Curvelets and Approximation Theory

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IMA Thematic Year on Imaging, Sep. 2005–Jun. 2006

Mathematicians are like Frenchmen: Whatever you say to them, they translate into their own language and forthwith it is something entirely different.

Johan Wolfgang von Goethe

Outline

1. Curvelet Transforms
   - Background and Motivation
   - Continuous Curvelet Transform
   - Discrete Curvelet Transform

2. Analysis with Curvelets
   - Curvelets and Singularities
   - Curvelets and Cartoons
   - Curvelets and Besov Spaces

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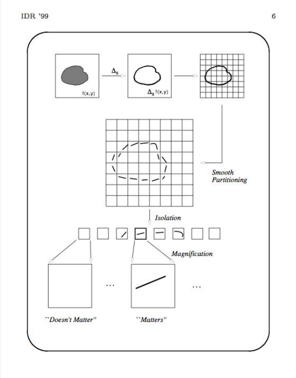
- Curvelets were introduced in 1999 by Candès and Donoho to address the edge representation problem. The definition they gave was based on windowed ridgelets.



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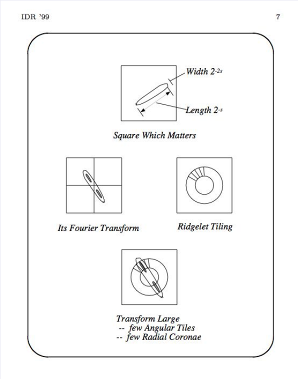

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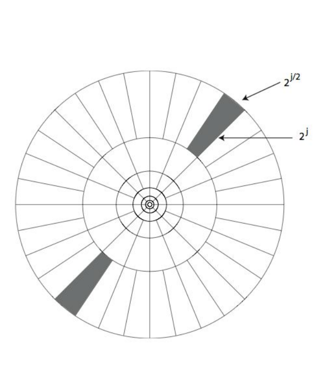

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- In 2003, they developed a Continuous Curvelet Transform.

$$f = \iiint \langle \Phi_{\alpha\beta\theta}, f \rangle \, \Phi_{\alpha\beta\theta} \, \frac{d\alpha}{\alpha^3} \, d\theta \, d\beta$$

$$\|f\|_2^2 = \iiint |\langle \Phi_{\alpha\beta\theta}, f \rangle|^2 \, \frac{d\alpha}{\alpha^3} \, d\theta \, d\beta$$

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$$f = \iiint \langle \Phi_{\alpha\beta\theta}, f\rangle \, \Phi_{\alpha\beta\theta} \, \tfrac{d\alpha}{\alpha^3} \, d\theta \, d\beta + \int \langle \gamma_\beta, f\rangle \, \gamma_\beta \, d\beta$$

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Some applications to Imaging

- **Candès, Donoho, Starck:**
  - Image Denoising.

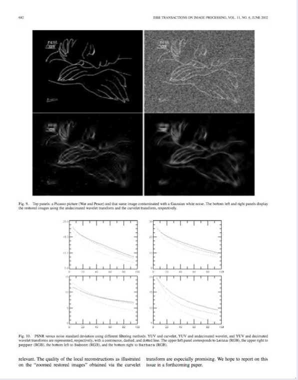
682 IEEE TRANSACTIONS ON IMAGE PROCESSING, VOL. 11, NO. 6, JUNE 2002

Fig. 9. Top panels: a Picasso picture (War and Peace) and that same image contaminated with a Gaussian white noise. The bottom left and right panels display the restored images using the undecimated wavelet transform and the curvelet transform, respectively.

Fig. 10. PSNR versus noise standard deviation using different filtering methods. YUV and curvelet, YUV and undecimated wavelet, and YUV and decimated wavelet transforms are represented, respectively, with a continuous, dashed, and dotted line. The upper left panel corresponds to Lenna (RGB), the upper right to pepper (RGB), the bottom left to Baboon (RGB), and the bottom right to Barbara (RGB).

relevant. The quality of the local reconstructions as illustrated on the "zoomed restored images" obtained via the curvelet transform are especially promising. We hope to report on this issue in a forthcoming paper.

Some applications to Imaging

- **Candès, Donoho, Starck**:
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  - Imaging in Astrophysics.

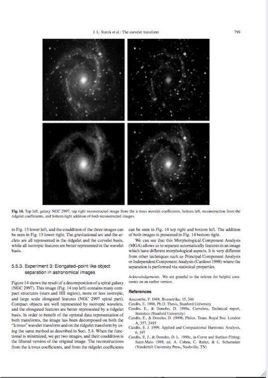

Fig. 14. Top left, galaxy NGC 2997, top right reconstructed image from the à trous wavelet coefficients, bottom left, reconstruction from the ridgelet coefficients, and bottom right addition of both reconstructed images.

in Fig. 13 lower left, and the coaddition of the three images can be seen in Fig. 13 lower right. The gravitational arc and the arclets are all represented in the ridgelet and the curvelet basis, while all isotropic features are better represented in the wavelet basis.

