ON THE SUBSYMMETRIC SEQUENCES IN S

G. ANDROULAKIS AND TH. SCHLUMPRECHT

Abstract We establish a sufficient condition for a block sequence in the Schlumprecht space S to have a subsymmetric subsequence, and prove the existence of an uncountable family of subsymmetric and mutually non-isomorphic block sequences in S.

1. Introduction

In [S1] the second named author constructed the first known example of an arbitrarily distortable Banach space. In the literature this space is denoted by S. In [S2] (see [AS1]) the space S was proved to be complementably minimal. This means that the space S embeds complementably in every infinite dimensional subspace of itself. We do not know whether S is a prime space. In [AS1] it is shown that every complemented block sequence in S has a subsequence which spans a space isomorphic to S. These results suggest that in the space S there are "no many isomorphically different structures". In the present paper we show that this is not correct. In fact our main theorem is the following surprising result:

Theorem 1.1. *There exist uncountably many non-isomorphic semi-normalized subsymmetric block sequences in S.*

The sequences that substantiate Theorem 1.1 are "*stabilizing sequences*" (see section 2 for the definition). Stabilizing sequences were used in [AS2] to construct strictly singular non-compact operators on the space S as well as on the Gowers-Maurey space [GM1]. Recall that a sequence (x_n) in S is called *subsymmetric* if there exists a constant $C \geq 1$ such that for any $(\lambda_n) \in c_{00}$ (the linear space of finitely supported real sequences) and for any increasing

Date: 11/26/2000.

1991 *Mathematics Subject Classification.* 46B03, 46B20.

Research supported by NSF, and the Pacific Institute of Mathematical Sciences.

sequence of positive integers (k_n), we have that

$$\|\sum \lambda_n x_n\| \overset{C}{\approx} \|\sum \lambda_n x_{k_n}\|$$

where $\|\cdot\|$ will always denote the norm of S and for two non-negative numbers a, b and for some $C \geq 1$ we write $a \overset{C}{\approx} b$ if $\frac{1}{C} \cdot b \leq a \leq C \cdot b$). Two sequences (x_n) and (y_n) will be called *non-isomorphic* if

$$0 = \inf_{(\lambda_n) \in c_{00}} \frac{\|\sum \lambda_n x_n\|}{\|\sum \lambda_n y_n\|} \text{ and } \sup_{(\lambda_n) \in c_{00}} \frac{\|\sum \lambda_n x_n\|}{\|\sum \lambda_n y_n\|} = \infty. \tag{1}$$

Note that use "and" rather than "or" in (1) which strengthens the statement of Theorem 1.1. We ask whether the spaces $[(x_n)]$ and $[(y_n)]$, which are generated by (x_n) and (y_n) respectively, are non-isomorphic. We also ask whether the spaces generated by the stabilizing sequences that substantiate Theorem 1.1 are complemented. The answers to these questions may reveal that S is not a prime space. In fact if the answers to the above questions are positive, then as it was observed by P.G. Casazza, [C], S will be a negative solution to the Schroeder-Bernstein problem for Banach spaces: if two spaces are isomorphic to a complemented subspace of each other, must they be isomorphic? The only known negative solutions to the Schroeder-Bernstein problem for Banach spaces are given by W.T. Gowers [G] and by W.T. Gowers and B. Maurey [GM2].

We recall the definition of the space S: Let f be the function $f(x) = \log_2(x+1)$. For $I, J \subseteq \mathbb{N}$ we write $I < J$ or $I \leq J$ if $\max I < \min J$ or $\max I \leq \min J$ respectively. Then the norm of S satisfies the following implicit equation:

$$\|x\| = \|x\|_\infty \vee \sup_{\substack{2 \leq \ell \in \mathbb{N} \\ E_1 < E_2 < \cdots < E_\ell}} \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \|E_j x\|$$

where E_j's are intervals of integers and $E_j x$ denotes the projection of x on E_j. Also, for $2 \leq \ell \in \mathbb{N}$ we define the equivalent norms $\|\cdot\|_\ell$ and $|||\cdot|||_\ell$ on S by

$$\|x\|_\ell = \sup_{E_1 < E_2 < \cdots < E_\ell} \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \|E_j x\|$$

and

$$|||x|||_\ell = \sup_{\ell \leq r \leq \infty} ||x||_r.$$

We denote by $(e_i)_i$ the unit vector basis of c_{00}, and by $(e_i^*)_i$ its dual basis. In order to prove Theorem 1.1 we first give a sufficient condition for a block sequence (x_i) of $(e_i)_i$ in S to have a subsymmetric subsequence. Recall that for $x = \sum x_i e_i \in c_{00}$ the *support* of x is defined to be $\text{supp}\,(x) = \{i \in \mathbb{N} : x_i \neq 0\}$. For $x, y \in c_{00}$ we write $x < y$ if $\text{supp}\,(x) < \text{supp}\,(y)$. A sequence $(x_n) \in c_{00}$ is a block sequence of (e_i) if $x_1 < x_2 < \cdots$. The main result of section 3 is the following:

Theorem 1.2. *Let (x_i) be a seminormalized block sequence in S such that $\lim_j \|x_i\|_j = 0$ uniformly in i. Then (x_i) has a subsymmetric subsequence.*

2. Preliminaries-Definitions and Notations about trees

We introduce some terminology about trees. If $\mathcal{T}$ is a partially ordered set then its elements will be called *nodes*. If $\prec$ denotes the order on $\mathcal{T}$ and $s, t \in \mathcal{T}$ with $s \prec t$ then we say that s is a *predecessor* of t and t is a *successor* of s. If $s \prec t$ we say that s is an *immediate predecessor* of t, or t is an *immediate successor* of s, if there is no u with $s \prec u \prec t$. An element is called *maximal* if it does not have successors. For $t, s \in \mathcal{T}$ we write $t \preceq s$ if $t \prec s$ or $t = s$. An element $t \in \mathcal{T}$ is called a *root* if $t \preceq s$ for all $s \in \mathcal{T}$. Usually we denote the root of a tree by $\emptyset$. A *tree* $\mathcal{T}$ is defined to be a partially ordered set such that

- $\mathcal{T}$ has a unique root.
- Every node other than the root has finitely many predecessors.
- Every non-maximal node has finitely many immediate successors.

If t is a node of a tree $\mathcal{T}$ then we define the *length* of t, which is denoted by $|t|$, to be the number of predecessors of t (if $\emptyset$ is the root of $\mathcal{T}$ then $|\emptyset| = 0$). If t is a non-maximal node of a tree $\mathcal{T}$ then k_t will denote the number of immediate successors of t. If $(\mathcal{T}, \prec)$ is a tree and $\tilde{\mathcal{S}} \subseteq \mathcal{T}$ then a subset $\mathcal{T}$ is called a *subtree of $\mathcal{T}$ generated by $\mathcal{T}\backslash\tilde{\mathcal{S}}$* and it is denoted by $\text{subtree}\,(\mathcal{T}\backslash\tilde{\mathcal{S}})$, if its order is induced by the order on $\mathcal{T}$, and it is obtained from $\mathcal{T}$ by eliminating all the nodes of $\mathcal{T}$ that either belong in $\tilde{\mathcal{S}}$ or they are successors of some node of $\tilde{\mathcal{S}}$, i.e.

$$\text{subtree}\,(\mathcal{T}\backslash\tilde{\mathcal{S}}) = \mathcal{T}\backslash\{s \in \mathcal{T} :\ \text{there exists } t \in \tilde{\mathcal{S}} \text{ with } t \preceq s\}.$$

Let T be a finite tree, and let $\emptyset$ denote its root. Let $(x_t)_{t\in\mathcal{T}} \subset S$, $x \in S$, $(x_t^*)_{t\in T} \subset S^*$ and $x^* \in S^*$. We say that $(x_t)_{t\in T}$ and x (respectively $(x_t^*)_{t\in\mathcal{T}}$ and x^*) are *roughly associated* to $\mathcal{T}$ if

(a) If t is a maximal node of $\mathcal{T}$ then $\|x_t\| \geq 1$ (respectively $\|x_t^*\| \leq 1$).

(b) If $t \in T$ is a non-maximal node of $\mathcal{T}$ then the vectors in $\{x_s : s$ is an immediate successor of $t\}$ (respectively $\{x_s^* : s$ is an immediate successor of $t\}$) form a block basis of (e_i) (respectively (e_i^*)). Furthermore if s_1, s_2 are non-maximal immediate successors of t with $k_{s_1} < k_{s_2}$ then $x_{s_1} < x_{s_2}$.

(c) If $t \in T$ is a non-maximal node of $\mathcal{T}$ then

$$x_t = \frac{f(k_t)}{k_t}\sum\{x_s : s \text{ is an immediate successor of } t\}$$
$$(\text{respectively } x_t^* = \frac{1}{f(k_t)}\sum\{x_s^* : s \text{ is an immediate successor of } t\}.$$

(d) $x = x_\emptyset$ (respectively $x^* = x_\emptyset^*$).

We say that $(x_t)_{t\in\mathcal{T}} \subset S$ and $x \in S$ (respectively $(x_t^*)_{t\in\mathcal{T}} \subset S^*$ and $x^* \in S^*$) are *associated* to $\mathcal{T}$ if (a'), (b), (c) and (d) are valid, where

(a') If t is a maximal node of $\mathcal{T}$ then x_t (respectively x_t^*) is an element of the unit vector basis.

We say that an element $x \in S$ is associated (respectively roughly associated) to a tree $\mathcal{T}$ if there exists a family $(x_t)_{t\in\mathcal{T}} \subset S$ such that $(x_t)_{t\in\mathcal{T}}$ and x are associated (respectively roughly associated) to $\mathcal{T}$. Similarly we define when an element $x^* \in S^*$ is associated or roughly associated to a tree $\mathcal{T}$.

