Computational Geometry Lab:
FINITE ELEMENTS

John Burkardt
Information Technology Department
Virginia Tech
http://people.sc.fsu.edu/~jburkardt/presentations/cg_lab_fem.pdf

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1 Introduction

This lab continues the topic of *Computational Geometry.* Having studied triangles and how triangles are used to create triangulations of a region, we will now turn to the use of triangulations in the finite element method.

The finite element method is a procedure for approximating and solving partial differential equations. In order to discretize the problem so that it is suitable for computational analysis, it is first necessary to take the domain over which the partial differential equation is defined, and construct a suitable triangulation.

One requirement for the triangulation is that it must well approximate the shape of the boundary. This is simple to do when the region is a simple convex polygon with no unusually small angles. However, it is obviously difficult for triangles to follow a curved boundary. It turns out that other difficulties may arise if the boundary has small angles, or inward corners; in this case the accuracy of the computed solution may be very poor unless the triangulation is properly adapted.

The second requirement for the triangulation concerns the elements in the interior of the region. Roughly speaking, the accuracy of the solution depends in part on the size (which we may take to be the length of the longest edge) of the triangles involved, typically denoted by h. So the triangulation must control the maximum triangle size. While all triangles must be somewhat small, there may be some triangles that will have to be very small, because they occur in a part of the region where the solution is difficult to capture. But while pursuing accuracy, these regions of very small triangles must be monitored carefully so that the total number of triangles does not become too large.

Thus, a student may begin by experiencing simple triangulations of rectangular regions in which all the triangles are the same size and essentially occur in pairs that form squares. In practice, the region to be meshed may have a complicated shape with curves, sharp angles and internal walls; the resulting mesh may have some subregions in which there are so many triangles that they plot as a single dark blur!

For the finite element method, the triangulation of the region is not simply a matter of where to choose the points of approximation. The triangles that make up the triangulation are a fundamental tool in building the set of basis functions which will be used to represent the solution. Simply stated, each node of the mesh corresponds to a basis function. That basis function has the property that it is 1 at the node, and zero at all other nodes. Moreover, over any triangular element, the basis function is linear. That is enough information to completely define the basis function. We will investigate the implications of this idea, and how it enables the finite element method to define and compute approximate solutions to partial differential equations.

2 The Finite Element Mesh

3 Finite Element Basis Functions

In finite element analysis, a region is simulated by a collection of triangles, and a function defined over that region might be simulated by a piecewise linear function, whose values are known at the vertices of the triangles. Knowing vertex values is logically enough to determine the value of the function inside the triangle. But the computational details can be a little perplexing.

If we want to evaluate a linear function in a triangle, given the location of the vertices and the values there, it's easy if we can construct three *basis functions*. Basis function $\phi_a(x,y)$ will be a linear function which is associated with node **Va**, where it has the value 1; it is 0 at the other two nodes. Again, this is enough to define $\phi_a(x,y)$. Basis functions $\phi_b(x,y)$ and $\phi_c(x,y)$ are defined similarly.

If we can find an arithmetic definition of these basis functions, then our linear function $f(x,y)$ is simply a linear combination of them:

$$f(x,y) = Wa * \phi_a(x,y) + Wb * \phi_b(x,y) + Wc * \phi_c(x,y)$$

for certain values **Wa, Wb, Wc**, and so our task will be done.

So let us find a formula for $\phi_a(x,y)$. Since it is 0 at nodes Vb and Vc, it must also be zero at all points (x,y) on the line between these two nodes. But since these points lie on a line, we already know one linear relationship they satisfy:

$$\frac{Vc.y - Vb.c}{Vc.x - Vb.x} = \frac{y - Vb.y}{x - Vb.x}$$

(It's customary to write the slope relationship this way. Should either denominator be zero, we could elminate the fractions, and have a valid, if less familiar, formula.)

If we subtract one side from the other, and call the result $g(x,y)$, we have that:

$$g(x,y) = (x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(y - Vb.y)$$

We know that $g(x,y) = 0$ for those points on the line between Vb and Vc. Assuming our triangle is not degenerate, then $g(x_i, y_i)$ is *nonzero* (because n_i does *not* lie on the line between Vb and Vc!). So $g(x,y)$ is almost a basis function, since it's zero at the right places, and nonzero at the other. But it's easy to scale a function so that a nonzero value is 1. How's this for a candidate for our basis function:

$$\phi_a(x,y) = \frac{g(x,y)}{g(Va.x, Vb.y)}$$

You should easily see that this formula is indeed zero at Vb and Vc, and sure enough, it's 1 at Va, and therefore, it represents our basis function. Of course, it would be nice to see this formula explicitly. All we have to do is substitute, to get:

$$\phi_a(x,y) = \frac{(x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(y - Vb.y)}{(Va.x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(Va.y - Vb.y)}$$

This is a linear function of two arguments. Putting in the values $(Va.x, Va.y)$, $(Vb.x, Vb.y)$, and $(Vc.x, Vc.y)$ gives you the values 1, 0 and 0, respectively, so we know it's right.

This means we have a way to construct and evaluate the linear function in the triangle, based on its values at the nodes.

4 Program #6: Finite Element Functions

Write a program which accepts three triangle vertices **Va, Vb, Vc** a set of three values associate with the vertices, **Wa, Wb, Wc** and a point **P**.

For the given point **P**, generate the barycentric coordinates $(\xi_a(P), \xi_b(P), \xi_c(P)$. Evaluate $f(P)$, the linear function which has the values **Wa, Wb, Wc** at the points **Va, Vb, Vc**.

Some simple checks include the following:

- setting **Wa, Wb, Wc** to (1,0,0) should mean $f(P) = \xi_a(P)$;
- setting **P = Va** should result in $f(P) = Va$;
- setting **P = (Va+Vb+Vc)/3** should result in $f(P) = (Wa + Wb + Wc)/3$;