5.5.3. Experiment 3: Elongated–point like object separation in astronomical images

Figure 14 shows the result of a decomposition of a spiral galaxy (NGC 2997). This image (Fig. 14 top left) contains many compact structures (stars and HII region), more or less isotropic, and large scale elongated features (NGC 2997 spiral part). Compact objects are well represented by isotropic wavelets, and the elongated features are better represented by a ridgelet basis. In order to benefit of the optimal data representation of both transforms, the image has been decomposed on both the "à trous" wavelet transform and on the ridgelet transform by using the same method as described in Sect. 5.4. When the functional is minimized, we get two images, and their coaddition is the filtered version of the original image. The reconstructions from the à trous coefficients, and from the ridgelet coefficients can be seen in Fig. 14 top right and bottom left. The addition of both images is presented in Fig. 14 bottom right.

We can see that this Morphological Component Analysis (MCA) allows us to separate automatically features in an image which have different morphological aspects. It is very different from other techniques such as Principal Component Analysis or Independent Component Analysis (Cardoso 1998) where the separation is performed via statistical properties.

Acknowledgements. We are grateful to the referee for helpful comments on an earlier version.

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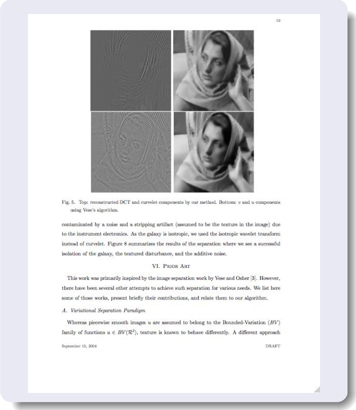
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Fig. 5. Top: reconstructed DCT and curvelet components by our method. Bottom: v and u components using Vese's algorithm.

contaminated by a noise and a stripping artifact (assumed to be the texture in the image) due to the instrument electronics. As the galaxy is isotropic, we used the isotropic wavelet transform instead of curvelet. Figure 8 summarizes the results of the separation where we see a successful isolation of the galaxy, the textured disturbance, and the additive noise.

VI. Prior Art

This work was primarily inspired by the image separation work by Vese and Osher [3]. However, there have been several other attempts to achieve such separation for various needs. We list here some of those works, present briefly their contributions, and relate them to our algorithm.

A. Variational Separation Paradigm

Whereas piecewise smooth images u are assumed to belong to the Bounded-Variation (BV) family of functions $u \in BV(\mathcal{R}^2)$, texture is known to behave differently. A different approach

September 15, 2004 DRAFT

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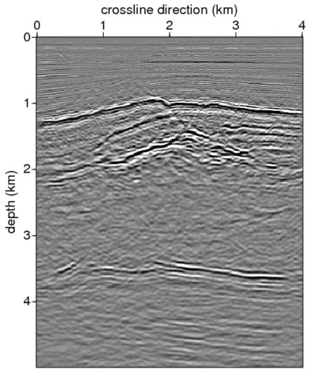
- **Donoho, Elad, Querre, Starck:** Morphological Component Analysis.
- **Douma, Herrmann, de Hoop. . . :** Seismic Imaging.



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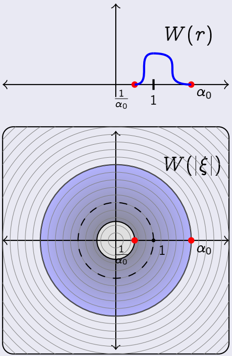
Curvelets: Construction in the frequency domain

Amplitude Window

$W \in C_0^\infty(0,\infty)$ nonnegative with support $[\frac{1}{\alpha_0}, \alpha_0]$ for some $\alpha_0 > 1$ (usually, $\alpha_0 = 2$), and $\int_0^\infty W(t)^2 \, \frac{dt}{t} = 1$.



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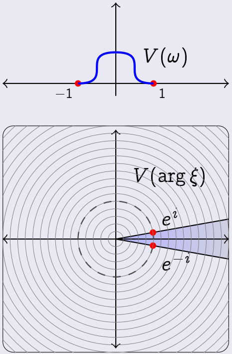
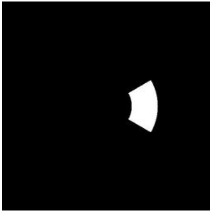
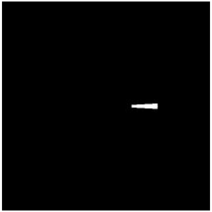
Phase Window

$V \in C_0^\infty(\mathbb{R})$ nonnegative with support in $[-1, 1]$ and $\| V \|_2 = 1$.



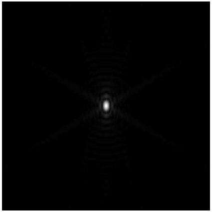
Dilations, Rotations, Shifts

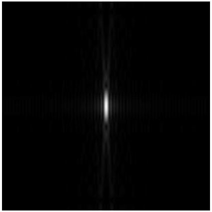




$W(|\xi|)\,V\left(\frac{6}{\pi}\arg\xi\right)$

$W(|\xi|)\,V\left(\frac{48}{\pi}\arg\xi\right)$





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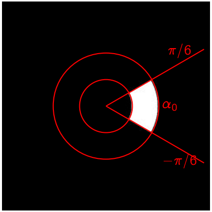

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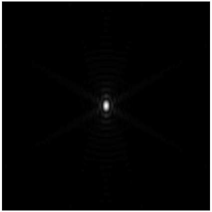

