

AN EXTENSION OF SCHREIER UNCONDITIONALITY

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ABSTRACT. The main result of the paper extends the classical result of E. Odell on Schreier unconditionality to arrays in Banach spaces. An application is given on the “multiple of the inclusion plus compact” problem which is further applied to a hereditarily indecomposable Banach space constructed by N. Dew.

1. INTRODUCTION

A finite subset F of \mathbb{N} is called a Schreier set if $|F| \leq \min(F)$ (where $|F|$ denotes the cardinality of F). The important notion of Schreier unconditionality was introduced by E. Odell [16] and has inspired rich literature on the subject (see for example [3], [8]). Earlier very similar results can be found in [15, page 77] and [19, Theorem 2.1']. A basic sequence (x_n) in a Banach space is defined to be **Schreier unconditional** if there is a constant $C > 0$ such that for all scalars $(a_i) \in c_{00}$ and for all Schreier sets F we have

$$\left\| \sum_{i \in F} a_i e_i \right\| \leq C \left\| \sum a_i e_i \right\|.$$

In this case (e_i) is called **C -Schreier unconditional**.

Theorem 1.1. [16] *Let (x_n) be a normalized weakly null sequence in a Banach space. Then for any $\varepsilon > 0$, (x_n) contains a $(2 + \varepsilon)$ -Schreier unconditional subsequence.*

Our main result is Theorem 1.2 where we extend Theorem 1.1 to arrays of vectors of a Banach space such that each row is a seminormalized weakly null sequence. Then Theorem 1.2 guarantees the existence of a subarray which preserves all the rows of the original array and has a Schreier type of unconditionality. We now define the notions of array, subarray and regular array in a Banach space. An **array** in a Banach space X is a sequence of vectors in $(x_{i,j})_{i \in \mathbb{N}; j \in J_i} \subseteq X$ where J_i is an infinite subsequence of \mathbb{N} for all $i \in \mathbb{N}$, say $J_i = \{j_{i,1} < j_{i,2} < \dots\}$ and $(x_{i,j_{i,k}})_{k \in \mathbb{N}}$ is a seminormalized weakly null sequence in X for all $i \in \mathbb{N}$. The reader is warned that the paper [12] also considers arrays of vectors with different properties. Let $<_{r\ell}$ denote the reverse lexicographical order on \mathbb{N}^2 . Let $(x_{i,j})_{i \in \mathbb{N}; j \in J_i}$ be an array in a Banach space X . A **subarray** of $(x_{i,j})_{i \in \mathbb{N}; j \in J_i}$ is an array $(y_{i,\ell})_{i \in \mathbb{N}; \ell \in L_i}$ in X which satisfies the following two properties:

$$(1) \quad \{y_{i,\ell} : \ell \in L_i\} \subseteq \{x_{i,j} : j \in J_i\} \text{ for all } i \in \mathbb{N}$$

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and

- if $J_i = \{j_{i,1} < j_{i,2} < \dots\}$, and $L_i = \{\ell_{i,1} < \ell_{i,2} < \dots\}$ for all $i \in \mathbb{N}$ then there
- (2) exists a $<_{rl}$ -order preserving map $H : \{(i, j_{i,k}) : i, k \in \mathbb{N}\} \rightarrow \{(i, j_{i,k}) : i, k \in \mathbb{N}\}$ such that $y_{i,\ell_{i,k}} = x_{H(i,j_{i,k})}$ for all $i, k \in \mathbb{N}$.

A **regular array** in a Banach space X is an array $(x_{i,j})_{i,j \in \mathbb{N}; i \leq j}$ which is a basic sequence when it is ordered with the reverse lexicographic order: $x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}, x_{1,4}, \dots$. For convenience, throughout this paper, we denote the index set of a regular array by I , i.e. $I = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$.

Theorem 1.2. *Let $(x_{i,j})_{(i,j) \in I}$ be an array in a Banach space X , $(M_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ be an increasing sequence of integers and $\varepsilon > 0$. Then there exists a regular subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for any finitely supported scalars $(a_{i,j})_{(i,j) \in I}$, $k_0 \in \mathbb{N}$ and $F \subseteq \mathbb{N}$ with $|F| \leq M_{\min(F)}$ and $k_0 \leq \min(F)$ we have*

$$\left\| \sum_{(i,j) \in I} a_{i,j} y_{i,j} \right\| \geq \frac{1}{2 + \varepsilon} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\|.$$

The “multiple of the inclusion plus compact problem” was asked by W.T. Gowers [9] and asks whether for every infinite dimensional Banach space X there exists a closed subspace Y of X and a bounded linear operator from Y to X which is not a compact perturbation of a multiple of the inclusion map from Y to X . Note that if a Banach space X contains an unconditional basic sequence $(x_n)_n$ then the operator $T \in \mathcal{L}([(x_n)_n], X)$ defined by $T(x_n) = (-1)^n x_n$ does not belong to $\mathbb{C}i_{Y \rightarrow X} + \mathcal{K}(Y, X)$ (where $\mathcal{L}(Y, X)$ and $\mathcal{K}(Y, X)$ denote respectively the space of all (linear bounded) operators and all compact operators from Y to X ; if Y is a subspace of X then $i_{Y \rightarrow X}$ denotes the inclusion from Y to X). Thus if a Banach space X contains an unconditional basic sequence then the “multiple of the inclusion plus compact” problem has an affirmative answer for X . Hence for the “multiple of the inclusion plus compact” problem we restrict our attention to Banach spaces with no unconditional basic sequences. Recall that by the Gowers’ dichotomy [9] every Banach space contains an unconditional basic sequence or a hereditarily indecomposable (HI) subspace. Recall that a Banach space X is called HI if no infinite dimensional closed subspace Y of X contains a complemented subspace Z which is of both infinite dimension and infinite codimension in Y [11]. Therefore for the “multiple of the inclusion plus compact problem” we only examine HI saturated Banach spaces. In Section 3 we prove Theorems 3.2 and 3.8 which give sufficient conditions on a Banach space X so that the “multiple of the inclusion plus compact” problem to have an affirmative answer in X , and extend results of [2] and [21]. In his 2000 dissertation N. Dew [7] introduced a new HI space which we refer to as space D . In Section 4 we examine some of the basic properties of D and we apply Theorem 3.8 to prove that the “multiple of the inclusion plus compact” problem has an affirmative answer in D .