Let $\mathcal{T}$ be an infinite tree with no maximal nodes, and for $i \in \mathbb{N}$ let $\mathcal{T}_i = \{t \in \mathcal{T} : |t| \leq i\}$. A block sequence $(x_i)_i$ is called *stabilizing according to* $\mathcal{T}$ if (x_i) is associated to $\mathcal{T}_i$ for all i.

Let $\mathcal{T}$ be a tree and and let vectors x_t of S for any maximal node t of $\mathcal{T}$ such that $\|x_t\| \leq 1$ for all such t, and for every subset T of the set of maximal nodes of $\mathcal{T}$ such that the elements of T are siblings we have that the vectors in $\{x_t : t \in T\}$ form a block basis of (e_i). Note then that this determines in a unique way a family of vectors $(x_t)_{t\in\mathcal{T}}$ which are associated to $\mathcal{T}$. This is true in S^* as well. Now let $\mathcal{T}$ be a tree, $\emptyset$ denote its root, and let

$t \in \mathcal{T}\backslash\emptyset$. Let $(x_s)_{s\in\mathcal{T}} \subset S$ and $x, y \in S$. Assume that $(x_s)_{s\in\mathcal{T}}$ and x are associated to $\mathcal{T}$. Let $\{x_s : s' = t'\} = \{z_i : i = 1, \ldots, n\}$ for some integer n, with $z_1 < z_2 < \cdots z_n$, and let $i_0 \in \{1, \ldots, n\}$ with $z_{i_0} = x_t$. Assume that $y < z_2$ if $i_0 = 1$; $z_{i_0-1} < y < z_{i_0+1}$ if $1 < i_0 < n$; $z_{n-1} < y$ if $i_0 = n$. Let $\mathcal{S} = \mathcal{T}\backslash\{s \in \mathcal{T} : t \prec s\}$. Define $(\tilde{x}_s)_{s\in\mathcal{S}}$ as follows: $\tilde{x}_t = y$ and $\tilde{x}_s = x_s$ if s is a maximal node of $\mathcal{S}$ different than t. Then we say that $\tilde{x}_\emptyset$ is obtained from x if we *replace* x_t *by* y. If $(x_s^*)_{s\in\mathcal{T}} \subset S^*$ and $x^*, y^* \in S^*$ with $(x_s^*)_{s\in\mathcal{T}}$ and x^* are associated to $\mathcal{T}$, then we define when we obtain $\tilde{x}_\emptyset^*$ from x^* by replacing x_t^* by y^* in a similar way.

Note that for a tree $\mathcal{T}$, if $(x_t)_{t\in\mathcal{T}} \subset S$, $x \in S$ as well as $(x_t^*)_{t\in\mathcal{T}} \subset S^*$, $x^* \in S^*$ are associated to $\mathcal{T}$, and $\operatorname{supp}(x_t) = \operatorname{supp}(x_t^*)$ for all $t \in \mathcal{T}$ then $x_t^*(x_t) = 1$ for all $t \in \mathcal{T}$. We have that $\|x_t^*\| \leq 1$ and $\|x_t\| \geq 1$ for all $t \in \mathcal{T}$.

Let $\mathcal{T}$ be a tree and let $\emptyset$ denote its root. Then for $t \in \mathcal{T}$ we define $\alpha_t, \beta_t \in \mathbb{R}$ as follows

$$\alpha_t = \prod_{\emptyset\preceq s\preceq t} \frac{f(s)}{k_s} \quad \text{and} \quad \beta_t = \prod_{\emptyset\preceq s\preceq t} \frac{1}{f(s)}.$$

Let $\mathcal{T}$ be a tree and $\mathcal{S} \subseteq \mathcal{T}$ that contains exactly one node from any maximal branch of $\mathcal{T}$. Observe that if $(x_t)_{t\in\mathcal{T}} \subset S$ and $x \in S$ (respectively $(x_t^*)_{t\in\mathcal{T}} \subset S^*$ and $x^* \in S^*$) are associated to $\mathcal{T}$ then

$$x = \sum_{t\in\mathcal{S}} \alpha_{t'} x_t, \quad \text{and} \quad x^* = \sum_{t\in\mathcal{S}} \beta_{t'} x_t^*.$$

The immediate successors of each non-maximal node of a tree $\mathcal{T}$ are called *siblings*. A *fast increasing* tree is a tree with the property that for every non-maximal node $t \in \mathcal{T}$ we have $k_t \geq n^{f(n)}$ where n is the number of siblings of t. We define the set

$$\begin{aligned}\mathcal{K} = \{ \quad x^* \quad &\in S^* \text{ there exists a fast increasing tree } \mathcal{T} \text{ such that} \\ x^* \quad &\text{is associated to } \mathcal{T}\}.\end{aligned}$$

Recall from [S2] (see [AS1]) the following

Lemma 2.1. *There exists a constant $d > 1$ such that for all $r \in R_+$ with $f(r) > d^2$ we have*

$$|||x|||_r \leq \left(\frac{1}{1 - \frac{d}{\sqrt{f(r)}}}\right) \sup_{\ell\geq r, E_1<E_2<\cdots<E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |||E_i x|||_{r^{f(r)}}$$

if $x \in c_{00}$ with $\|x\| \neq \|x\|_\infty$

By the properties of the function f, the product

$$c = \prod_{k=0}^{\infty} \left(\frac{1}{1 - \frac{d}{\sqrt{f(r_k)}}} \right)$$

is finite, where $f(r_0) > d^2$ and $r_{k+1} = r_k^{f(r_k)}$. Therefore $\sup\{|x^*(x)| : x^* \in \mathcal{K}\} \overset{c}{\approx} \|x\|$ for all $x \in S$.

3. A Sufficient Condition for a Block Sequence in S to have a Subsymmetric Subsequence

The main result of this section is Theorem 1.2. We first give the following stabilization result.

Proposition 3.1. *Let (x_i) be a seminormalized block sequence in S such that*

$$\lim_{j\to\infty} \|x_i\|_j = 0$$

uniformly in i. Then for every $\varepsilon > 0$ and $K \in \mathbb{N}$ there exists a finite family $\mathcal{T}$ of fast increasing trees with the following property: For every $i \in \mathbb{N}$, fast increasing tree T, and $x^* \in \mathcal{K}$ which is associated to T, there exists $\Delta \in \mathcal{T}$ and there exists $y^* \in \mathcal{K}$ which is associated to Δ such that $\{t \in \Delta : |t| \le K\} \subseteq \{t \in \mathcal{T} : |t| \le K\}$ and $x^*(x_i) - \varepsilon \le y^*(x_i)$.

Proof. Let $\varepsilon > 0$ and $K \in \mathbb{N}$. Choose $\varepsilon_1 > 0$ and $L_1 \in \mathbb{N}$ such that

$$\varepsilon_1 = \frac{\varepsilon}{2} \text{ and } \|x_i\|_j < \varepsilon_1 \text{ for all integers } i, j \text{ with } j > L_1. \tag{2}$$

Choose $\varepsilon_2 > 0$ and $L_2 \in \mathbb{N}$ such that

$$L_1\varepsilon_2 < \frac{\varepsilon}{2^2} \text{ and } \|x_i\|_j < \varepsilon_2 \text{ for all integers } i, j \text{ with } j > L_2. \tag{3}$$

Choose $\varepsilon_3 > 0$ and $L_3 \in \mathbb{N}$ such that

$$L_1L_2\varepsilon_3 < \frac{\varepsilon}{2^3} \text{ and } \|x_i\|_j < \varepsilon_3 \text{ for all integers } i, j \text{ with } j > L_3. \tag{4}$$

Continue in the same manner and finally choose $\varepsilon_K > 0$ and $L_K \in \mathbb{N}$ such that

$$L_1L_2\cdots L_{K-1}\varepsilon_K < \frac{\varepsilon}{2^K} \text{ and } \|x_i\|_j < \varepsilon_K \text{ for all integers } i, j \text{ with } j > L_K. \tag{5}$$

Let $\mathcal{T}$ be the finite set of fast increasing trees whose maximal nodes have length at most K and such that if $\Delta \in \mathcal{T}$ and $t \in \Delta$ with $|t| < K$ then $k_t \leq L_{|t|+1}$.

Let $i \in \mathbb{N}$, T be fast increasing tree $\mathcal{T}$ with $\emptyset$ denoting its root, and $x^* \in \mathcal{K}$ be associated to T. Let $(x_t^*)_{t\in T} \subset S^*$ such that $(x_t^*)_{t\in T} \subset S^*$ and $x^* \in S^*$ are associated to T. If $k_\emptyset > L_1$ then the statement holds for $\Delta = \{\emptyset\}$. Thus assume that $k_\emptyset \leq L_1$. Let T_1 be the subtree of T which is obtained if we eliminate all the nodes $s \in T$ of length at least 2 that are successors to nodes $t \in T$ with $|t| = 1$ and $k_t > L_2$ i.e.

$$T_1 = T \backslash \{s \in T\colon \text{ there exists } t \in T \text{ with } |t| = 1, k_t > L_2 \text{ and } t \prec s\}.$$

Let x_1^* be the functional which is obtained from x^* if we replace x_t^*'s by elements of the unit vector basis of S^* for every $t \in T$ with $|t| = 1$ and $k_t > L_2$. Then

$$x^*(x_i) - \frac{\varepsilon}{2^2} < x^*(x_i) - L_1\varepsilon_2 \leq x_1^*(x_i)$$

by (3) since there are at most L_1 many nodes $t \in T$ with $|t| = 1$. Let T_2 be the subtree of T_1 which is obtained if we eliminate all the nodes $s \in T_1$ of length at least 3 that are successors to nodes $t \in T_1$ with $|t| = 2$ and $k_t > L_3$ i.e.

$$T_2 = T_1 \backslash \{s \in T_1\colon \text{ there exists } t \in T_1 \text{ with } |t| = 2, k_t > L_3 \text{ and } t \prec s\}.$$

