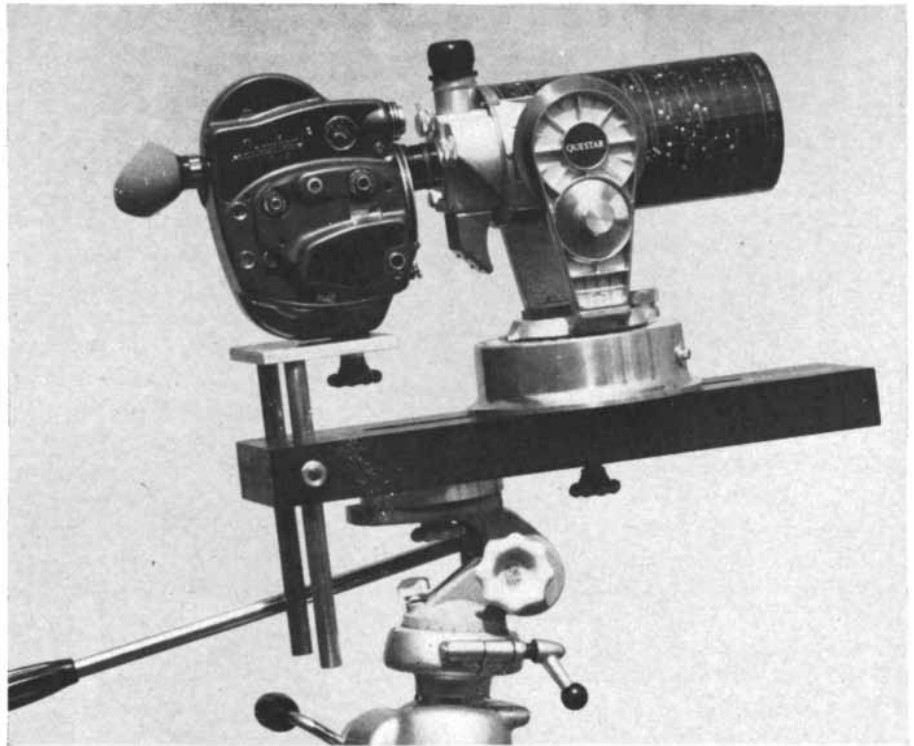
If the reader can solve the following five elementary problems, all to be answered in this department next month, he will find his understanding of the triangle's structure pleasantly enriched:

1. What formula gives the sum of all numbers above row *n*?
2. How many odd numbers are there in row 256?
3. How many numbers in row 67 (in honor of the coming year) are evenly divisible by 67?
4. If a checker is placed on one of the four black squares in the first row of an otherwise empty checkerboard, it can move (by standard checker moves) to any of the four black squares on the last (eighth) row by a variety of different paths. One pair of starting and ending squares is joined by a maximum number of different routes. Identify the two squares and give the number of different ways the checker can move from one to the other.
5. Given an initial row of *n* cards, in the pyramid trick described at the beginning, how can one obtain from Pascal's triangle simple formulas for calculating the value of the apex card?

The answers to last month's problems follow:

A tesseract of side *x* has a hypervolume of x^4 . The volume of its hypersurface is $8x^3$. If the two magnitudes are