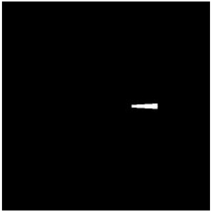

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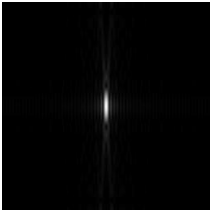

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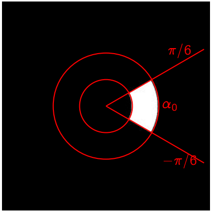

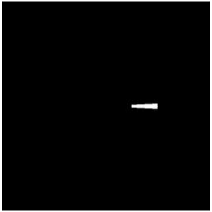

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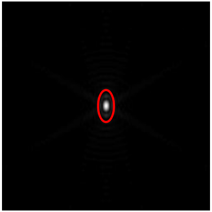

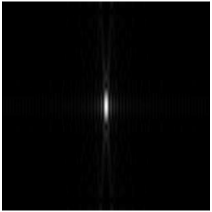

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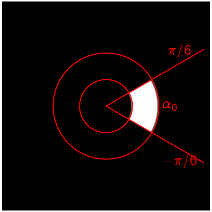

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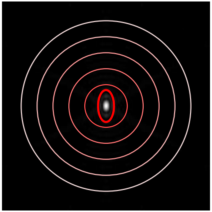

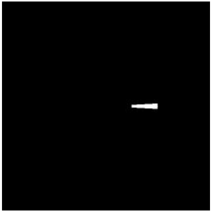

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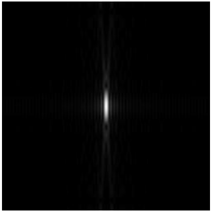

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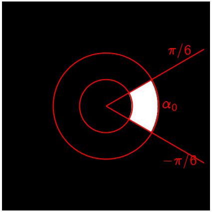

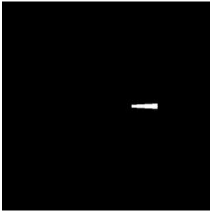

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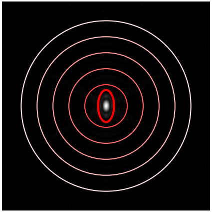

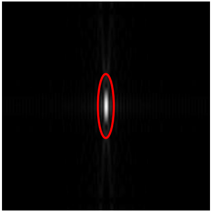

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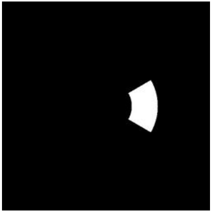
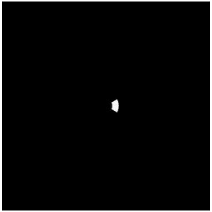
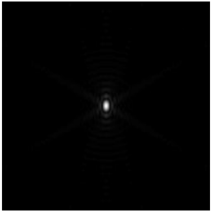
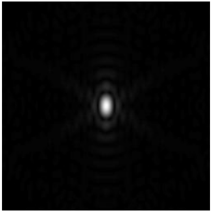




$$W(|\xi|)\,V\left(\tfrac{6}{\pi}\arg\xi\right) \qquad W(3|\xi|)\,V\left(\tfrac{6}{\pi}\arg\xi\right)$$





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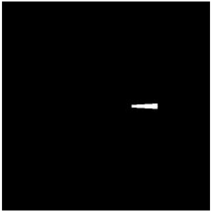

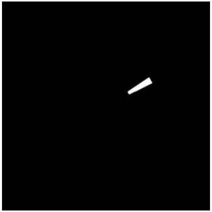

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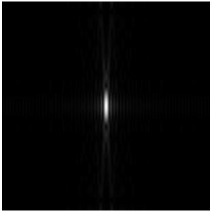

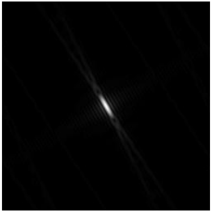

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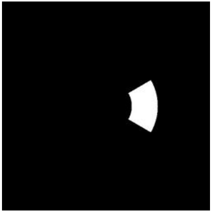
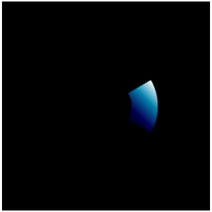
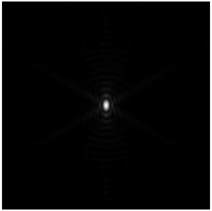
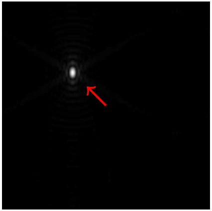




$$W(|\xi|)\,V\left(\tfrac{6}{\pi}\arg\xi\right) \qquad W(|\xi|)\,V\left(\tfrac{6}{\pi}\arg\xi\right) e^{2\pi i[(-1,1)\cdot\xi]}$$





Putting it all together

Definition (Curvelets)

$\mathbf{\Phi}_{\alpha\beta\theta} \colon \mathbb{R}^2 \to \mathbb{C}$ with parameters $\alpha \in (0, \infty)$ (shape AND scaling), $\beta \in \mathbb{R}^2$ (location), and $\theta \in \mathbb{S}^1$ (direction).