2. EXTENSION OF ODELL’S SCHREIER UNCONDITIONALITY

In this section we will prove our main Theorem, 1.2. We need several remarks and lemmas until we reach its proof.

Remark 2.1. If $(x_{i,j})_{(i,j) \in I}$ is a regular array and $(y_{i,j})_{(i,j) \in I}$ is a subarray of $(x_{i,j})_{(i,j) \in I}$ then $(y_{i,j})_{(i,j) \in I}$ is also regular.

The proof of the following remark can be found in functional analysis text books such as [1, Theorem 1.5.2] or [14, Lemma 1.a.5]).

Remark 2.2. Let $(x_i)_{i=1}^N$ be a finite basic sequence in some infinite dimensional Banach space X having basis constant C . Let (y_i) be a seminormalized weakly null sequence X and $\varepsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $(x_1, x_2, \dots, x_N, y_n)$ is a basic sequence with constant $C(1 + \varepsilon)$.

By repeated application of Remark 2.2 we obtain the following.

Remark 2.3. Let X be a Banach space and $(x_{i,j})_{(i,j) \in I}$ be an array in X . Then there exists a subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ which is regular. Moreover, the basis constant of $(y_{i,j})_{(i,j) \in I}$ can be chosen to be arbitrarily close to 1.

For $p \in \mathbb{N}$ any element $\vec{a} = (a_i)_{i=1}^p$ of \mathbb{R}^p will be called a **p -pattern** and for such \vec{a} define $|\vec{a}| := p$. Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X . Let $f \in X^*$, $k \in \mathbb{N}$, $\vec{a} = (a_i)_{i=1}^p$ a p -pattern and $F = \{j_1, j_2, \dots, j_p\} \subseteq \mathbb{N}$. We say that f has pattern \vec{a} on (k, F) with respect to $(x_{i,j})_{(i,j) \in I}$ if $f(x_{k,j_i}) = a_i$ for all $i \in \{1, 2, \dots, p\}$.

Lemma 2.4. Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X , \vec{a} be a p -pattern, $\mathcal{F} \subseteq 2Ba(X^*)$, $\delta > 0$ and $i_0, j_0, k_0 \in \mathbb{N}$ with $j_0 \geq i_0$. Then there exists a subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for any $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$, with $(i_0, j_0) <_{rel} (k_0, \min(F))$ and $|F| = p$ we have the following:

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g(y_{i_0, j_0})| < \delta$.

Additionally, $(y_{i,j})_{(i,j) \in I}$ can be chosen to satisfy $y_{i,j} = x_{i,j}$ for all $(i, j) <_{rel} (i_0, j_0)$.

Proof. First note that there exists $m \in \mathbb{N}$ such that for all $f \in 2Ba(X^*)$ there exists $j' \in \{j_0, j_0 + 1, j_0 + 2, \dots, m\}$ with $|f(x_{i_0, j'})| < \delta$.

Otherwise assume that for all $m \in \mathbb{N}$ there exists $x_m^* \in 2Ba(X^*)$ with $|x_m^*(x_{i_0, j})| \geq \delta$ for $j \in \{j_0, \dots, m\}$. By passing to a subsequence and relabeling assume that (x_m^*) converges weak* to some $x^* \in 2Ba(X^*)$. Then $|x^*(x_{i_0, j})| \geq \delta$ for all $j \geq j_0$, which contradicts that each row, in particular $(x_{i_0, j})_{j=i_0}^\infty$, is weakly null.

Let $\mathbb{N} = \{m + 1, m + 2, \dots\}$. Divide the set $[N]^p$ of all p -element subsets of \mathbb{N} as follows: $[N]^p = \cup_{j=j_0}^{m+1} \mathcal{A}_j$ where for $j \in \{j_0, j_0 + 1, j_0 + 2, \dots, m\}$ we set

$$\mathcal{A}_j = \left\{ F \in [N]^p : \begin{array}{l} \text{there exists } f \in \mathcal{F} \text{ having pattern } \vec{a} \text{ on } (k_0, F) \\ \text{with respect to } (x_{i,j})_{(i,j) \in I} \text{ and } |f(x_{i_0, j})| < \delta \end{array} \right\}$$

and

$$\mathcal{A}_{m+1} = \left\{ F \in [N]^p : \text{there is no } f \in \mathcal{F} \text{ having pattern } \vec{a} \text{ on } (k_0, F) \text{ with respect to } (x_{i,j})_{(i,j) \in I} \right\}.$$

By Ramsey's theorem there exist a subsequence $(m_i)_{i=1}^\infty \in [N]$, and $j' \in \{j_0, j_0 + 1, \dots, m + 1\}$ such that $[(m_i)_{i=1}^\infty]^p \subset \mathcal{A}_{j'}$ (where for an infinite subset M of \mathbb{N} , $[M]$ denotes the set of

all infinite subsequences of M). We then can pass to a subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ by setting

$$y_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) <_{r\ell} (i_0, j_0) \\ x_{i,j'} & \text{if } (i,j) = (i_0, j_0) \\ x_{i,m_j} & \text{if } (i_0, j_0) <_{r\ell} (i,j). \end{cases}$$