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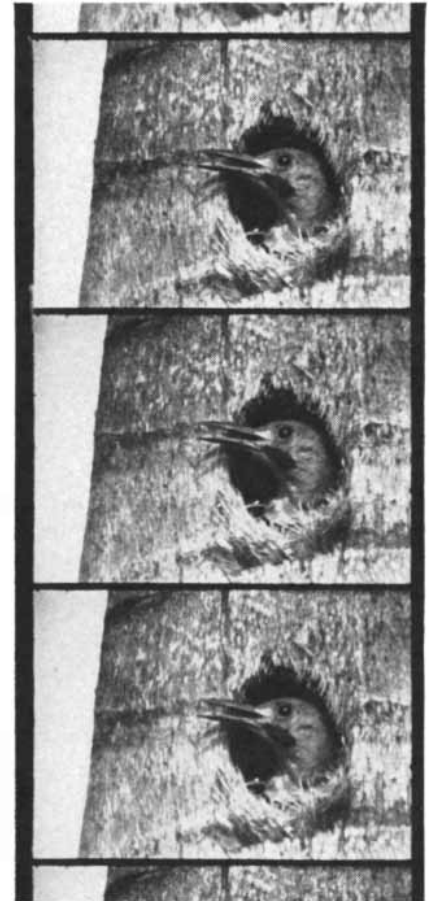
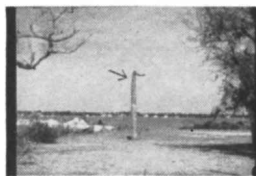
They approve our Camera Cradle solution to the heavy camera problem and suggest that telescope and camera be coupled as closely as possible, also that pictures be taken in bright sunlight, with Plus-X and Ektachrome E.M.S. the most satisfactory films, at 16 and 24 frames per second for general work. Further, they point out that a precision-made tripod designed for cine photography is mandatory for such high powers.

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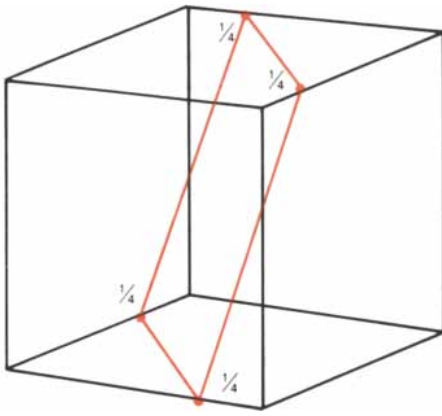
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Packing a square in a cube

equal, the equation gives x a value of 8. In general an n -space "cube" with an n -volume equal to the $(n - 1)$ -volume of its "surface" is an n -cube of side $2n$.

The largest square that can be fitted inside a unit cube is the square shown in the illustration above. Each corner of the square is a distance of $1/4$ from a corner of the cube. The square has an area of exactly $9/8$ and a side that is three-fourths of the square root of 2. Readers familiar with the old problem of pushing the largest possible cube through a square hole in a smaller cube will recognize this square as the cross section of the limiting size of the square hole. In other words, a cube of side not quite three-fourths of the square root of 2 can be pushed through a square hole in a unit cube.

The illustration at the right shows the 11 different hexominoes that fold into a cube. They form a frustrating set of the 35 distinct hexominoes, because they will not fit together to make any of the rectangles that contain 66 unit squares, but perhaps there are some interesting patterns they *will* form.

So many letters are still being received on the September problem of packing squares of sides 1, 2, 3...24 into a square of side 70 that I shall postpone publishing the best solution until next month.

A large number of readers sent comments on the science puzzles of August. Several believed the impact of falling sand in the hourglass paradox might play a role in keeping a buoyant hourglass at the bottom, but this is not true. The force of impact of a falling grain is balanced by its loss of weight in free fall, with a zero net effect.

Maya and Nicolas Slater wrote from London to say that, in the boat-carriage problem, if one abandons Lewis Carroll's

proviso that elliptical wheels opposite each other must have their major axes at right angles, there is a way to make the carriage pitch and roll without any wheel's leaving the ground. The wheels must be geared so that *diagonally* opposite wheels keep their major axes at right angles. Regardless of the angle between the two front wheels, all four wheels remain on the ground at all times. If the front wheels are at a 90-degree angle, the carriage rolls without pitching; if the angle is zero, it pitches without rolling. All intermediate angles combine rolling and pitching. "Our preference is for 45 degrees," the Slaters wrote. "Our only problem is keeping our coachman."

Many alternate solutions for the rope-stealing problem were received; some made use of knots that could be shaken loose from the ground, others involved cutting a rope partway through so that it would just support the thief's weight and later could be snapped by a sudden pull. Several readers doubted that the thief would get *any* rope because the bells would start ringing.