$$\mathcal{F}(\mathbf{\Phi}_{\alpha\beta\theta})(\xi) = W_\alpha(|\xi|)\, V_{\varphi(\alpha)}(\arg_\theta \xi - \arg\theta)\, e^{2\pi i(\beta\cdot\xi)}$$

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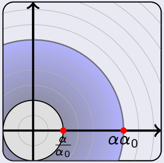

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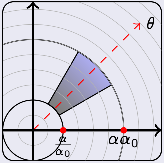

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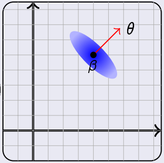
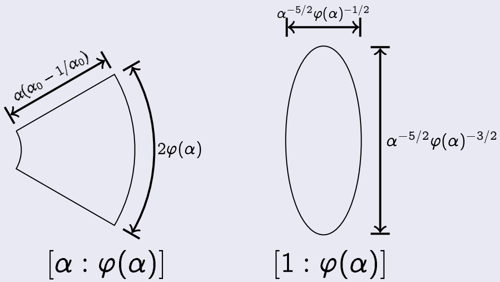
- $V_{\varphi(\alpha)}(\arg_\theta \xi - \arg\theta) = \frac{1}{\varphi(\alpha)^{1/2}} V\big(\frac{\arg_\theta \xi - \arg\theta}{\varphi(\alpha)}\big)$
- $\Phi_{\alpha\beta\theta}(x) = \Phi_{\alpha 0\theta}(x - \beta)$



A word about the *aspect-ratio* weight function φ

The width and length of a curvelet obey the anisotropy scaling relation $\text{width}_\alpha/\text{length}_\alpha \asymp \varphi(\alpha)$.



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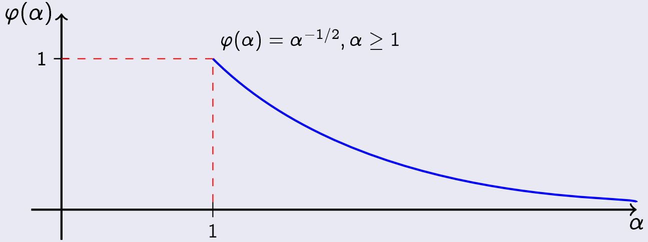

- Candès–Donoho, 1999—2002: width $\approx$ length^2.

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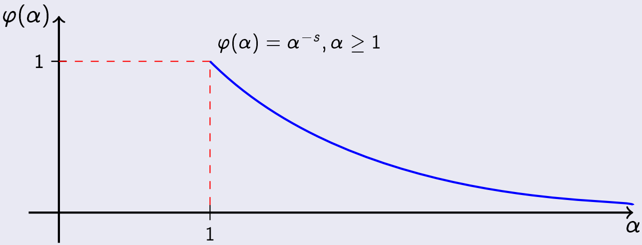

- Candès–Donoho, 1999—2002: width $\approx$ length^2.
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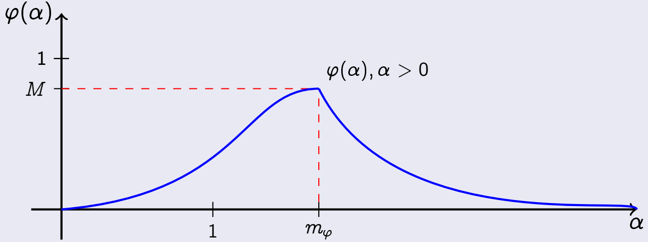

- Candès–Donoho, 1999—2002: width $\approx$ length^2.
- Candès–Donoho, 2003: width $\approx$ length$^{1/s}$, any $0 < s < 1$.
- width$_\alpha$/length$_\alpha \asymp \varphi(\alpha)$, where $\varphi\colon (0,\infty) \to (0, \frac{\pi}{4})$ satisfies:
  - Non-decreasing in $(0, m_\varphi)$ and non-increasing in (m_φ, ∞).
  - $\varphi(m_\varphi) = M < \frac{\pi}{4}$, $\lim_{\alpha\to 0}\varphi(\alpha) = 0$ and $\lim_{\alpha\to\infty}\varphi(\alpha) = 0$.
  - **Neither** $\varphi(\cdot)|_{(m_\varphi,\infty)}$ **nor** $\varphi(1/\cdot)|_{(0,m_\varphi)}$ decrease rapidly.

Gathering Information

Definition (Curvelet Coefficient)

For each choice of parameters $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$, the inner product

$$\langle f, \boldsymbol{\Phi}_{\alpha\beta\theta} \rangle = \int f(x) \overline{\boldsymbol{\Phi}_{\alpha\beta\theta}(x)}\, dx$$

offers local information of a function $f \in L_2(\mathbb{R}^2)$ at the location β, in the direction θ, and frequency $(\alpha/\alpha_0, \alpha_0\alpha)$.

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For each choice of parameters $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}^2$ and $\theta \in \mathbb{S}^1$,

$$\langle \nu, \mathbf{\Phi}_{\alpha\beta\theta} \rangle$$

offers local information of a tempered distribution $\nu \in \mathcal{S}'(\mathbb{R}^2)$ at the location β, in the direction θ, and frequency $(\alpha/\alpha_0, \alpha_0\alpha)$.