Then $(y_{i,j})_{(i,j) \in I}$ satisfies the conclusion of the lemma, since if for some $F \subseteq \{k_0, k_0 + 1, \dots\}$ with $(i_0, j_0) <_{r\ell} (k_0, \min(F))$ and $|F| = p$ there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$, then the integer j' that was obtained by Ramsey's theorem could not be equal to $m + 1$, hence $j' \in \{j_0, j_0 + 1, \dots, m\}$ and the definition of $A_{j'}$ gives the conclusion. \square

Lemma 2.5. *Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X , \vec{A} be a finite set of patterns, $\mathcal{F} \subseteq 2Ba(X^*)$, $\delta > 0$ and $i_0, k_0 \in \mathbb{N}$. Then there exists some subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for any \vec{a} in \vec{A} , $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$, with $|F| = |\vec{a}|$ and $j_0 \in \mathbb{N}$ with $j_0 \geq i_0$ and $(i_0, j_0) <_{r\ell} (k_0, \min(F))$, we have the following:*

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g(y_{i_0, j_0})| < \delta$.

Additionally, we can assume that $x_{i,j} = y_{i,j}$ for all $(i,j) <_{r\ell} (i_0, i_0)$.

Proof. We begin by fixing one particular element \vec{a} in \vec{A} . Now apply Lemma 2.4 for $(x_{i,j})_{(i,j) \in I}$, \vec{a} , \mathcal{F} , δ , i_0 , k_0 and $j_0 = i_0$ to obtain some subarray $(y_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$ of $(x_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$ with the property that for any $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$, with $(i_0, j_0) <_{r\ell} (k_0, \min(F))$ and $|F| = |\vec{a}|$ we have the following. If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$ and $|g(y_{i_0, j_0}^{\vec{a}, i_0})| < \delta$. Moreover, $y_{i,j}^{\vec{a}, i_0} = x_{i,j}$ for all $(i,j) <_{r\ell} (i_0, j_0)$.

We repeat inductively on j_0 counting upward from i_0 . Thus we next apply Lemma 2.4 to $(y_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$, \vec{a} , \mathcal{F} , δ , i_0 , k_0 , $j_0 = i_0 + 1$, to obtain some subarray $(y_{i,j}^{\vec{a}, i_0+1})_{(i,j) \in I}$ of $(y_{i,j}^{\vec{a}, i_0})_{(i,j) \in I}$ with the property that for any $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $(i_0, j_0) <_{r\ell} (k_0, \min(F))$ and $|F| = |\vec{a}|$ we have the following. If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}, i_0+1})_{(i,j) \in I}$ then

- there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}, i_0+1})_{(i,j) \in I}$ and $|g(y_{i_0, i_0}^{\vec{a}, i_0+1})| < \delta$ (since $y_{i_0, i_0}^{\vec{a}, i_0+1} = y_{i_0, i_0}^{\vec{a}, i_0}$) and
- there exists $h \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}, i_0+1})_{(i,j) \in I}$ and $|h(y_{i_0, i_0+1}^{\vec{a}, i_0+1})| < \delta$.

Moreover, $y_{i,j}^{\vec{a}, i_0+1} = y_{i,j}^{\vec{a}, i_0}$ for all $(i,j) <_{r\ell} (i_0, i_0 + 1)$.

Continue in this manner for each $j_0 \in \{i_0 + 2, i_0 + 3, \dots\}$. Note that by fixing the elements of the subarray for $(i,j) <_{r\ell} (i_0, j_0)$ at each step j_0 , there exists a subarray after infinitely many steps which possesses the properties of all the previous subarrays. We call this "limit" subarray $(y_{i,j}^{\vec{a}})_{(i,j) \in I}$ and notice it has the property that for any $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = |\vec{a}|$ and for all $j_0 \in \mathbb{N}$ with $(i_0, j_0) <_{r\ell} (k_0, \min(F))$, we have the following:

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{\vec{a}})_{(i,j) \in I}$ and $|g(y_{i_0, j_0}^{\vec{a}})| < \delta$.

Notice also that any further subarray of $(y_{i,j}^{\vec{a}})_{(i,j) \in I}$ has this same property. Then repeat the above process for each $\vec{a} \in \vec{A}$ to obtain the desired array. \square

Notice that if $(y_{i,j})_{(i,j) \in I}$ is the result of applying Lemma 2.4 to some regular array and $(z_{i,j})_{(i,j) \in I}$ is a subarray of $(y_{i,j})_{(i,j) \in I}$ then $(z_{i,j})_{(i,j) \in I}$ may not satisfy the conclusion of Lemma 2.4. However, if $(y_{i,j})_{(i,j) \in I}$ is the result of applying Lemma 2.5 to some regular array and $(z_{i,j})_{(i,j) \in I}$ is a regular subarray of $(y_{i,j})_{(i,j) \in I}$ then $(z_{i,j})_{(i,j) \in I}$ does satisfy the conclusion of Lemma 2.5. This idea is summarized in the following remark.