Let x_2^* be the functional which is obtained from x_1^* if we replace x_t^*'s by elements of the unit vector basis of S^* for every $t \in T_1$ with $|t| = 2$ and $k_t > L_3$. Then

$$x_1^*(x_i) - \frac{\varepsilon}{2^3} < x_1^*(x_i) - L_1L_2\varepsilon_2 \leq x_2^*(x_i)$$

by (4) since there are at most L_1L_2 many nodes $t \in T_1$ with $|t| = 2$. Continue similarly, and define T_{K-1} to be the subtree of T_{K-2} which is obtained if we eliminate all the nodes $s \in T_{K-2}$ of length at least $k-1$ that are successors to nodes $t \in T_{k-2}$ with $|t| = K-1$ and $k_t > L_K$, i.e.

$$T_{K-1} = T_{K-2} \backslash \{s \in T_{K-2}\colon \text{ there exists } t \in T_{K-2} \text{ with } |t| = K-1, k_t > L_K \text{ and } t \prec s\}.$$

Let x^*_{K-1} be the functional which is obtained from x^*_{K-2} if we replace x^*_t's by elements of the unit vector basis of S^* for every $t \in T_{K-2}$ with $|t| = K-1$ and $k_t > L_K$. Then

$$x^*_{K-2}(x_i) - \frac{\varepsilon}{2^K} < x^*_{K-2}(x_i) - L_1 L_2 \dots L_{K-1}\varepsilon_K \leq x^*_{K-1}(x_i)$$

by (5) since there are at most $L_1 L_2 \dots L_{K-1}$ many nodes $t \in T_{K-2}$ with $|t| = K-1$. Set $\Delta = T_{K-1}$ and $y^* = x_{K-1}$. Then

$$x^*(x_i) - \varepsilon < x^*(x_i) - \sum_{i=2}^{k} \frac{\varepsilon}{2^i} < y^*(x_i),$$

and obviously $\{t \in \Delta : |t| \leq K\} \subseteq \{t \in T : |t| \leq K\}$. This finishes the proof of the proposition. $\square$

For the proof of the main result of this section we will need the following definition: Let T be a tree and K be an integer. For $x \in c_{00}$ we define

$$\begin{aligned}\|x\|_{T,K} = \sup\{|x^*(x)| : {}& \text{there exists } (x^*_t)_{t\in T} \subseteq S^* \text{ such that } (x^*_t)_{t\in T} \text{ and } x^* \text{ are roughly} \\ & \text{associated to } T, \text{ and } x^*_t \text{ is an element of the unit vector basis of } S^* \\ & \text{for all maximal nodes } t \in T \text{ with } |t| \leq K\}.\end{aligned}$$

We can now prove the main result of this section:

Theorem 3.2. *Let (x_i) be a semi-normalized block sequence in S such that*

$$\lim_{j\to\infty} \|x_i\|_j = 0$$

uniformly in i. Then there is a subsequence of (x_i) which is subsymmetric.

Proof. Define an increasing sequence of integers (K_j) such that if $P_1 = 1$ and

$$P_I = \prod_{i=1}^{I-1} \frac{f(K_i+1)}{f(K_i)} \text{ for } I > 1$$

then $P := \lim_{I\to\infty} P_I < \infty$. For every integer j, there exists a finite family $\mathcal{T}_j$ fast increasing special trees which satisfy the Proposition 3.1 for $\varepsilon = \frac{1}{2^j}$ and $K = K_j$. Now, by compactness there exists a subsequence (x^1_i) of (x_i) such that

$$\big|\|x^1_i\|_{T,K_1} - \|x^1_{i'}\|_{T,K_1}\big| < \frac{1}{2}$$

for all i, i' and for all $T \in \mathcal{T}_1$. Then choose (x_i^2) a subsequence of (x_i^1) such that

$$\big| \|x_i^2\|_{T,K_2} - \|x_{i'}^2\|_{T,K_2} \big| < \frac{1}{2^2}$$

for all i, i' and for all $T \in \mathcal{T}_2$. Continue in the same way, and finally set $y_i = x_i^i$ for all i. Then for every integer j and for every $i, i' \geq j$, we have that

$$\big| \|y_i\|_{T,K_j} - \|y_{i'}\|_{T,K_j} \big| < \frac{1}{2^j}$$

for all $T \in \mathcal{T}_j$.

We claim that (y_i) is subsymmetric. Indeed, fix a strictly increasing sequence of integers (n_i). We will show that

$$\| \sum_I \alpha_i y_i \| \overset{P}{\approx} \| \sum_i a_i y_{n_i} \| \text{ for all } (a_i) \in c_{00}.$$

Since the unit vector basis of S is 1-unconditional, assume without loss of generality that each y_i can be written as a linear combination of the unit vector basis of S using only non-negative coefficients. It is enough to prove the following two statements:

(A) For all $x^* \in \mathcal{K}$ there exists $y^* \in S^*$, $\|y^*\| \leq P$, such that $x^*(y_{n_i}) - \frac{1}{2^{i-1}} \leq y^*(y_i)$ for all i.

(B) For all $x^* \in \mathcal{K}$ there exists $y^* \in S^*$, $\|y^*\| \leq P$, such that $x^*(y_i) - \frac{1}{2^{i-1}} \leq y^*(y_{n_i})$ for all i.

The proof of these statements is similar, so we only show statement (A). Let $x^* \in \mathcal{K}$ and assume that x^* is associated to T for some fast increasing special tree T. Let $I \in \mathbb{N}$ such that $x^* \leq \max\operatorname{supp}(y_{n_I})$. Fix a sequence of intervals $(A_i)_{i\in\mathbb{N}}$ such that $A_1 < A_2 < \ldots$, their union is $\mathbb{N}$ and the non-zero coordinates of the vector y_i belong to A_i for all i. We prove by induction on $I \in \mathbb{N}$, the following

Claim There exists $y^* \in S^*$ such that

(a) $\operatorname{supp} y^* \subseteq \cup_{\ell=1}^{I} A_\ell$.

(b) $\|y^*\| \leq P_I$.

(c) There exists a fast increasing tree $\mathcal{S}$ so that $\frac{1}{P_I} y^*$ is roughly associated to $\mathcal{S}$. For each i let $x_i^* = A_{n_i} x^*$ and $y_i^* = A_i \frac{1}{P_I} y^*$, and let T_i and $\mathcal{S}_i$ be fast increasing trees, so that x_i^* is associated to T_i and y_i^* is roughly associated to $\mathcal{S}_i$. Let $(y_{i,t}^*)_{t\in\mathcal{S}_i} \subset S^*$ such that $(y^*_{i,t})_{t\in\mathcal{S}_i} \subset S^*$ and y^*_i are associated to $\mathcal{S}_i$. Then for all i, $\mathcal{S}_i \subseteq T_{n_i}$ and $y^*_{i,t}$ is an element of the unit vector basis of S^* for all nodes t that are maximal in S_i with $|t| \leq K_i$.

(d) $x^*(y_{n_i}) - \frac{1}{2^{i-1}} \leq y^*(y_i)$ for all i.

Then obviously, y^* satisfies statement (A).

If $I = 1$, then by Proposition 3.1 that there exists $\Delta \in \mathcal{T}_1$, $\Delta \subseteq T_1$ and there exists $y^* \in S^*$ which is roughly associated to Δ via $(y^*_t)_{t\in\Delta}$ such that $x^*(y_{n_1}) - \frac{1}{2} \leq y^*(y_{n_1})$ and y^*_t is an element of the unit vector basis of S^* for every maximal node $t \in \Delta$ with $|t| \leq K_1$. Since $1 \leq n_1$ we have that

$$|\|y_1\|_{\Delta,K_1} - \|y_{n_1}\|_{\Delta,K_1}| < \frac{1}{2}.$$

Thus

$$y^*(y_{n_1}) \leq \|y_{n_1}\|_{\Delta,K_1} < \|y_1\|_{\Delta,K_1} + \frac{1}{2}.$$

Hence

$$x^*(y_{n_1}) - \frac{1}{2^0} < \|y_1\|_{\Delta,K_1}.$$

Let $(z^*_t)_{t\in\Delta} \subset S^*$ and $z^* \in S^*$ such that $(z^*_t)_{t\in\Delta}$ and z^* are roughly associated to Δ, such that z^*_t is an element of the unit vector basis of S^* for all maximal nodes $t \in \Delta$ with $|t| \leq K_1$, and $z^*(y_1) = \|y_1\|_{\Delta,K_1}$. Thus z^* satisfies the Claim.