Resolution of the identity for CCT in $L_2(\mathbb{R}^2)$

Calderón formula for CCT

For any function $f \in L_2(\mathbb{R}^2)$,

$$f(x) = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \langle f, \boldsymbol{\Phi}_{\alpha\beta\theta} \rangle \boldsymbol{\Phi}_{\alpha\beta\theta}(x) \, d\beta \, d\sigma(\theta) d\alpha$$

Parseval's Formula for CCT in $L_2(\mathbb{R}^2)$

Inner product identity

$$\langle f, g \rangle = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} \langle f, \mathbf{\Phi}_{\alpha\beta\theta} \rangle \overline{\langle g, \mathbf{\Phi}_{\alpha\beta\theta} \rangle} \, d\beta \, d\sigma(\theta) \, d\alpha$$

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In particular,

Parseval's Formula for CCT

$$\|f\|^2_{L_2(\mathbb{R}^2)} = \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{R}^2} |\langle f, \mathbf{\Phi}_{\alpha\beta\theta} \rangle|^2 \, d\beta \, d\sigma(\theta) \, d\alpha$$

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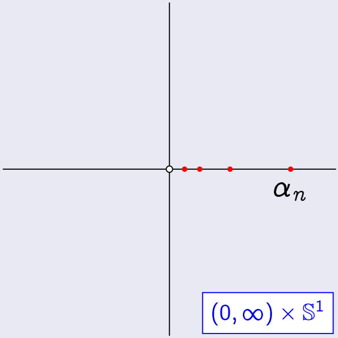
Discretization of the Curvelet Transform I



Discretization

- $(0,\infty)$: For each $n \in \mathbb{Z}$, $\alpha_n = \alpha_0^n$.

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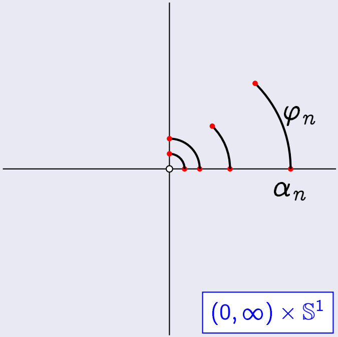

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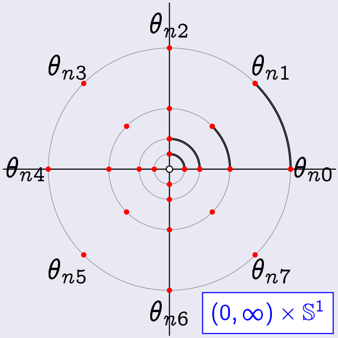

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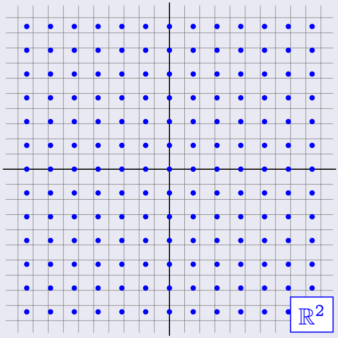

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- $\mathbb{R}^2$: Chosen n, for each $z \in \mathbb{Z}^2$, $\beta_{nz} = \frac{\pi}{\alpha_{n+1}} z$.

Discretization of the Curvelet Transform II

Amplitude and Phase Windows

- $W \in C_0^\infty(0,\infty)$ nonnegative with supp $W = [\frac{1}{\alpha_0}, \alpha_0]$, and $W(u)^2 + W(\alpha_0 u)^2 = 1$ for $\frac{1}{\alpha_0} \leq u \leq 1$.

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Discretization of the Curvelet Transform to obtain tight frames in $L_2(\mathbb{R}^2)$

Theorem

$$\{\phi_{nkz} : n \in \mathbb{Z}; k = 1, \dots, 2\pi/\varphi_n; z \in \mathbb{Z}^2\}$$

is a tight frame in $L_2(\mathbb{R}^2)$ *with frame bound* 1.

$$\|f\|^2_{L_2(\mathbb{R}^2)} = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{2\pi/\varphi_n} \sum_{z\in\mathbb{Z}^2} |\langle f, \phi_{nkz}\rangle|^2$$

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$$f = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{2\pi/\varphi_n} \sum_{z\in\mathbb{Z}^2} \langle f, \phi_{nkz}\rangle\, \phi_{nkz}.$$

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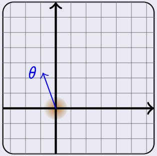
Watch your step!

$$\langle \delta, g \rangle \;=\; g(0)$$

- $\langle \delta, \Phi_{\alpha 0 \theta} \rangle = \Theta\left(\frac{1}{\varphi(\alpha)}\right)$ for all $\theta \in \mathbb{S}^1$ as $\alpha \to \infty$.



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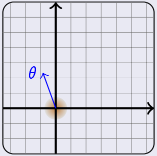

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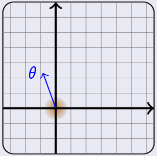

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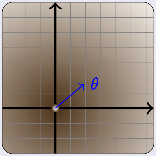

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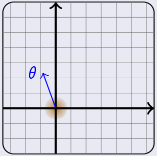

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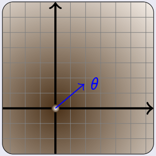

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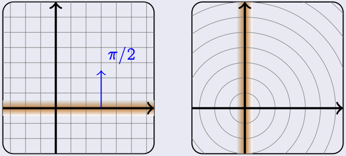

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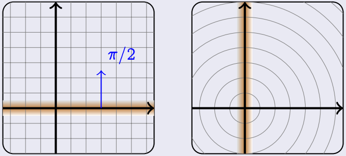

$$H(x,y) \;=\; \mathbf{1}_{\{y\geq 0\}}$$

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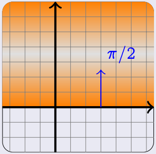

Microlocal Analysis

Theorem (Candès, Donoho)

The $\alpha \to \infty$ asymptotics of the Continuous Curvelet Transform precisely resolve the wavefront set of tempered distributions.