Remark 2.6. *Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X , \vec{A} be a finite set of patterns, $\mathcal{F} \subseteq 2Ba(X^*)$, $i_0, k_0 \in \mathbb{N}$ and $\delta > 0$. Then there exists a subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that if $(z_{i,j})_{(i,j) \in I}$ is any subarray of $(y_{i,j})_{(i,j) \in I}$, then for any \vec{a} in \vec{A} , $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$, with $|F| = |\vec{a}|$ and $j_0 \in \mathbb{N}$ with $(i_0, j_0) <_{r\ell} (k_0, \min(F))$, we have the following:*

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(z_{i,j})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(z_{i,j})_{(i,j) \in I}$ and $|g(z_{i_0, j_0})| < \delta$.

Additionally, we can assume that $x_{i,j} = y_{i,j}$ for all $(i, j) <_{r\ell} (i_0, i_0)$.

Lemma 2.7. *Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X , \vec{A} be a finite set of patterns, $\mathcal{F} \subseteq 2Ba(X^*)$ and $\delta > 0$. Then there exists some subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for all \vec{a} in \vec{A} , $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = |\vec{a}|$ and $(i_0, j_0) \in I$ with $(1, k_0) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, \min(F))$ we have the following:*

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g(y_{i_0, j_0})| < \delta$.

Proof. We will apply Remark 2.6 inductively with the subarray changing at each step, but \vec{A} , \mathcal{F} and δ remaining as in the hypothesis and (i_0, k_0) cycling through \mathbb{N}^2 . We create the final subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ inductively one column at a time. At the j_0 step of the induction we create a subarray $(y_{i,j}^{j_0})_{(i,j) \in I}$ of $(y_{i,j+j_0-1}^{j_0-1})_{(i,j) \in I}$ (where for $j_0 = 1$, $(y_{i,j}^0)_{(i,j) \in I} = (x_{i,j})_{(i,j) \in I}$) and we set $y_{i,j_0} = y_{i,j_0}^{j_0}$ for $i \in \{1, 2, \dots, j_0\}$.

COLUMN 1: Apply Remark 2.6 to $(x_{i,j})_{(i,j) \in I}$, \vec{A} , \mathcal{F} , δ , $i_0 = 1$, and $k_0 = 1$ to obtain a subarray $(y_{i,j}^1)_{(i,j) \in I}$ with the property that for all \vec{a} in \vec{A} , $F \subseteq \{1, 2, \dots\}$ with $|F| = |\vec{a}|$ such that $(1, 1) <_{r\ell} (k_0, \min(F))$ we have the following:

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^1)_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^1)_{(i,j) \in I}$ and $|g(y_{1,1}^1)| < \delta$.

We then fix column 1 of $(y_{i,j})_{(i,j) \in I}$ by setting $y_{1,1} := y_{1,1}^1$.

COLUMN 2: Apply Remark 2.6 to $(y_{i,j+1}^1)_{(i,j) \in I}$, \vec{A} , \mathcal{F} , δ , successively for each $(i_0, k_0) \in \{(1, 2), (2, 1), (2, 2)\}$ to obtain a subarray $(y_{i,j}^2)_{(i,j) \in I}$ with the property that for all \vec{a} in \vec{A} ,

$F \subseteq \{2, 3, \dots\}$ with $|F| = |\vec{a}|$, $i_0 \in \{1, 2\}$ and $k_0 \in \{1, 2\}$ such that $(i_0, 2) <_{r\ell} (k_0, \min(F))$ we have the following:

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^2)_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^2)_{(i,j) \in I}$ and $|g(y_{i_0,2}^2)| < \delta$.

We then fix column 2 of $(y_{i,j})_{(i,j) \in I}$ by setting $y_{i,2} := y_{i,2}^2$ for $i \in \{1, 2\}$.

COLUMN j_0 : Apply Remark 2.6 to $(y_{i,j+j_0-1}^{j_0-1})_{(i,j) \in I}$, \vec{A} , \mathcal{F} , δ , successively for each $(i_0, k_0) \in \{(i, j_0) : 1 \leq i < j_0\} \cup \{(j_0, j) : 1 \leq j \leq j_0\}$ to obtain a subarray $(y_{i,j}^{j_0})_{(i,j) \in I}$ with the property that for all \vec{a} in \vec{A} , $F \subseteq \{j_0, j_0+1, \dots\}$ with $|F| = |\vec{a}|$, $i_0 \in \{1, 2, \dots, j_0\}$ and $k_0 \in \{1, 2, \dots, j_0\}$ such that $(i_0, j_0) <_{r\ell} (k_0, \min(F))$ we have the following:

If there exists $f \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{j_0})_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^{j_0})_{(i,j) \in I}$ and $|g(y_{i_0,j_0}^{j_0})| < \delta$.

We then fix column j_0 of $(y_{i,j})_{(i,j) \in I}$ by setting $y_{i,j_0} := y_{i,j_0}^{j_0}$ for $i \in \{1, 2, \dots, j_0\}$.

Let \vec{a} in \vec{A} , $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0+1, k_0+2, \dots\}$ with $|F| = |\vec{a}|$ and $(i_0, j_0) \in I$ with $(1, k_0) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, \min(F))$ all be given. Since $(1, k_0) \leq_{r\ell} (i_0, j_0)$ we have that $k_0 \leq j_0$. Since $(i_0, j_0) <_{r\ell} (k_0, \min(F))$ we have that there exists a set $G \subseteq \mathbb{N}$ with $|G| = |F|$, $\min(G) \geq k_0$ and $(y_{k_0,j})_{j \in F} = (y_{k_0,j}^{j_0})_{j \in G}$. Thus if there exists $f \in \mathcal{F}$ which has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ then f has pattern \vec{a} on (k_0, G) with respect to $(y_{i,j}^{j_0})_{(i,j) \in I}$, therefore by the property of $(y_{i,j}^{j_0})_{(i,j) \in I}$ we obtain that there exists $g \in \mathcal{F}$ which has pattern \vec{a} on (k_0, G) with respect to $(y_{i,j}^{j_0})_{(i,j) \in I}$ and $|g(y_{i_0,j_0}^{j_0})| < \delta$. Hence g has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g(y_{i_0,j_0})| < \delta$. \square