Assume that the Claim has been proved for all integers less than or equal to I, and consider $x^* \in \mathcal{K}$ such that $x^* \leq \max\operatorname{supp}(y_{n_{I+1}})$. Write $x^* = \tilde{x}^*_1 + \tilde{x}^*_2$ where $\tilde{x}^*_1 = \cup_{i=1}^I A_{n_i} x^*$ and $\tilde{x}^*_2 = A_{n_{I+1}} x^*$. Use the induction hypothesis for $\tilde{x}^*_1$ to obtain $\tilde{y}^*_1$ which satisfies the Claim for $y^* = \tilde{y}^*_1$. Then as in the case $I = 1$ obtain $\tilde{y}^*_2 \in S^*$ and $(\tilde{y}^*_{2,t})_{t\in\tilde{\mathcal{S}}} \subset S^*$ such that $\operatorname{supp}\tilde{y}^*_2 \subseteq A_{I+1}$, $\tilde{y}^*_2$ and $(\tilde{y}^*_{2,t})_{t\in\tilde{\mathcal{S}}}$ are roughly associated to $\tilde{\mathcal{S}} \subseteq T_{I+1}$, $\tilde{y}^*_{2,t}$ is an element of the unit vector basis of S^* for all maximal nodes t of $\tilde{\mathcal{S}}$ with $|t| \leq K_{I+1}$, and

$$x^*(y_{n_{I+1}}) - \frac{1}{2^I} \leq \tilde{y}^*_2(y_{I+1}).$$

Let $y^* = \tilde{y}^*_1 + \tilde{y}^*_2$. In order to prove the Claim, we only need to establish that

$$\|y^*\| \leq P_{I+1}. \tag{6}$$

By the induction hypothesis there exists a fast increasing tree $\mathcal{S} \subseteq T_I$ and $(\tilde{y}^*_{1,t})_{t\in\mathcal{S}} \subset S^*$ so that $\tilde{y}^*_{1,t}$ is an element of the unit vector basis of S^* for every $t \in \mathcal{S}$ with $|t| \leq K_I$ and satisfying that $A_I \frac{1}{P_I}\tilde{y}^*_1$ and $(\tilde{y}^*_{1,t})_{t\in\mathcal{S}}$ are roughly associated to $\mathcal{S}$. Thus there is at most one common node t for the trees $\mathcal{S}$ and $\tilde{\mathcal{S}}$, and t has length at least K_I. Since both trees $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are subsets of the fast increasing tree T we have that

$$\left\| \frac{f(K_I)}{f(K_I+1)} \left(\frac{1}{P_I}\tilde{y}^*_1 + \tilde{y}^*_2 \right) \right\| \leq 1$$

which gives (6) and finishes the proof of the Claim. □

4. The construction of uncountably many non-isomorphic subsymmetric sequences in S

This section is devoted to the proof of Theorem 1.1. First we shall construct an infinite tree $(\Delta, \prec)$ with no maximal nodes. $\emptyset$ will denote its root, and for $t \in \Delta$, k_t will denote the number of immediate successors of t. The construction will reveal that for $s, t \in \Delta$ with $s \neq t$ we have $k_s \neq k_t$. The tree Δ will be completely determined by the family of numbers $(k_t)_{t\in\Delta}$ since we assume that if $t_1, t_2, t_3, t_4 \in \Delta$ with $|t_1| = |t_2|$, $k_{t_1} < k_{t_2}$, $t_1 = t'_3$ and $t_2 = t'_4$ then $k_{t_3} < k_{t_4}$. We shall equip Δ with the lexicographic order $<_\ell$ defined by

$$s <_\ell t \text{ if and only if } k_s < k_t, \text{ for } s, t \in \Delta.$$

For $t \in \Delta$ let $t+1$ denote the immediate $<_\ell$-successor of t. Also, for $n \in \mathbb{N}$, if $t+n \in \Delta$ has been defined then set $t+(n+1)$ to be $(t+n)+1$ where the parentheses denote the order of the operations. For $t \in \Delta\backslash\{\emptyset\}$ let $t-1$ denote the immediate $<_\ell$-predecessor of t. Similarly, for $n \in \mathbb{N}$, if $t-n \in \Delta\backslash\{\emptyset\}$ has been defined then we define $t-(n+1)$ to be $(t-n)-1$. Define

$$\mathcal{M} = \{t \in \Delta\backslash\{\emptyset\} : t \text{ is } <_\ell\text{-maximum among its siblings in } \Delta\}$$

Then we prove

Theorem 4.1. *For $j = 1, 2$ let $M^j \subseteq \mathcal{M}$, $T^j = subtree(\Delta\backslash M^j)$, and a block sequence $x^j_i)_i$ of $(e_i)_i$ which is stabilizing according to T_jIf $M^1\backslash M^2$ is infinite then*

$$\limsup_{N\to\infty} \frac{\|\sum_{i=1}^N x^1_i\|}{\|\sum_{i=1}^N x^2_i\|} = \infty.$$

and

Theorem 4.2. *Let $M \subseteq \mathcal{M}$, $T = subtree(\Delta \backslash M)$, and a block sequence $(x_i)_i$ of $(e_i)_i$ which is stabilizing according to T. Then for $i \in \mathbb{N}$ and $j \geq 2$,*

$$\|x_i\|_j \leq 27 \frac{f(k_\emptyset)}{f(j)}$$

Note that Theorem 1.1 follows immediately from Theorems 4.1, 4.2 and 1.2.

We break this section into two parts. In the first part we will define the family $(k_t)_{t \in \Delta}$. In the second part we will prove Theorems 4.1 and 4.2.

Part I: The construction of $(k_t)_{t \in \Delta}$

We enumerate $\mathcal{M} \cup \{\emptyset\}$ as $(s_n)_{n \in \mathbb{N} \cup \{0\}}$ where for integers $n < m$ we have that $s_n <_\ell s_m$ (we set $s_0 = \emptyset$). Along with the numbers $(k_t)_{t \in \Delta}$ we will also define a summable decreasing sequence $(\eta_m)_{m=0}^\infty$ of positive numbers. The numbers $(k_t)_{t \in \Delta}$ are defined inductively on m: We first define $k_\emptyset$ and $\eta_\emptyset$. Then for every $m \in \mathbb{N} \cup \{0\}$, we define the numbers k_t and η_{m+1} and for all siblings t of s_{m+1}. For the definition of the numbers $(k_t)_{t \in \Delta}$ we use the following

Remark 4.3. *For every $E > 0$ and $M \in \mathbb{N}$ there exists $K > 0$ and $\eta > 0$ such that for every $\alpha_1 > \alpha_2 > \cdots > \alpha_M > 0$ and $K \leq k_1 < k_2 < \cdots < k_M$ we have that*

$$F_i'(x) \overset{\sqrt{1+\eta}}{\approx} \alpha_i \frac{f(k_i)}{f(k_i x)} \text{ for all } x \in [1, \infty), \text{ and for all } i = 1, \ldots, M, \tag{7}$$

and for all $\varepsilon \geq E$

$$(1+\eta)(1+\varepsilon) G_-'(y) \leq \frac{(1+\varepsilon)G(y) - G(x)}{y - x} \text{ for all } 0 < x < y, \tag{8}$$

where $F_i(x) = \alpha_i \frac{f(k_i)x}{f(k_i x)}$ $(i = 1, \ldots, M)$ and $G = \max_{1 \leq i \leq M} F_i$ are defined on $[1, \infty)$.

Also, note that with the assumptions of the previous Remark, if $n < m \leq M$ and N is the unique solution of the equation $F_n(x) = F_m(x)$ then

$$F_n'(N) \overset{\sqrt{1+\eta}}{\approx} \alpha_n \frac{f(k_n)}{f(k_n N)} = \frac{F_n(N)}{N} = \frac{F_m(N)}{N} = \alpha_m \frac{f(k_m)}{f(k_m N)} \overset{\sqrt{1+\eta}}{\approx} F_m'(N)$$

Thus

$$F_n'(N) \overset{(1+\eta)}{\approx} F_m'(N). \tag{9}$$

For $t \in \Delta$ we define the concave functions F_t and F and the function G on $[1, \infty)$ by

$$F_t(x) = \alpha_{t'} \frac{f(k_t)x}{f(k_t x)},$$

$$G(x) = \sup_{t \in \Delta} F_t(x) \text{ and}$$

$$F = \text{ the concave envelope of the function } G.$$

Now the definition of $(k_t)_{t\in\Delta}$ proceeds as follows: Fix a decreasing summable sequence of positive numbers $(\varepsilon_m)_{m=0}$, $m = 0, 1, 2\ldots$, with $\varepsilon_0 < 1/6$. We define η_0 and $k_\emptyset = k_{s_0}$ as follows: Let $K \in \mathbb{N}$ and $\eta > 0$ that satisfy the statement of Remark 4.3 for $E = \varepsilon_0$ and $M = 1$. Set $k_\emptyset = K \vee (2^9 - 1)$ and $\eta_0 = \eta$. Assume that for some $m \in \mathbb{N} \cup \{0\}$ we have defined k_t for all $t \leq_\ell s_m$. Let $t \in \Delta$ such that k_t has been defined. Also, for some $m \in \mathbb{N} \cup \{0\}$ assume that k_{s_m} has been defined. Assume that k_{t+1} and $k_{s_{m+1}}$ are siblings. Then we define $k_{t+1} > k_t$ so that the following conditions (10), (11), (12) , (13) and (14) are satisfied. If $t + 1 = s_{m+1}$ then we require that the conditions (15), (16) and (19) are satisfied as well.

First we recall from [S1] (Lemma 4 and Lemma 6) how to construct finite block sequences in S which are almost isometric to the unit vector basis:

Remark 4.4. *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exist integers $k_1 < k_2 < \cdots < k_N$ such that if $I_1 < I_2 < \cdots < I_N$ are intervals of integers with $|I_i| = k_i$ for all i and $y_i = (f(k_i)/k_i)\sum_{j\in I_i} e_j$ then $(y_i)_{i=1}^N$ is $(1+\varepsilon)$-equivalent to the first N unit basis vectors of S.*

Using Remark 4.4 we ensure that if for $n < N$ in $\mathbb{N}$ we set $\Delta_N = \{t \in \Delta : |t| \leq N\}$ and we assume that $(x_t)_{t\in\Delta_N} \subset S$ and $x \in S$ are associated to Δ_N then

$$(10) \qquad (x_u)_{\{u\in\Delta_N : |u|=n\}} \overset{1+\varepsilon_n}{\approx} \text{ to the first } \#\{u \in \Delta_N : |u| = n\} \text{ unit basis vectors of } S.$$

Assume that k_{t+1} is sufficiently large so that

$$(11) \qquad \frac{f(k_{t+1})}{f(k_{t+1}/k_v)} \leq 9/8 \text{ for all } v <_\ell t + 1.$$

Also assume that k_{t+1} is sufficiently large to ensure that

$$(12) \qquad 108f(ca) \leq f\Big(\Big(\frac{k_{t+1}}{2\prod\{k_s : \emptyset \preceq s \preceq t+1'}a\Big) \vee b\Big) f\Big(\frac{ca}{b}\Big) \text{ for all } a > 0 \text{ and } b, c \geq 1.$$