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Theorem (Candès, Donoho)

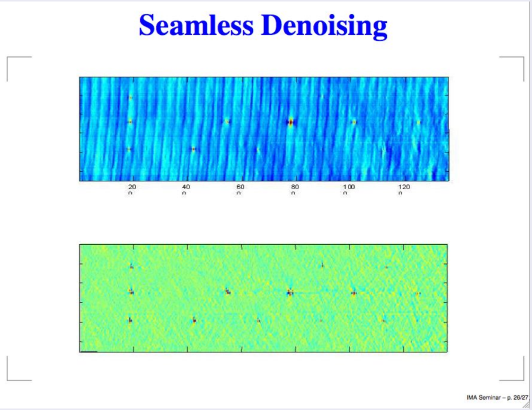
The $\alpha \to \infty$ asymptotics of the Continuous Curvelet Transform precisely resolve the wavefront set of tempered distributions.

Given a tempered distribution $\nu \in \mathcal{S}'(\mathbb{R}^2)$, let

$$\mathcal{R} = \{(\beta_0, \theta_0) \in \mathbb{R}^2 \times \mathbb{S}^1 : \langle \nu, \mathbf{\Phi}_{\alpha\beta\theta} \rangle \text{ decays rapidly near } (\beta_0, \theta_0) \text{ as } \alpha \to \infty\}$$

Then $WF(\nu)$ is the complement of $\mathcal{R}$.

Who cares?



Peter Massopust. **Mathematical Problems Associated with a Class of Non-destructive Evaluations.**

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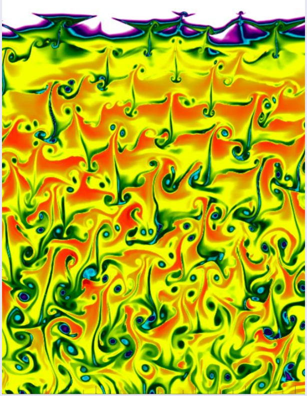
Nonlinear Approximation to "cartoon" functions



A. C. Calder *et al.* **High-Performance Reactive Fluid Flow Simulations Using Adaptive Mesh Refinement on Thousands of Processors.**

Silicon abundances (ashes)

- Approximation by selecting the N largest terms in the Fourier series:

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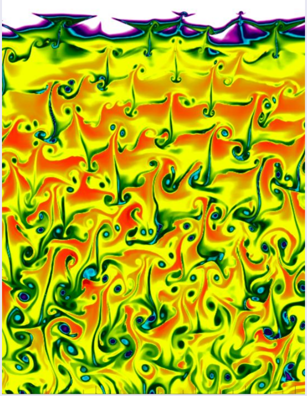

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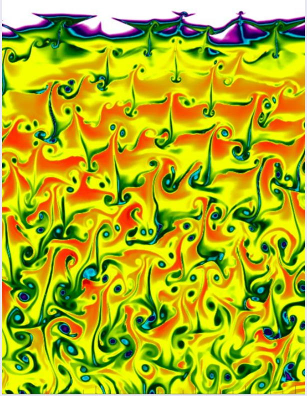

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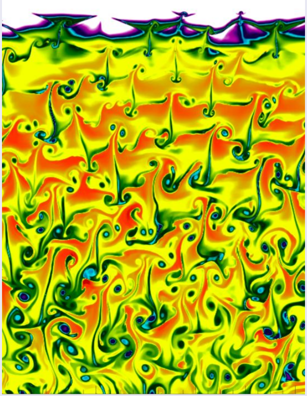

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$$\|f - f_N^{\mathcal{C}}\|_{L_2(\mathbb{R}^2)} = \Theta(N^{-1}(\log N)^{3/2})$$

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How smooth is this function?

Besov Spaces

Definition

Given $f\colon \mathbb{R}^d \to \mathbb{R}$, for $h \in \mathbb{R}^d$, set for any $n \in \mathbb{N}$,

$$\Delta_h^n f(x) = \Delta_h^{n-1}\Delta_h f(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{\eta}{k} f(x+kh).$$

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For $\eta > 0$, set $\omega_\eta(f,t)_r = \sup_{|h|<t} \big\| \Delta_h^{\lceil \eta \rceil} f \big\|_{L_r(\mathbb{R}^d)}$.