Lemma 2.8. *Let $(x_{i,j})_{(i,j) \in I}$ be a regular array in a Banach space X , $\varepsilon > 0$, and $k \in \mathbb{N}$. Then there exists some subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for any pattern \vec{a} in $[-1, 1]^p$ for some $p \leq k$, for any $k_0 \in \mathbb{N}$ and any $F \subseteq \{k_0, k_0+1, k_0+2, \dots\}$ with $|F| = |\vec{a}|$ we have the following:*

If there exists $f \in BaX^$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ then there exists $g \in (1 + \varepsilon)BaX^*$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and*

$$\sum_{\{(i,j) \in I: j \geq k_0\} \setminus \{(k_0, \ell): \ell \in F\}} |g(y_{i,j})| < \varepsilon$$

Proof. Let $(\delta_j)_{j=0}^\infty \subseteq (0, 1)$ such that

$$(3) \quad \frac{1}{\inf_{i,j} \|x_{i,j}\|} \left(4Ck\delta_0 + \sum_{j=1}^\infty 4Cj\delta_j \right) < \varepsilon$$

where C is the basis constant for the regular array $(x_{i,j})_{(i,j) \in I}$. Let A_0 be a δ_0 net for $[-1, 1]$ containing zero and for each $j \in \mathbb{N}$ choose a δ_j net B_j for $[-1, 1]$ with $\{0\} \subseteq A_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$. Let

$$(4) \quad \vec{A} = \{\vec{a} = (a_i)_{i=1}^p \in A_0^p : \text{where } 1 \leq p \leq k\}$$

and

$$(5) \quad \mathcal{F} = \{f \in (1 + \frac{\varepsilon}{2})Ba(X^*) : f(x_{i,j}) \in B_j \text{ for all } (i,j) \in I\},$$

where with out loss of generality we assume $\varepsilon < 2$ so $\mathcal{F} \subseteq 2Ba(X^*)$. Since $0 \in A_0$ the zero functional is in \mathcal{F} therefore \mathcal{F} is nonempty .

We construct the subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ inductively. First we will construct a subarray $(y_{i,j}^1)_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$, then for $j \in \mathbb{N}$ for $j \geq 2$ we will construct a subarray $(y_{i,k}^j)_{(i,k) \in I}$ of $(y_{i,k+j-1}^{j-1})_{(i,k) \in I}$. Once the subarray $(y_{i,k}^j)_{(i,k) \in I}$ has been constructed we set $y_{i,j} := y_{i,j}^j$ for $1 \leq i \leq j$. Since $(y_{i,k}^j)_{(i,k) \in I}$ is a subarray of $(y_{i,k+j-1}^{j-1})_{(i,k) \in I}$ and $y_{i,j} = y_{i,j}^j$ for $1 \leq i \leq j$, we have that $(y_{i,j})_{(i,j) \in I}$ is a subarray of $(x_{i,j})_{(i,j) \in I}$.

Apply Lemma 2.7 to $(x_{i,j})_{(i,j) \in I}$, \vec{A} , δ_1 , \mathcal{F} to obtain a subarray $(y_{i,j}^1)_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$ such that for all $\vec{a} \in \vec{A}$, $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = |\vec{a}|$ and $(i_0, j_0) \in I$ with $(1, k_0) \leq_{rl} (i_0, j_0) <_{rl} (k_0, \min(F))$ we have: if there exists $f \in \mathcal{F}$ having pattern $\vec{a} \in \vec{A}$ on (k_0, F) with respect to $(y_{i,j}^1)_{(i,j) \in I}$ then there exists $g \in \mathcal{F}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^1)_{(i,j) \in I}$ and $|g(y_{i_0, j_0}^1)| < \delta_1$. Define the elements of the first column of $(y_{i,j})_{(i,j) \in I}$ by setting $y_{1,1} := y_{1,1}^1$. Define for each $b \in B_1$ the set

$$(6) \quad \mathcal{F}_b = \{f \in \mathcal{F} : f(y_{1,1}) = b\}.$$

Apply Lemma 2.7 to $(y_{i,j+1}^1)_{(i,j) \in I}$, \vec{A} , δ_2 , \mathcal{F}_b successively for each $b \in B_1$ to obtain a subarray $(y_{i,j}^2)_{(i,j) \in I}$ of $(y_{i,j+1}^1)_{(i,j) \in I}$ such that for all $\vec{a} \in \vec{A}$, $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = |\vec{a}|$ and $(i_0, j_0) \in I$ with $(1, k_0) \leq_{rl} (i_0, j_0) <_{rl} (k_0, \min(F))$ we have for all $b \in B_1$: if there exists $f \in \mathcal{F}_b$ having pattern $\vec{a} \in \vec{A}$ on (k_0, F) with respect to $(y_{i,j}^2)_{(i,j) \in I}$ then there exists $g \in \mathcal{F}_b$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^2)_{(i,j) \in I}$ and $|g(y_{i_0, j_0}^2)| < \delta_2$.