We make sure that k_{t+1} sufficiently large to satisfy

$$1-\frac{1}{2^{|t+1|+1}}<\frac{(k_{t+1}-1)f(k_{t+1})}{k_{t+1}f(k_{t+1}-1)}. \tag{13}$$

We finally ensure that k_{t+1} is chosen large enough so that

$$\frac{\alpha_{t+1}}{\alpha_t}\to 0 \text{ as } t \text{ becomes arbitrarily large in the } <_\ell \text{ order.} \tag{14}$$

Assume that $t+1=s_{m+1}$. Let $K\in\mathbb{N}$ and $\eta>0$ that satisfy the statement of Remark 4.3 for $E=\varepsilon_{m+1}$ and $M=m+1$. Then set

$$k_{s_{m+1}}\geq K \tag{15}$$

and $\eta_{m+1}=\eta$. We make sure that

$$\frac{f(k_{s_{M-1}})}{f(k_{s_{M-1}}k_{s_M})}\to 0 \text{ as } M\to\infty. \tag{16}$$

Let's explain the role of conditions (14) and (16):

Remark 4.5. *We obtain the following limits:*

$$\frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+1}-1}(k_{s_{m+1}})}\to\infty \text{ as } m\to\infty, \tag{17}$$

and

$$\frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+2}}(k_{s_{m+1}})}\to\infty \text{ as } m\to\infty. \tag{18}$$

Indeed,

$$\frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+1}-1}(k_{s_{m+1}})}\geq\frac{1}{2}\frac{f(k_{s_{m+1}-1}k_{s_{m+1}})}{f(k_{s_{m+1}-1})}\to\infty \text{ as } m\to\infty,$$

(by (16)), and

$$\frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+2}}(k_{s_{m+1}})}\geq\frac{1}{2}\frac{\alpha_{s'_{m+1}}}{\alpha_{s'_{m+2}}}\to\infty \text{ as } m\to\infty,$$

(by (14)), which finishes the proof of Remark 4.5.

Finally we ensure that k_{t+1} is chosen so that $F_{s_{m+1}}(k_{s_m})<F_{s_m}(k_{s_m})$ and $F_{s_m}(k_{s_{m+1}})<F_{s_{m+1}}(k_{s_{s+1}})$. Thus

$$F(k_{s_{m+1}})=F_{s_{m+1}}(k_{s_{m+1}}). \tag{19}$$

This completes the definition of k_{t+1} and the inductive definition of $(k_u)_{u\in\Delta}$.

Part II: The main estimates

In this part we give the proof of Theorems 4.1 and 4.2. For the proof of Theorem 4.1 we will need Theorems 4.6 and 4.7, while Theorem 4.2 will be a byproduct of the proof of Theorem 4.7 (indeed, note that Theorem 4.2 follows immediately from Lemma 4.10).

Theorem 4.6. *There exists a constant C such that $G(x) \leq F(x) \leq CG(x)$ for all $x \in [1, \infty)$.*

Before we state Theorem 4.7 we need to introduce some terminology. Let $M \subseteq \mathcal{M}$, $T = \text{subtree}\,(\Delta \backslash M)$ and let $(x_i)_i$ be a block sequence of $(e_i)_i$ which is stabilizing according to T. For $i \in \mathbb{N}$ let $T_i = \{t \in T : |t| \leq i\}$ and let $(x_{i,t})_{t \in T_i} \subset S$ such that $(x_{i,t})_{t \in T_i}$ and x_i are associated to T_i. Let E be a finite interval of positive integers. We define $|E|$ to be an "approximation" of $\#\{i \in \mathbb{N} : E \cap \text{supp}\,(x_i) \neq \emptyset\}$ in such a way that if $E_1 < E_2$ are two finite intervals whose union is an interval then $|E_1| + |E_2| = |E_1 \cup E_2|$. For that we define some auxiliary vectors y_i and $y_{i,t}$ for $i \in \mathbb{N}$ and $t \in T_i$. The definition of the vectors y_i and $y_{i,t}$ proceeds as follows: For $i \in \mathbb{N}$ and t a maximal node of T_i we define $y_{i,t} = x_{i,t}$ (which is an element of the unit vector basis of S). For every $i \in \mathbb{N}$ and $t \in T$ with $|t| < i$ we define $y_{i,t}$ by induction on $|t| = i-1, i-2, \ldots, 0$ by

$$y_{i,t} = \frac{1}{k_t} \sum_{\{s \in T : t = s'\}} y_{i,s}.$$

Then $y_i = y_{i,\emptyset}$. Now if E is a finite interval of positive integers we define

$$|E| = \|E(\sum_{i=1}^{\infty} y_i)\|_1$$

where $\|\cdot\|_1$ denotes the norm of the space ℓ_1. The second ingredient of the proof of Theorem 4.1 is the following result:

Theorem 4.7. *For every finite interval E with $1 \leq |E|$ we have that*

$$\left\| E \sum_{i=1}^{\infty} x_i \right\| \leq 28F(|E|). \tag{20}$$

We postpone the proof of Theorems 4.6 and 4.7 in order to present the

Proof of Theorem 4.1. Assume that some $m \in \mathbb{N} \cup \{0\}$ we have that $s_{m+1} \in M_1 \backslash M_2$. Note that by (13),

$$\| \sum_{i=1}^{k_{s_{m+1}}} x_i^1 \| \geq \| \sum_{i=1}^{k_{s_{m+1}}} \alpha_{s'_{m+1}} x_{i,s_{m+1}}^1 \| \geq \prod_{k=|s_{m+1}|}^{\infty} (1 - \frac{1}{2^{k+1}}) F_{s_{m+1}}(k_{s_{m+1}}),$$

Also, by (19) and Theorems 4.6 and 4.7,

$$\| \sum_{i=1}^{k_{s_{m+1}}} x_i^2 \| \leq 28 F(k_{s_{m+1}}) \leq 28 G(k_{s_{m+1}}) \leq 28 C \left(F_{s_{m+1}-1}(k_{s_{m+1}}) \vee F_{s_{m+2}}(k_{s_{m+1}}) \right).$$

Thus

$$\frac{\| \sum_{i=1}^{k_{s_{m+1}}} x_i^1 \|}{\| \sum_{i=1}^{k_{s_{m+1}}} x_i^2 \|} \geq \frac{\prod_{k=0}^{\infty} (1 - \frac{1}{2^{k+1}})}{28C} \left(\frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+1}-1}(k_{s_{m+1}})} \wedge \frac{F_{s_{m+1}}(k_{s_{m+1}})}{F_{s_{m+2}}(k_{s_{m+1}})} \right)$$

which gives the result by Remark 4.5. □

First we concentrate on the proof of Theorem 4.6.

Proposition 4.8. *Let $(\varepsilon_m)_{m=0}^{\infty}$ and $(\eta_m)_{m=0}^{\infty}$ be sequences of positive numbers and $(F_m)_{m=0}^{\infty}$ be a sequence of positive differentiable concave functions defined on $[1, \infty)$. Assume that for all $m \in \mathbb{N} \cup \{0\}$, for all $\varepsilon \geq \varepsilon_m$ and for all $1 \leq x \leq y$.*

$$(1 + \eta_m)(1 + \varepsilon) F'_{m+1-}(y) \leq \frac{(1 + \varepsilon) F_{m+1}(y) - \max_{0 \leq i \leq m+1} F_i(x)}{y - x} \tag{21}$$

For all $n < m$ assume that there exists a unique $N \in (1, \infty)$ such that $F_n(x) > F_m(x)$ for all $1 < x < N$ and $F_n(x) < F_m(x)$ for $N < x$. Also assume that

$$F'_n(N) \overset{1+\eta_n}{\approx} F'_m(N). \tag{22}$$

Then for every $m \geq 0$, for every integers $i_0 < i_1 < \cdots < i_{m+1}$, for every $(t_j)_{j=0}^{m+1} \subset (0, 1]$ with $\sum_{j=0}^{m+1} t_j = 1$ and for all $(x_j)_{j=0}^{m+1}$ with $N_j \leq x_j < N_{j+1}$ we have that

$$\sum_{j=0}^{m} \prod_{k=0}^{j-1} (1 + \varepsilon_k) t_j G(x_j) + (1 + \varepsilon) \prod_{k=0}^{m} (1 + \varepsilon_k) t_{m+1} G(x_{m+1}) \leq (1 + \varepsilon) \prod_{k=0}^{m} (1 + \varepsilon_k) G(\sum_{j=0}^{m+1} t_j x_j)$$

where $G = \max_{0 \leq j \leq m+1} F_{i_j}$, $N_0 = 1$, N_j is the unique solution of the equation $F_{i_{j-1}}(x) = F_{i_j}(x)$ for $j = 1, \ldots, m+1$ on $(1, \infty)$, $N_{m+2} = \infty$ and we adopt the convention that a product is equal to 1 whenever the lower index is larger than the upper index.