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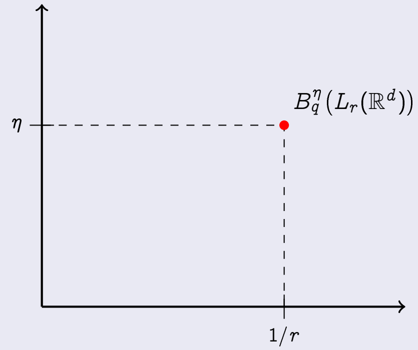
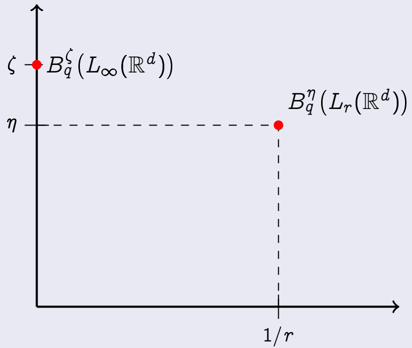
For $\eta > 0$, set $\omega_\eta(f,t)_r = \sup_{|h|<t} \big\| \Delta_h^{\lceil \eta \rceil} f \big\|_{L_r(\mathbb{R}^d)}$.
$f \in B_q^\eta(L_r(\mathbb{R}^d))$ if

$$\|f\|_{L_r(\mathbb{R}^d)} + \left\{ \int_0^\infty \big(t^{-\eta}\omega_\eta(f,t)_r\big)^q \frac{dt}{t} \right\}^{1/q} < \infty$$

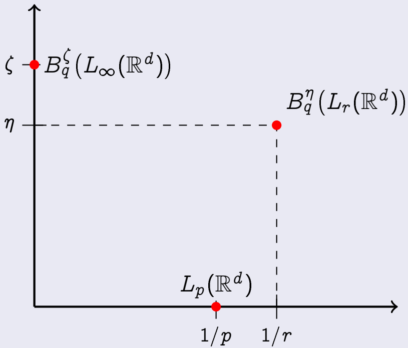
The (η, r) plane



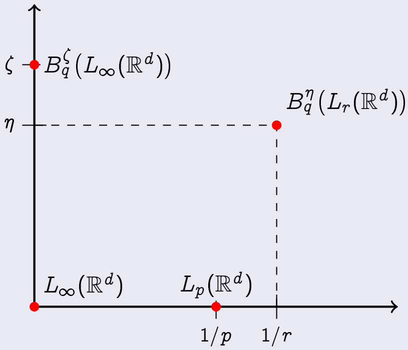
The (η, r) plane



The (η, r) plane



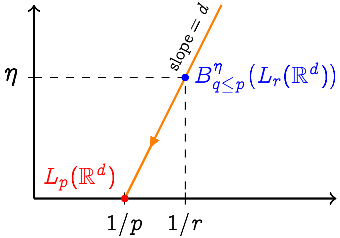
The (η, r) plane



Embedding Theorems

Theorem (DeVore, Popov)

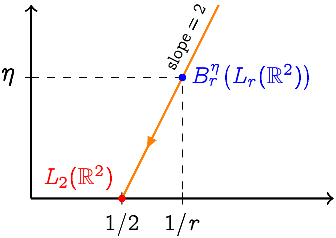
If $\eta, r, p > 0$ are related by $\frac{1}{r} = \frac{\eta}{d} + \frac{1}{p}$, then $B_p^\eta(L_r(\mathbb{R}^d))$ is continuously embedded in $L_p(\mathbb{R}^d)$.



Embedding Theorems

Corollary (DeVore, Popov)

If $\eta, r > 0$ are related by $\frac{1}{r} = \frac{\eta}{2} + \frac{1}{2}$, then $B_r^\eta(L_r(\mathbb{R}^2))$ is continuously embedded in $L_2(\mathbb{R}^2)$.



Approximation Theorems

Theorem

$$f \in B_r^{\eta}(L_r(\mathbb{R}^2)) \text{ if and only if } \|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta(N^{-\eta/2}).$$

Approximation by selecting the N largest terms in the Wavelet decomposition

Approximation Theorems

Theorem

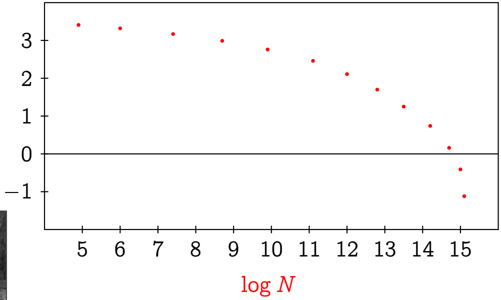
$$f \in B_r^{\eta}(L_r(\mathbb{R}^2)) \text{ if and only if } \|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta(N^{-\eta/2}).$$

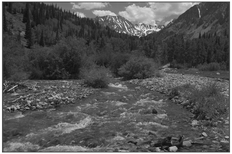
Approximation by selecting the N largest terms in the Wavelet decomposition

or equivalently, $\log\|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta\big(-\frac{\eta}{2}\log N\big).$

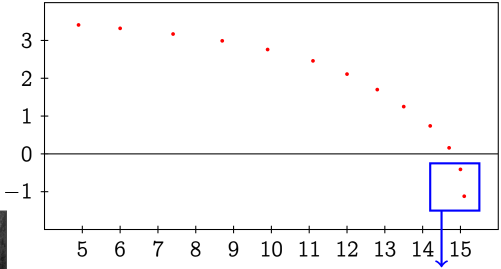
Computation of Smoothness via Nonlinear Approximation with Wavelets

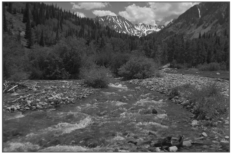
$\log \|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)}$





Computation of Smoothness via Nonlinear Approximation with Wavelets

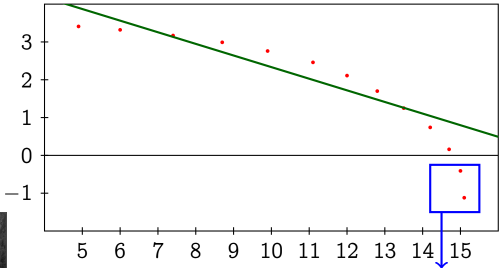




If the number of nonzero coefficients is close to the number of all possible coefficients, the error of non-linear approximation will resemble the error of linear approximation.