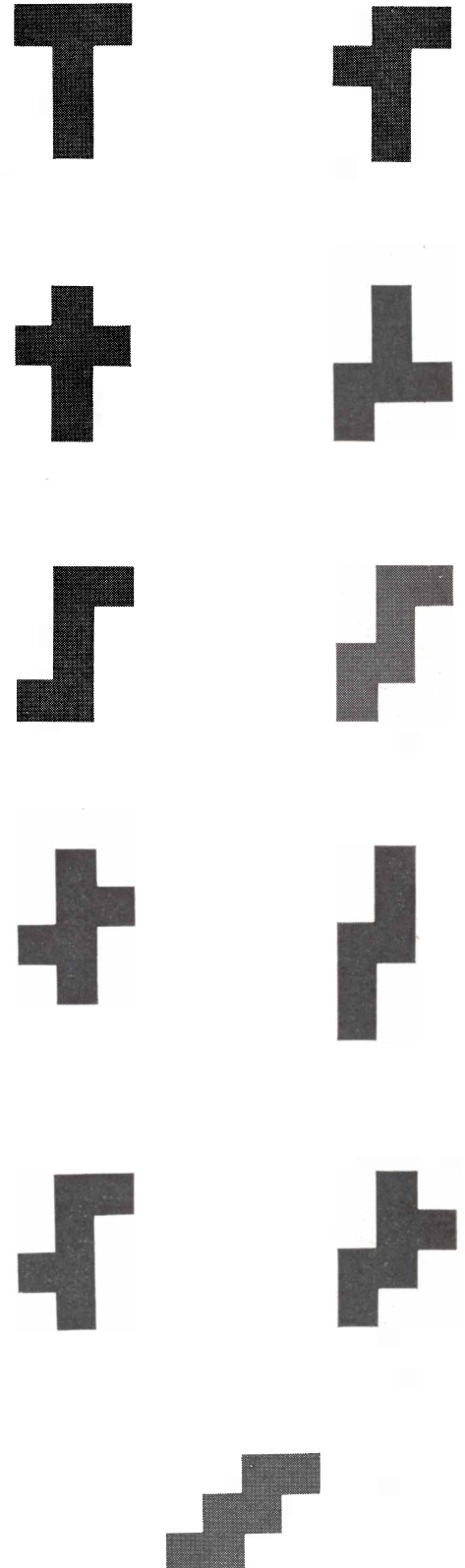
A number of readers corrected the statement that a peeled hard-boiled egg is drawn into a milk bottle by a vacuum created by the loss of oxygen when matches are burned inside. Oxygen is indeed used up, but the loss is compensated by the production of carbon dioxide and water vapor. The vacuum is created solely by the quick cooling and contracting of the air after the flames go out.

The coiled-hose paradox is more complicated than was indicated. If the funnel end of the empty hose is high enough, water poured into it will be forced over more than one winding to form a series of "heads" in each coil. The maximum height of each head is about equal to the coil's diameter. The diameter, times the number of coils, gives the approximate height the water column at the funnel end must be to force water out at the other end. (This was pointed out by John C. Bryner, Jan Lundberg and J. M. Osborne.) W. N. Goodwin, Jr., noted that for hoses with an outside diameter of $5/8$ inch or less the funnel end can be as low as twice the height of the coils and water will flow all the way through a series of many coils. The reason for this is not yet clear.

Several readers thought of a second way, albeit a temporary one, to make a cork float at the center of the surface of a glass of water. Create a vortex with a spoon, then drop in the cork.

John Friedlein, who teaches mathe-

tics at a high school in St. Charles, Ill., observed that not only does Christmas equal Halloween, as pointed out last month (Dec. 25 = Oct. 31 when the abbreviations are taken for the decimal and octal systems), but also Thanksgiving if it falls, as it sometimes does, on Novem. 27.



The 11 hexominoes that fold into cubes

A handful of people like Mary Carnwath are trying to keep our promise to the Indians. But they won't make it without you.