Proof. We use induction on m. For the inductive step, let $\varepsilon \geq 0$, $(t_j)_{j=0}^{m+1} \subset (0,1]$ with $\sum_{j=0}^{m+1} t_j = 1$ and $(x_j)_{j=0}^{m+1}$ with $N_j \leq x_j < N_{j+1}$ for $j = 0, 1, \ldots, m+1$ and $F_{i_{j-1}}(x_j) \neq F_{i_j}(x_j)$ for $j = 1, \ldots, m+1$ We claim that

$$\begin{aligned}(23)\quad & \sum_{j=0}^{m} \left(\prod_{k=0}^{j-1}(1+\varepsilon_k)\right) t_j G(x_j) + (1+\varepsilon)\left(\prod_{k=0}^{m}(1+\varepsilon_k)\right) t_{m+1} G(x_{m+1}) \\ \leq\ & \sum_{j=0}^{m-1} \left(\prod_{k=0}^{j-1}(1+\varepsilon_k)\right) t_j G(x_j) + (1+\varepsilon)\left(\prod_{k=0}^{m}(1+\varepsilon_k)\right) (t_m + t_{m+1}) G(\frac{t_m x_m + t_{m+1} x_{m+1}}{t_m + t_{m+1}})\end{aligned}$$

(if $m = 1$ then $\sum_{j=1}^{m-1}$ is zero). To simplify the notation, let $\tilde{t}_m = t_m/(t_m + t_{m+1})$, $\tilde{t}_{m+1} = t_{m+1}/(t_m + t_{m+1})$ and $\tilde{\varepsilon} = (1+\varepsilon)(1+\varepsilon_m) - 1$. If (23) were false then

$$(1+\tilde{\varepsilon})G(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) < \tilde{t}_m G(x_m) + \tilde{t}_{m+1}(1+\tilde{\varepsilon})G(x_{m+1}).$$

which can be written equivalently as

$$(24)\quad \frac{(1+\tilde{\varepsilon})G(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) - G(x_m)}{(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) - x_m} < (1+\tilde{\varepsilon}) \frac{G(x_{m+1}) - F_I(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1})}{x_{m+1} - (\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1})}.$$

Note that by the concavity of $F_{i_{m+1}}$,

$$(25)\quad \frac{G(x_{m+1}) - G(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1})}{x_{m+1} - (\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1})} \leq (F_I)'_+(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}).$$

If $\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1} \neq N_{m+1}$ then

$$(26)\quad G'_+(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) = G'_-(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}).$$

If $\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1} = N_{m+1}$ then by (9) we have

$$(27)\quad G'_+(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) \leq (1+\eta_m) G'_-(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}).$$

Combining (24), (25), (26), and (27), we obtain

$$\frac{(1+\tilde{\varepsilon})G(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) - G(x_m)}{(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1}) - x_m} < (1+\tilde{\varepsilon})(1+\eta_m) G'_-(\tilde{t}_m x_m + \tilde{t}_{m+1} x_{m+1})$$

which contradicts (8). This finishes the proof of (23). Now the use of the induction hypothesis completes the proof of Proposition 4.8. ☐

Now Theorem 4.6 follows immediately from Proposition 4.8 since the sequence (ε_i) which is used in the construction of Δ in Part I is summable. Note that the assumptions (21) and (22) of Theorem 4.6 are satisfied by (8) and (9) respectively (using (15).

Now we concentrate on the proof of Theorem 4.7. We will need Lemmas 4.9-4.11, and the following notation: Let E be a finite interval of positive integers and $t \in T$. We define $|E|_t = \|E \sum_{i=1}^{\infty} y_{i,t}\|_1$.

Lemma 4.9. *Assume that there exists a unique $i \in \mathbb{N}$ such that $E \cap supp(x_i) \neq \emptyset$. Then for $\ell \in \mathbb{N}$,*

$$\|Ex_{i,t}\|_\ell \leq 8 \frac{f(k_t)|E|_t}{f(\ell) f(k_t|E|_t/\ell)}$$

Proof. Let $E_1 < E_2 < \cdots < E_\ell$ be subintervals of E such that

$$\|Ex_{i,t}\|_\ell = \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \|E_j x_{i,t}\|.$$

Split each E_j into at most three non-empty intervals $E_j = E_j^1 \cup E_j^2 \cup E_j^3$ where $E_j^1 < E_j^2 < E_j^3$ and E_j^2 is the largest subinterval of E_j which satisfies the following properties:

- If $E_j^2 \neq \emptyset$ then $E_j^2 \cap \mathrm{supp}(x_i) \neq \emptyset$.
- If for some $u \in T$ with $t = u'$ we have that $E_j^2 \cap \mathrm{supp}(x_{i,u}) \neq \emptyset$ then $\mathrm{supp}(x_{i,u}) \subseteq E_j^2$.

Let

$$U = \{u \in T\colon\ t = u' \text{ and } \mathrm{supp}(x_{i,u}) \subseteq E\}$$

and for $j = 1, \ldots, \ell$, let

$$U_j = \{u \in T\colon\ t = u' \text{ and } \mathrm{supp}(x_{i,u}) \subseteq E_j^2\}.$$

Then

$$\begin{aligned}
\|Ex_{i,t}\|_\ell &\leq \frac{1}{f(\ell)} \sum_{m \in \{1,3\}} \sum_{j=1}^{\ell} \|E_j^m x_{i,t}\| + \frac{1}{f(\ell)} \frac{f(k_t)}{k_t} \sum_{\{j\colon\ E_j^2 \neq \emptyset\}} \frac{\#U_j}{f(\#U_j)} \left\| \frac{f(\#U_j)}{\#U_j} E_j^2 x_{i,t} \right\| \\
&\leq \frac{1}{f(\ell)} 2\ell \frac{2 f(k_t)}{k_t} + \frac{1}{f(\ell)} \frac{f(k_t)}{k_t} \sum_{\{j\colon\ E_j^2 \neq \emptyset\}} \frac{\#U_j}{f(\#U_j)} 2.
\end{aligned}$$

Since the function $(0,\infty) \ni x \mapsto x/f(x)$ is concave, we have

$$\begin{aligned}\|Ex_{i,t}\|_\ell &\leq 4\frac{f(k_t)}{k_t}\frac{\ell}{f(\ell)} + \frac{2}{f(\ell)}\frac{f(k_t)}{k_t}\frac{\sum\limits_{\{j:\ E_j^2\neq\emptyset\}} \#U_j}{f\left(\sum\limits_{\{j:\ E_j^2\neq\emptyset\}} \#U_j/\#\{j:\ E_j^2 \neq \emptyset\}\right)} \\ &\leq 4\frac{f(k_t)}{k_t}\frac{\ell}{f(\ell)} + \frac{2}{f(\ell)}\frac{f(k_t)}{k_t}\frac{\#U}{f(\frac{\#U}{\ell})}\end{aligned}$$

since $\sum\{\#U_j:\ E_j^2 \neq \emptyset\} \leq \#U$ and $\#\{j:\ E_j^2 \neq \emptyset\} \leq \ell$. Now note that $\#U/k_t \leq |E|_t$ i.e. $\#U \leq k_t|E|_t$ and thus

$$\|Ex_{i,t}\|_\ell \leq 4\frac{f(k_t)}{k_t}\frac{\ell}{f(\ell)} + \frac{2}{f(\ell)}\frac{f(k_t)}{k_t}\frac{k_t|E|_t}{f(\frac{k_t|E|_t}{\ell})}.$$

Note that if $k_t|E|_t/\ell \geq 1$ then $k_t|E|_t/\ell \geq f(k_t|E|_t/\ell)$. Also if $0 < k_t|E|_t/\ell < 1$ then $k_t|E|_t/\ell \geq (\ln 22)f(k_t|E|_t/\ell)$. Thus we always have $k_t|E|_t/\ell \geq (\ln 2)f(k_t|E|_t/\ell)$. Hence

$$\begin{aligned}\|Ex_{i,t}\|_\ell &\leq \frac{4}{\ln_2}\frac{f(k_t)|E|_t}{f(\ell)f(k_t|E|_t/\ell)} + 2\frac{f(k_t)|E|_t}{f(\ell)f(k_t|E|_t/\ell)} \\ &\leq 8\frac{f(k_t)|E|_t}{f(\ell)f(k_t|E|_t/\ell)}.\end{aligned}$$

□

Lemma 4.10. *Assume that there exists a unique $i \in \mathbb{N}$ such that $E \cap supp(x_i) \neq \emptyset$. Then for $t \in T$ and $\ell \leq k_{t+1}|E|$ we have*

$$\|Ex_{i,t}\|_\ell \leq 27\frac{f(k_t)|E|_t}{f(\ell)}.$$

Proof. We write $x_{i,t} = (f(k_t)/k_t)\sum_{t=u'} x_{i,u}$. There exists $u_1,\ u_2 \in T$, with $t = u_1' = u_2'$, $|E|_{u_1} \leq 1$, $|E|_{u_2} \leq 1$, $u_1 <_\ell u_2$ and $|E|_u = 1$ for all $u \in T$ with $t = u'$ and $u_1 <_\ell u <_\ell< u_2$. Then

$$\|Ex_{i,t}\|_\ell \leq \frac{f(k_t)}{k_t}\|Ex_{i,u_1}\|_\ell + \frac{f(k_t)}{k_t}\sum_{u_1<_\ell u<_\ell u_2}\|Ex_{i,u}\|_\ell + \frac{f(k_t)}{k_t}\|Ex_{i,u_2}\|_\ell.$$

For $u \in \{u_1, u_2\}$, by Lemma 4.9 we have

$$\begin{aligned}\frac{f(k_t)}{k_t}\|Ex_{i,u}\|_\ell &\le 8\frac{f(k_t)}{k_t}\frac{f(k_u)|E|_u}{f(\ell)f(k_u|E|_u/\ell)}\\ &\le 8\frac{f(k_t)}{k_t}\frac{f(k_u)k_t|E|_t}{f(\ell)f(k_uk_t|E|_t/k_{t+1})}\\ &\quad(\text{since } \frac{1}{k_t}|E|_u \le |E|_t \text{ and } \ell \le k_{t+1})\\ &\le 9\frac{f(k_t)|E|_t}{f(\ell)}.\end{aligned}$$

For $u_1 <_\ell u <_\ell u_2$, by Lemma 4.9 we have

$$\begin{aligned}\frac{f(k_t)}{k_t}\sum_{u_1<_\ell u<_\ell u_2}\|Ex_{i,u}\|_\ell &\le \frac{f(k_t)}{k_t}\sum_{u_1<_\ell u<_\ell u_2} 8\frac{f(k_u)|E|_u}{f(\ell)f(k_u|E|_u/\ell)}\\ &= \frac{f(k_t)}{k_t}\sum_{u_1<_\ell u<_\ell u_2} 8\frac{f(k_u)}{f(\ell)f(k_u/\ell)}\\ &\le \frac{f(k_t)}{k_t}\sum_{u_1<_\ell u<_\ell u_2} 8\frac{f(k_u)}{f(\ell)f(k_u/k_{t+1})}\\ &\quad(\text{since } \ell \le k_{t+1}|E| \le k_{t+1})\\ &\le \frac{9}{f(\ell)}\frac{f(k_t)}{k_t}\#\{u\colon\ u_1 <_\ell u <_\ell u_2\} \text{ (by (11))}\\ &\le 9\frac{f(k_t)|E|_t}{f(\ell)}.\end{aligned}$$

By putting together the previous estimates, the result follows immediately. $\square$

Lemma 4.11. *Assume that there exist a unique i such that $E \cap supp(x_i) \neq \emptyset$. Then for $t \in T$ and $\ell \in \mathbb{N}$ with $\ell \le k_{t+1}|E|$ we have*

$$\|Ex_i\|_\ell \le 28\frac{A_t|E|}{f(\ell)} + 24\frac{A_{t+1}|E|}{f(\ell)f(B_{t+1}|E|/\ell)}.$$

where $A_t = \prod\{f(k_s)\colon \emptyset \preceq s \preceq t\}$ and $B_t = \prod\{k_s\colon \emptyset \preceq s \preceq t\}$.