Define the elements of the second column of $(y_{i,j})_{(i,j) \in I}$ by setting $y_{1,2} := y_{1,2}^2$, and $y_{2,2} := y_{2,2}^2$. For each $\vec{b} = (b_1, b_2, b_3) \in B_1 \times B_2 \times B_2$ set

$$(7) \quad \mathcal{F}_{\vec{b}} = \{f \in \mathcal{F} : f(y_{1,1}) = b_1, f(y_{1,2}) = b_2 \text{ and } f(y_{2,2}) = b_3\}.$$

Apply Lemma 2.7 to $(y_{i,j+2}^2)_{(i,j) \in I}$, \vec{A} , δ_3 , $\mathcal{F}_{\vec{b}}$ successively for each $\vec{b} \in B_1 \times B_2 \times B_2$ to obtain a subarray $(y_{i,j}^3)_{(i,j) \in I}$ of $(y_{i,j+2}^2)_{(i,j) \in I}$ such that for all $\vec{a} \in \vec{A}$, $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = |\vec{a}|$ and $(i_0, j_0) \in I$ with $(1, k_0) \leq_{rl} (i_0, j_0) <_{rl} (k_0, \min(F))$ we have for all $\vec{b} \in B_{p_{1,1}} \times B_{p_{1,2}} \times B_{p_{2,2}}$: if there exists $f \in \mathcal{F}_{\vec{b}}$ having pattern $\vec{a} \in \vec{A}$ on (k_0, F) with respect to $(y_{i,j}^3)_{(i,j) \in I}$ then there exists $g \in \mathcal{F}_{\vec{b}}$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j}^3)_{(i,j) \in I}$ and $|g(y_{i_0, j_0}^3)| < \delta_3$. Define the elements in the third column $(y_{i,j})_{(i,j) \in I}$ by setting $y_{1,3} := y_{1,3}^3$, $y_{2,3} := y_{2,3}^3$ and $y_{3,3} := y_{3,3}^3$. Continue in this manner to create the subarray $(y_{i,j})_{(i,j) \in I}$ of $(x_{i,j})_{(i,j) \in I}$.

Let $\tilde{f} \in Ba(X^*)$, \vec{c} be a p -pattern for $p \leq k$, $k_0 \in \mathbb{N}$ and $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| = p$ such that \tilde{f} has pattern \vec{c} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$.

First it is easy to see using (3) that since $\tilde{f} \in Ba(X^*)$ there is $f \in \mathcal{F}$ (as defined in (5)) such that

- for all $j \in F$ we have $f(y_{k_0,j}) \in A_0$ and $|\tilde{f}(y_{k_0,j}) - f(y_{k_0,j})| \leq \delta_0$, and
- if we define the finite set $G \subset \mathbb{N}$ by $(y_{k_0,j})_{j \in F} = (x_{k_0,j})_{j \in G}$, then for all $(i,j) \in I \setminus \{(k_0, j) : j \in G\}$ we have that $|\tilde{f}(x_{i,j}) - f(x_{i,j})| \leq \delta_j$ and $f(x_{i,j}) \in B_j$.

Let $\vec{a} := (f(y_{k_0,j}))_{j \in F}$ and note that f has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$. We will find a functional $g \in (1 + \frac{\varepsilon}{2})Ba(X^*)$ such that g has pattern \vec{a} on F with respect to $(y_{i,j})_{(i,j) \in I}$ and $\sum_{I' \setminus \{(k_0,j): j \in F\}} |g(y_{i,j})| < \varepsilon$.

We proceed to find such functional g . But first a bit of notation, for a p -pattern $\vec{\alpha} = (\alpha_i)_{i=1}^p$ we define its derivative $\vec{\alpha}' = (\alpha_{i-1})_{i=2}^p$.

We will walk through the index set $I' = \{(i, j) \in I : j \geq k_0\}$ proceeding through this set in $<_{r\ell}$ -order and at each step find a functional $g_{i,j}$ with the property that if $(i, j) \notin \{(k_0, j) : j \in F\}$ then $|g_{i,j}(y_{i,j})|$ will be small and “agree” with the previous functional on $\{(i', j') \in I' : (i', j') <_{r\ell} (i, j)\}$. If $(i, j) \in \{(k_0, j) : j \in F\}$ then we will not change the previously defined functional. We will assume $k_0 \geq 3$ for purposes of demonstrating the construction, but if $k_0 = 1$ or 2 then we proceed similarly.

STEP $(1, k_0)$: Note $(1, k_0) \notin \{(k_0, j) : j \in F\}$ (since $k_0 \geq 3$). Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), f(y_{3,1}), \dots, f(y_{k_0-1, k_0-1})).$$

Then $f \in \mathcal{F}_{\vec{b}}$, f has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$, $(y_{i,j})_{(i,j) \in I'}$ is a subarray of $(y_{(i,j)}^{k_0})_{(i,j) \in I'}$ and $(1, k_0) \leq_{r\ell} (1, k_0) <_{r\ell} (k_0, \min(F))$, (the last inequality is valid since $k_0 \geq 3$). Thus there exists $g_{1,k_0} \in \mathcal{F}_{\vec{b}}$ such that g_{1,k_0} has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{1,k_0}(y_{1,k_0})| < \delta_{k_0}$. Set $F_{1,k_0} = F$ and $\vec{a}_{1,k_0} = \vec{a}$.

STEP $(2, k_0)$: Note that $(2, k_0) \notin \{(k_0, j) : j \in F_{1,k_0}\}$ (since $k_0 \geq 3$). Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), \dots, f(y_{k_0-1, k_0-1}), g_{1,k_0}(y_{1,k_0})).$$

Then $g_{1,k_0} \in \mathcal{F}_{\vec{b}}$, g_{1,k_0} has pattern \vec{a}_{1,k_0} on (k_0, F_{1,k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$ and $(1, k_0) <_{r\ell} (2, k_0) <_{r\ell} (k_0, \min(F))$, (the last inequality is valid because $k_0 \geq 3$). Thus there exists $g_{2,k_0} \in \mathcal{F}_{\vec{b}}$ such that g_{2,k_0} has pattern \vec{a}_{1,k_0} on (k_0, F_{1,k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{2,k_0}(y_{2,k_0})| < \delta_{k_0}$. Set $F_{2,k_0} = F_{1,k_0}$ and $\vec{a}_{2,k_0} = \vec{a}_{1,k_0}$.