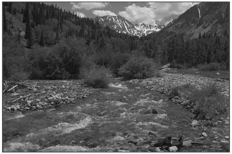
Computation of Smoothness via Nonlinear Approximation with Wavelets

$\log \|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)}$



$\log N$

If the number of nonzero coefficients is close to the number of all possible coefficients, the error of non-linear approximation will resemble the error of linear approximation.

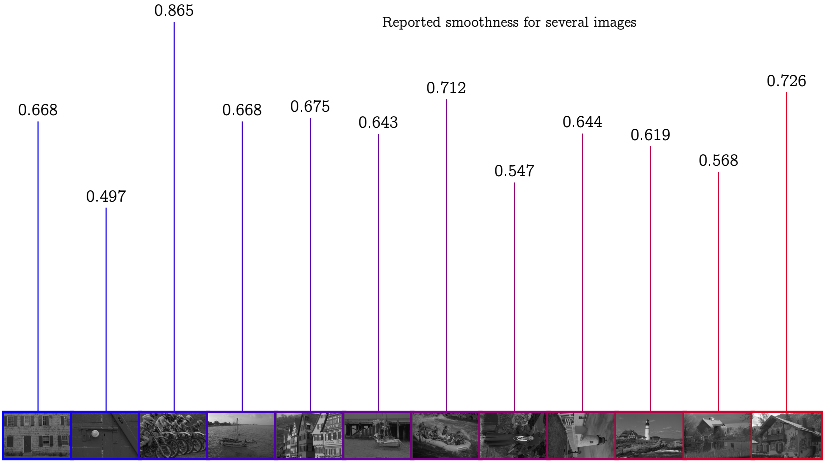


Computation of Smoothness via Nonlinear Approximation with Wavelets

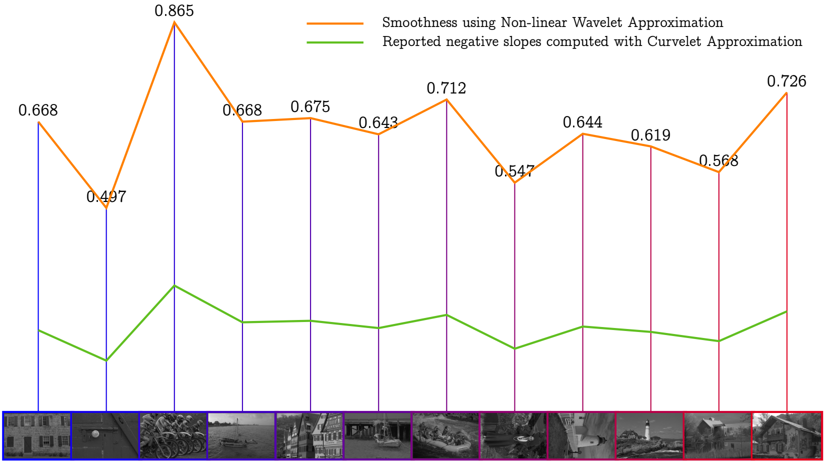
slope ≈ -0.3072

$\eta \approx 0.6144$

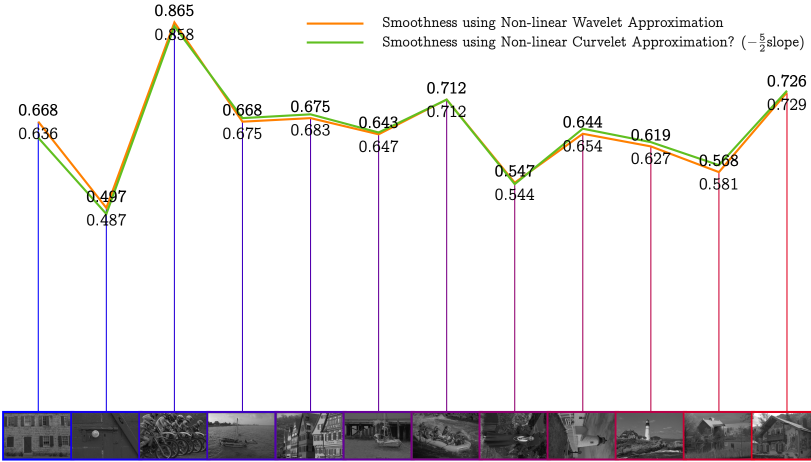
Experiments



Experiments



Experiments



Which one is the one?

slope ≈ -0.3072

$\eta \approx 0.6144$

slope ≈ -0.2031

$\eta \approx 0.5077$