Proof. If $t_1 \in T$ is the $<_\ell$-maximum element of T with $t_1' = t-1$, let

$$\mathcal{A}_t = \{u \in T\colon\ t+1 <_\ell u <_\ell t_1 \quad \text{and} \quad E \cap \operatorname{supp}(x_{i,u}) \neq \emptyset\}.$$

There exist $u_1, u_2 \in \mathcal{A}_t \cup \{t, t+1\}$ such that $|E|_{u_1} \leq 1$, $|E|_{u_2} \leq 1$ and $|E|_u = 1$ for all $u \in (\mathcal{A}_1 \cup \{t, t+1\}) \backslash \{u_1, u_2\}$. Let $\tilde{\mathcal{A}}_t = \mathcal{A}_t \backslash \{u_1, u_2\}$. By the triangle inequality we have

$$\|Ex_i\|_\ell \leq \alpha_{t'}\|Ex_{i,t}\|_\ell + \alpha_{t+1'}\|Ex_{i,t+1}\|_\ell + \alpha_{u'_1}\|E_{i,u_1}\|_\ell + \alpha_{u'_2}\|Ex_{i,u_2}\|_\ell + \sum_{u \in \tilde{\mathcal{A}}_t} a_{u'}\|Ex_{i,u}\|_\ell. \tag{28}$$

By Lemma 4.10 we have

$$\alpha_{t'}\|Ex_{i,t}\|_\ell \leq 27\alpha_{t'}\frac{f(k_t)|E|_t}{f(\ell)} \leq 27\frac{A_t|E|}{f(\ell)} \tag{29}$$

since $(1/B_{t'})|E|_t \leq |E|$. By Lemma 4.9 we have

$$\alpha_{t+1'}\|Ex_{i,t+1}\|_\ell \leq 8\alpha_{t+1'}\frac{f(k_{t+1})|E|_{t+1}}{f(\ell)f(k_{t+1}|E|_{t+1}/\ell)} \leq 8\frac{A_{t+1}|E|}{f(\ell)f(B_{t+1}|E|/\ell)} \tag{30}$$

(since $|E|_{t+1} \leq B_{t+1'}|E|$). For $u \in \{u_1, u_2\}$, by Lemma 4.9 we have as in (30),

$$\alpha_{u'}\|Ex_{i,u}\|_\ell \leq 8\frac{A_{u+1}|E|}{f(\ell)f(B_{u+1}|E|/\ell)} \leq 8\frac{A_{t+1}|E|}{f(\ell)f(B_{t+1}|E|/\ell)} \tag{31}$$

since $t+1 <$ and $\ell \leq k_{t+1}|E| \leq B_{t+1}|E|$. Finally since $|E|_u = 1$ for $u \in \tilde{\mathcal{A}}_t$ we have by Lemma 4.9,

$$\begin{aligned}
\sum_{u \in \tilde{\mathcal{A}}_t} \alpha_{u'}\|Ex_{i,u}\|_\ell &\leq 8\sum_{u \in \tilde{\mathcal{A}}_t} \alpha_{u'}\frac{f(k_u)}{f(\ell)f(k_u/\ell)} \\
&\leq \frac{8}{f(\ell)}\sum_{u \in \tilde{\mathcal{A}}_t} \alpha_{u'}\frac{f(k_u)}{f(k_u/k_{t+1})} \quad (\text{since } \ell \leq k_{t+1}) \\
&\leq \frac{9}{f(\ell)}\sum_{u \in \tilde{\mathcal{A}}_t} \alpha_{u'} \quad (\text{by } (11)) \\
&\leq \frac{9}{f(\ell)}\sum_{u \in \tilde{\mathcal{A}}_t \cup \{t, t+1, u_1, u_2\}} \alpha_{u'}|E|_u \\
&\leq \frac{9}{f(\ell)}A_{t'}\sum_{u \in \tilde{\mathcal{A}}_t \cup \{t, t+1, u_1, u_2\}} \frac{\alpha_{u'}}{A_{u'}}|E|_u \\
&= \frac{9}{f(\ell)}A_{t'}|E| \\
&\leq \frac{A_t|E|}{f(\ell)}
\end{aligned} \tag{32}$$

since we assume that $9 \leq f(k_t)$ for all $t \in T$. Combining the estimates (28)–(32) we obtain the result. $\square$

Corollary 4.12. *Assume that there exists a unique $i \in \mathbb{N}$ such that $E \cap supp(x_i) \neq \emptyset$. Let $t \in T$ with $\frac{1}{k_{t+1}} \leq |E| < \frac{1}{k_t}$.*

$$\text{If} \quad E \subseteq supp(x_{i,t}) \quad \text{then} \quad \|Ex_i\| \leq 27\frac{A_t|E|}{f(B_t|E|)}. \tag{33}$$

$$\text{If} \quad E \subseteq supp(x_{i,t+1}) \quad \text{then} \quad \|Ex_i\| \leq 8\frac{A_{t+1}|E|}{f(B_{t+1}|E|)}. \tag{34}$$

$$\text{If} \quad E \cap supp(x_{i,t} + x_{i,t+1}) = \emptyset \quad \text{then} \quad \|Ex_i\| \leq 9\alpha_{t''}\frac{f(k_{t'})|E|}{f(k_{t'}|E|)}. \tag{35}$$

Proof. All follow immediately from the proof of Lemma 4.11. Especially for (35), note that for $\ell > k_t|E|$ we have $f(\ell) > f(k_t|E|) > B_{t''}f(k_{t'}|E|)$ by assumption. $\square$

Corollary 4.13. *Assume that there exists a unique $i \in \mathbb{N}$ such that $E \cap supp(x_i) \neq \emptyset$. Let $t \in T$ with $\frac{1}{k_{t+1}} \leq |E| < \frac{1}{k_t}$. Then*

$$\|Ex_i\| \leq \max\left\{27\frac{A_t|E \cap supp\ x_{i,t}|}{f(B_t|E \cap supp\ x_{i,t}|)}, 8\frac{A_{t+1}|E \cap supp\ x_{i,t+1}|}{f(B_{t+1}|E \cap supp\ x_{i,t+1}|)},\right. \\ \left. 18a_{t''}\frac{f(k_{t'})|E \backslash supp(x_{i,t} + x_{i,t+1})|}{f(k_{t'}|E \backslash (supp\ x_{i,t} + x_{i,t+1})|}\right\}. \tag{36}$$

Finally we are ready to give the

Proof of Theorem 4.7. We prove the statement by induction on $\#E$. If $|E| = 1$ then the statement is obviously true. Let a finite interval E with $1 \leq |E|$ and let $\ell \in \mathbb{N}$ with

$$\left\|E\sum_{i=1}^{\infty} x_i\right\| = \left\|E\sum_{i=1}^{\infty} x_i\right\|_{\ell}.$$

Thus there exist intervals $E_1 < E_2 < \cdots < E_\ell$ with $E = \bigcup_{j=1}^{\ell} E_j$ and

$$\left\|E\sum_{i=1}^{\infty} x_i\right\| = \frac{1}{f(\ell)}\sum_{j=1}^{\ell}\left\|E_j\sum_{i=1}^{\infty} x_i\right\|.$$

For $t \in T$ let $I_t = \{j\colon\ 1/k_{t+1} \leq |E_j| < 1/k_t\}$ and let $I = \{j\colon\ 1 \leq |E_j|\}$. Obviously, $\{1,\ldots,\ell\} \subset \bigcup_{t\in T} I_t \cup I$, and the sets I_t's (for $t \in T$) and I are disjoint. Since F is concave we have by the induction hypothesis

$$\frac{1}{f(\ell)}\sum_{j\in I}\left\|E_j\sum_{i=1}^{\infty} x_i\right\| \leq \frac{1}{f(\ell)}\sum_{j\in I} 28F(|E_j|) \leq \frac{\#I}{f(\ell)}28F\left(\sum_{j\in I}|E_j|/\#I\right) \tag{37}$$

Also for $t \in T$ with $I_t \neq \emptyset$ we have by Corollary 4.13 that

$$\frac{1}{f(\ell)}\sum_{j\in I_t}\left\|E_j\sum_{i=1}^{\infty}x_i\right\| \leq$$

$$\frac{1}{f(\ell)}\sum_{j\in I_t}\max\left\{27\frac{A_t|E_j\cap\operatorname{supp}\sum_i x_{i,t}|}{f(B_t|E_j\cap\operatorname{supp}\ \sum_i x_{i,t}|)}, 8\frac{A_{t+1}|E_j\cap\operatorname{supp}\sum_i x_{i,t+1}|}{f(B_{t+1}|E_j\cap\operatorname{supp}\sum_i x_{i,t+1}|)},\right.$$
$$\left.18\alpha_{t''}\frac{f(k_{t'})|E_j\backslash\operatorname{supp}\sum_i(x_{i,t}+x_{i,t+1})|}{f(k_{t'}|E\backslash\operatorname{supp}\sum_i(x_{i,t}+x_{i,t+1})|)}\right\}$$

Let

$$I_t^1 = \left\{j\in I_t\colon\ \left\|E_j\sum_{i=1}^{\infty}x_i\right\| \leq 27\frac{A_t|E_j\cap\operatorname{supp}\sum_i x_{i,t}|}{f(B_t|E_j\cap\operatorname{supp}\sum_i x_{i,t}|)}\right\},$$
$$I_t^2 = \left\{j\in I_t\colon\ \left\|E_j\sum_{i=1}^{\infty}x_i\right\| \leq 8\frac{A_{t+1}|E_j\cap\operatorname{supp}\sum_i x_{i,t+1}|}{f(B_{t+1}|E_j\cap\operatorname{supp}\sum_i x_{i,t+1}|)}\right\}$$

and $I_t^3 = I_t\backslash(I_t^1\cup I_t^2)$.