We continue similarly until the $(k_0 - 1, k_0)$ step. The step (k_0, k_0) is slightly different. We separate this step into two different cases depending on whether or not $(k_0, k_0) \in \{(k_0, j) : j \in F_{k_0-1, k_0}\}$.

STEP (k_0, k_0) : If $(k_0, k_0) \in \{(k_0, j) : j \in F_{k_0-1, k_0}\}$ then set $g_{k_0, k_0} = g_{k_0-1, k_0}$, $F_{k_0, k_0} = F_{k_0-1, k_0} \setminus \{k_0\}$ and $\vec{a}_{k_0, k_0} = \vec{a}'_{k_0-1, k_0}$.

If $(k_0, k_0) \notin \{(k_0, j) : j \in F_{k_0-1, k_0}\}$ then $(k_0, k_0) <_{r\ell} (k_0, \min(F))$. Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), \dots, f(y_{k_0-1, k_0-1}), g_{1,k_0}(y_{1,k_0}), g_{2,k_0}(y_{2,k_0}), \dots, g_{k_0-1, k_0}(y_{k_0-1, k_0})).$$

Then $g_{k_0-1, k_0} \in \mathcal{F}_{\vec{b}}$, g_{k_0-1, k_0} has pattern \vec{a}_{k_0-1, k_0} on (k_0, F_{k_0-1, k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$, $(y_{i,j})_{(i,j) \in I'}$ is a subarray of $(y_{i,j}^{k_0})_{(i,j) \in I}$ and $(1, k_0) \leq_{r\ell} (k_0, k_0) <_{r\ell} (k_0, \min(F))$. Thus there exists $g_{k_0, k_0} \in \mathcal{F}_{\vec{b}}$ such that g_{k_0, k_0} has pattern \vec{a}_{k_0-1, k_0} on (k_0, F_{k_0-1, k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{k_0, k_0}(y_{k_0, k_0})| < \delta_{k_0}$. In this case set $F_{k_0, k_0} = F_{k_0-1, k_0}$ and $\vec{a}_{k_0, k_0} = \vec{a}_{k_0-1, k_0}$.

Then start again with the first entry $(1, k_0 + 1)$ of the next column as in steps $(1, k_0)$ and $(2, k_0)$.

STEP $(1, k_0 + 1)$: Note that $(1, k_0 + 1) \notin \{(k_0, j) : j \in F_{k_0, k_0}\}$, (since $k_0 \geq 3$). Let

$$\vec{b} = (f(y_{1,1}), f(y_{2,1}), f(y_{2,2}), \dots, g_{1,k_0}(y_{1,k_0}), g_{2,k_0}(y_{2,k_0}), g_{k_0, k_0}(y_{k_0, k_0})).$$

Then $g_{k_0, k_0} \in \mathcal{F}_{\vec{b}}$, g_{k_0, k_0} has pattern \vec{a}_{k_0, k_0} on (k_0, F_{k_0, k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$, $(y_{i,j})_{(i,j) \in I'}$ is a subarray of $(y_{i,j}^{k_0})_{(i,j) \in I}$ and $(1, k_0) \leq_{r\ell} (1, k_0 + 1) <_{r\ell} (k_0, \min(F))$, (the last inequality is valid since $k_0 \geq 3$). Thus there exists $g_{1, k_0+1} \in \mathcal{F}_{\vec{b}}$ such that g_{1, k_0+1} has pattern \vec{a}_{k_0, k_0} on

(k_0, F_{k_0, k_0}) with respect to $(y_{i,j})_{(i,j) \in I}$ and $|g_{1, k_0+1}(y_{1, k_0+1})| < \delta_{k_0+1}$. Set $F_{1, k_0+1} = F_{k_0, k_0}$ and $\vec{a}_{1, k_0+1} = \vec{a}_{k_0, k_0}$.

Continue in this manner to generate a sequence of functionals $(g_{i,j})_{(i,j) \in I'}$. We only need to distinguish two cases every time we reach the k_0 row as in step (k_0, k_0) . Let $g \in (1 + \frac{\varepsilon}{2})Ba(X^*)$ be a weak*-accumulation point of sequence $(g_{i,j})_{(i,j) \in I'}$. Note g has the following two properties:

- g has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$, and
- $\sum_{(i,j) \in I' \setminus \{(k_0, \ell): \ell \in F\}} |g(y_{i,j})| < \varepsilon$.

Since $g(y_{k_0, \ell}) = f(y_{k_0, \ell}) \in A_0$ for all $\ell \in F$ and $|f(y_{k_0, \ell}) - \tilde{f}(y_{k_0, \ell})| < \delta_0$ for all $\ell \in F$, (3) implies that there exists $\tilde{g} \in X^*$ such that $\|\tilde{g}\| \leq \|g\| + \frac{\varepsilon}{2} \leq 1 + \varepsilon$ and $\tilde{g}(y_{k_0, \ell}) = \tilde{f}(y_{k_0, \ell}) \in A_0$ for all $\ell \in F$ (thus \tilde{g} has pattern \vec{c} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$) and

$$\sum_{(i,j) \in I' \setminus \{(k_0, \ell): \ell \in F\}} |\tilde{g}(y_{i,j})| < \varepsilon$$

completing the proof. □

Finally we arrive to the

Proof of Theorem 1.2. First, by Remark 2.3 assume without loss of generality that $(x_{i,j})_{(i,j) \in I}$ is regular. Let $\eta > 0$ such that

$$(8) \quad 2(2\eta + 1) \leq 2 + \frac{\varepsilon}{2}$$

where C is the basis constant of the regular array $(x_{i,j})_{(i,j) \in I}$. Apply Lemma 2.8 to $(x_{i,j})_{(i,j) \in I}$, η and M_1 to get $(y_{i,j}^1)_{(i,j) \in I}$. Define $y_{1,1} := y_{1,1}^1$.