The Hopi Indians' village of Shipaulovi in Arizona sits on land so poor, infertile and inhospitable that so far nobody has tried to take it away from them.

Electricity has not yet reached the Hopis. Water must be hauled from three miles away. Jobs are few and far away. Only poverty and despair are close-by and in abundance.

Yet for the first time in generations, Mary Carnwath and people like her are stirring hope among the Hopis.

Mary Carnwath works and lives two thousand miles away, in Manhattan. Her own daughter is now grown-up, and through *Save the Children Federation* she is sponsoring one of the village girls, 8-year-old Grace Mahtewa.

The Mahtewas (two parents, three children, one grandmother and a sister-in-law) live tightly packed in a tiny rock and mud house. The father who knows ranch work but can't find any most of the year, isn't able to provide the family with even the bare necessities.



Grace, bright, ambitious and industrious, would possibly have had to quit school as soon as she was old enough to do a day's work. But, because of Mary

Carnwath, that won't be necessary. The \$12.50 a month contributed by Mary Carnwath is providing a remarkable number of things for Grace and her family.

Grace will have a chance to continue schooling. The family has been able to make its home a little more livable. And with the money left over, together with funds from other sponsors, the village has been able to renovate a dilapidated building for use as a village center. The center now has two manual sewing machines that are the beginnings of a small income-producing business. It's only a small beginning. More money and more people like Mary Carnwath are needed. With your help, perhaps this village program

will produce enough money to end the Hopi's need for help. That is what *Save the Children* is all about.

Although contributions are deductible, it's not a charity. The aim is not merely to buy one child a few hot meals, a warm coat and a new pair of shoes. Instead, your contribution is used to give the child, the family and the village a little boost that may be all they need to start helping themselves.

Sponsors are desperately needed for other American Indian children — who suffer the highest disease rate and who look forward to the shortest life span of any American group.

As a sponsor you will receive a photo of the child, regular reports on his progress and, if you wish, a chance to correspond with him and his family.

Mary Carnwath knows that she can't save the world for \$12.50 a month. Only a small corner of it. But, maybe that is the way to save the

world. If there are enough Mary Carnwaths. How about you?

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New Alcoa research paper evaluates methods of testing weldments for susceptibility to stress-corrosion cracking.

Practically all commercial aluminum alloy weldments are resistant to stress-corrosion cracking. However, the search for ever-stronger aluminum alloys, such as for armor plate, leads investigators into complex alloy systems for which the resistance to stress-corrosion cracking is a necessary consideration.

What causes stress-corrosion cracking?

Stress-corrosion cracking is cracking that is initiated by localized corrosion synergized by sustained tensile stress at the surface.

According to the theories developed by Alcoa in 1940 and 1944, stress-corrosion cracking of aluminum alloys begins at the grain boundaries of the metal. Hence, stress-corrosion cracking of aluminum alloys is characteristically intergranular. Stress-corrosion cracking can occur when three conditions are present:

Susceptible composition and metallurgical structure • High tensile stress at the surface • Specific environment

Alcoa's basic research in stress corrosion and tests of new alloys and applications have employed various methods of testing weldments. Our evaluation of these methods, as well as test results, is presented in a new 17-page paper. The investigated techniques for applying tension

include the use of both constant-load and constant-deformation methods. Four types of specimen are evaluated: simple beam, U-bend, tensile specimen and residual-stress specimen. Besides evaluating test methods, the paper also includes the results of testing seven structural alloys, including three from the 7000 series, as well as five weld-filler alloys. This paper documents another addition to the thousands of man-years that Alcoa has spent on aluminum research.

When you want authoritative answers about aluminum, come to Alcoa. Would you like to learn more about testing weldments for stress corrosion? Write Aluminum Company of America, 904-M Alcoa Building, Pittsburgh, Pa. 15219. Ask for the paper *Evaluation of Various Techniques for Stress-Corrosion Testing Welded Aluminum Alloys* by M. B. Shumaker, R. A. Kelsey, D. O. Sprowls and J. G. Williamson.



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