Since for fixed $t\in T$, the functions on (0,1)

$$x\mapsto 18\frac{A_t x}{f(B_t x)},\quad x\mapsto 8\frac{A_{t+1}x}{f(B_{t+1}x)},\quad x\mapsto 27\alpha_{t''}\frac{f(k_{t'})x}{f(k_{t'}x)}$$

are concave, we have if $I_t^1\neq\emptyset$,

$$\frac{1}{f(\ell)}\sum_{j\in I_t^1}27\frac{A_t|E_j\cap\operatorname{supp}\sum_i x_{i,t}|}{f(B_t|E_j\cap\operatorname{supp}\sum_i x_{i,t}|)}$$
$$\leq \frac{\#I_t^1}{f(\ell)}27\frac{A_t\sum_{j\in I_t^1}|E_j\cap\operatorname{supp}\sum_i x_{i,t}|/\#I_t^1}{f(B_t\sum_{j\in I_t^1}|E_j\cap\operatorname{supp}\sum_i x_{i,t}|/\#I_t^1)}$$
$$= \frac{27}{f(\ell)}\frac{A_t\sum_{j\in I_t^1}|E_j\cap\operatorname{supp}\sum_i x_{i,t}|}{f(B_t\sum_{j\in I_t^1}|E_j\cap\operatorname{supp}\sum_i x_{i,t}|/\#I_t^1)}.$$

Let E_t^1 be an interval with

$$\left|E_t^1 \cap \sum_i x_{i,t}\right| = \sum_{j\in I_t^1} \left|E_j \cap \operatorname{supp} \sum_i x_{i,t}\right|.$$

Then $|E_t^1 \cap \operatorname{supp} \sum_i x_{i,t}| = |E_t^1|/B_{t'}$. Thus

$$\frac{27}{f(\ell)} \frac{A_t \sum\limits_{j\in I_t^1} |E_j \cap \operatorname{supp} \sum\limits_i x_{i,t}|}{f(B_t \sum\limits_{j\in I_t^1} |E_j \cap \operatorname{supp} \sum\limits_i x_{i,t}|/\#I_t^1)} = \frac{27}{f(\ell)} \frac{A_t |E_t^1|/B_t'}{f(B_t|E_t^1|/(B_{t'}\#I_t^1))} \tag{38}$$

$$= \frac{27}{f(\ell)} \alpha_{t'} \frac{f(k_t)|E_t^1|}{f(k_t|E_t^1|/\#I_t^1)} =: \frac{27}{f(\ell)} e_t^1$$

Similarly, if $I_t^2 \neq \emptyset$ there exists an interval E_t^2 such that

$$\frac{1}{f(\ell)} \sum_{j\in I_t^2} \left\| E_j \sum_{i=1}^\infty x_i \right\| \leq \frac{8}{f(\ell)} \alpha_{t+1'} \frac{f(k_{t+1})|E_t^2|}{f(k_{t+1}|E_t^2|/\#I_t^2)} =: \frac{8}{f(\ell)} e_t^2 \tag{39}$$

and if $I_t^3 \neq \emptyset$ there exists an interval E_t^3 such that

$$\frac{1}{f(\ell)} \sum_{j\in I_t^3} \left\| E_j \sum_{i=1}^\infty x_i \right\| \leq \frac{18}{f(\ell)} \alpha_{t''} \frac{f(k_{t'})|E_t^3|}{f(k_{t'}|E_t^3|/\#I_t^3)} =: \frac{18}{f(\ell)} e_t^3. \tag{40}$$

Finally note that for all $t_1, t_2 \in T$, $t_1 \neq t_2$ and $m_1, m_2 \in \{1,2,3\}$ with $I_{t_1}^{m_1} \neq \emptyset$ and $I_{t_2}^{m_2} \neq \emptyset$,

$$\text{If} \quad |E_{t_1}^{m_1}| \geq \frac{1}{2}|E_{t_2}^{m_2}| \quad \text{then} \quad \frac{27}{f(\ell)} e_{t_2}^{m_2} \leq \frac{1}{4} F(|E_{t_1}^{m_1}|). \tag{41}$$

(41) is valid since for $j \in I_{t_1}^{m_1}$ we have $|E_j| < \frac{1}{k_{t_1}}$ while for $i \in \mathbb{N}$, $|x_{i,t_1}| = 1/B_{t_1'}$, thus $\#I_{t_1}^{m_1} \geq (k_{t_1}/B_{t_1'})|E_{t_1}^{m_1}|$, hence

$$f(\ell) \geq f(\#I_{t_1}^{m_1} \vee \#I_{t_2}^{m_2}) \geq f\left(\frac{k_{t_1}}{B_{t_1'}}|E_{t_1}^{m_1}| \vee \#I_{t_2}^{m_2}\right) \geq f\left(\frac{k_{t_1}}{2B_{t_1'}}|E_{t_2}^{m_2}| \vee \#I_{m_2}^{m_2}\right).$$

Therefore, assuming that $m_2 = 1$ (similarly if $m_2 = 2$ or $m_2 = 3$)

$$\begin{aligned}
\frac{27}{f(\ell)} e_{t_2}^{m_2} &\leq \frac{27}{f(\frac{k_{t_1}}{2B_{t_1'}}|E_{t_2}^{m_2}| \vee \#I_{t_2}^{m_2})} \alpha_{t_2'} \frac{f(k_{t_2})|E_{t_2}^1|}{f(k_{t_2}|E_{t_2}^1|/\#I_{t_2}^{m_2})} \\
&\leq \frac{1}{4} \alpha_{t_2'} \frac{f(k_{t_2})|E_{t_2}^1|}{f(k_{t_2}|E_{t_2}^1|)} \text{ (by (12) for } a = |E_{t_2}^{m_2}|,\ b = \#I_{t_2}^{m_2},\ c = k_{t_2}) \\
&\leq \frac{1}{4} F(|E_{t_2}^1|).
\end{aligned}$$

This finishes the proof of (41).

(41) and (37) imply that there exists $t_0 \in T$ and $m_0 \in \{1,2,3\}$ such that

$$
\begin{aligned}
(42)\quad \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \left\| E_j \sum_{i=1}^{\infty} x_i \right\| &\leq \frac{1}{f(\ell)} \sum_{j\in I} \left\| E_j \sum_{i=1}^{\infty} x_i \right\| + \frac{27}{f(\ell)} \sum_{m\in\{1,2,3\}} \sum_{t\in T} e_t^m \\
&\leq \frac{\#I}{f(\ell)} 28F\left(\sum_{j\in I} |E_j| / \#I \right) + \frac{27}{f(\ell)} e_{t_0}^{m_0} + \frac{1}{2} F(|E_{t_0}^{m_0}|) \\
&\leq \frac{\#I}{f(\ell)} 28F\left(\sum_{j\in I} |E_j| / \#I \right) + \frac{\#I_{t_0}^{m_0}}{f(\ell)} 27F\left(\frac{|E_{t_0}^{m_0}|}{\#I_{t_0}^{m_0}} \right) + \frac{1}{2} F(|E_{t_0}^{m_0}|).
\end{aligned}
$$

Since $\sum_{j=1}^{\ell} |E_j| = |E|$, we have $\sum_{j\in I} |E_j| + |E_{t_0}^{m_0}| \leq |E|$. Therefore, since F is concave and $(\ell/f(\ell))F(x/\ell) \leq F(x)$ for $1 \leq x, x/\ell, \ell$, we have that the maximum of the right hand side of (42) is obtained when $E_{t_0}^{m_0} = \emptyset$ and $\sum_{j\in I} |E_j| = |E|$. Hence the right hand side of (42) is at most $28F(|E|)$ which finishes the proof of Theorem 4.7. $\square$

References

[AS1] G. Androulakis, Th. Schlumprecht, *S is complementably minimal and subsequentially block prime*, preprint.

[AS2] G. Androulakis, Th. Schlumprecht, *Strictly singular, non-compact operators exist on the Gowers-Maurey space*, preprint.

[C] P.G. Casazza, *The Schroeder-Bernstein property for Banach spaces*, Contemporary Math. **85** (1989), 61-77.

[G] W.T. Gowers, *A solution to the Schroeder-Bernstein problem for Banach spaces*, Bull. London Math. Soc. **28** (1996), no. 3, 297–304.

[GM1] W.T. Gowers, B. Maurey, *The unconditional basic sequence problem* J. Amer. Math. Soc. **6** (1993), no. 4, 851–874

[GM2] W.T. Gowers, B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. **307** (1997), no. 4, 543–568.

[S1] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81-95.

[S2] Th. Schlumprecht, *A complementably minimal Banach space not containing c_0 or ℓ_p*, Seminar notes in Functional Analysis and Partial Differential Equations, Baton Rouge, Louisiana, (1992).

Department of Mathematics, University of South Carolina, Columbia, SC 29208. giorgis@math.sc.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843. schlump@math.tamu.edu