Apply Lemma 2.8 to $(y_{i,j+1}^1)_{(i,j) \in I}$, η and M_2 to get $(y_{i,j}^2)_{(i,j) \in I}$. Define $y_{i,2} := y_{i,2}^2$ for $i = 1, 2$.

Assuming that $(y_{i,j}^{\ell-1})_{(i,j) \in I}$ has been defined (and thus $(y_{i,j})_{(i,j) \in I; j < \ell}$) has also been defined) apply Lemma 2.8 to $(y_{i,j+\ell-1}^{\ell-1})_{(i,j) \in I}$, η and M_ℓ to get $(y_{i,j}^\ell)_{(i,j) \in I}$. Define $y_{i,\ell} := y_{i,\ell}^\ell$ for $i = 1, 2, \dots, \ell$.

Inductively construct the entire array $(y_{i,j})_{(i,j) \in I}$ and notice $(y_{i,j})_{(i,j) \in I}$ is regular by Remark 2.1.

Let $k_0 \in \mathbb{N}$, $F \subseteq \{k_0, k_0 + 1, k_0 + 2, \dots\}$ with $|F| \leq M_{\min(F)}$ and finitely supported scalars $(a_{i,j})_{(i,j) \in I}$ be given. We can assume without loss of generality that

$$\left\| \sum_{(i,j) \in I} a_{i,j} y_{i,j} \right\| = 1.$$

Then $|a_{i,j}| \leq 2C$ for $(i,j) \in I$. Let $f \in Ba(X^*)$ such that

$$f \left(\sum_{j \in F} a_{k_0, j} y_{k_0, j} \right) = \left\| \sum_{j \in F} a_{k_0, j} y_{k_0, j} \right\|.$$

Let $\vec{a} = (f(y_{k_0, j}))_{j \in F}$ be a p -pattern where $p = |F|$. Obviously f has pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$. Then by considering the subarray $(y_{i,j})_{(i,j) \in I, j \geq \min(F)}$ of $(y_{i,j}^{\min(F)})_{(i,j) \in I, j \geq \min(F)}$ we obtain by the above that there exists $g \in (1 + \eta)Ba(X^*)$ having pattern \vec{a} on (k_0, F) with respect to $(y_{i,j})_{(i,j) \in I}$ and $\sum_{\{(i,j) \in I: j \geq \min(F)\} \setminus \{(k_0, \ell): \ell \in F\}} |g(y_{i,j})| < \eta$. Thus

$$\begin{aligned}
(9) \quad 1 &= \left\| \sum a_{i,j} y_{i,j} \right\| \geq \frac{1}{2C} \left\| \sum_{j \geq \min(F)} a_{i,j} y_{i,j} \right\| \geq \frac{1}{2C(1+\eta)} \left| \sum_{j \geq \min(F)} a_{i,j} g(y_{i,j}) \right| \\
&\geq \frac{1}{2C(1+\eta)} \left| \sum_{j \in F} a_{k_0,j} g(y_{k_0,j}) \right| - \frac{1}{2C(1+\eta)} \left| \sum_{\{(i,j) \in I: j \geq \min(F)\} \setminus \{(k_0,j): j \in F\}} a_{i,j} g(y_{i,j}) \right| \\
&\geq \frac{1}{2C(1+\eta)} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\| - \frac{1}{2C(1+\eta)} \sum_{\{(i,j) \in I: j \geq \min(F)\} \setminus \{(k_0,j): j \in F\}} |a_{i,j}| |g(y_{i,j})| \\
&\geq \frac{1}{2C(1+\eta)} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\| - \frac{1}{1+\eta} \sum_{\{(i,j): j \geq \min(F)\} \setminus \{(k_0,j): j \in F\}} |g(y_{i,j})| \\
&\geq \frac{1}{2C(1+\eta)} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\| - \frac{1}{1+\eta} \eta.
\end{aligned}$$

Thus by (8) and (9) we have

$$\left\| \sum_{(i,j) \in I} a_{i,j} y_{i,j} \right\| = 1 \geq \frac{1}{2C(2\eta+1)} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\| \geq \frac{1}{C(2+\frac{\varepsilon}{2})} \left\| \sum_{j \in F} a_{k_0,j} y_{k_0,j} \right\|$$

Since we can choose C , the basis constant of our regular array, arbitrarily close to 1 (see Remark 2.3) we have shown the result. \square

3. EXISTENCE OF NON-TRIVIAL OPERATORS

In this section we give an application of Theorem 1.2 to the "multiple of the inclusion plus compact problem". The problem has been previously studied in [9], [2] and [21]. The main results of this section are Theorems 3.2 and 3.8. In order to formulate these results we need some definitions.

Recall that [4, 5, 6] for every seminormalized basic sequence (y_n) in a Banach space X and for every $(\varepsilon_n) \searrow 0$ there exists a subsequence (x_n) of (y_n) and a seminormalized basic sequence (\tilde{x}_n) (not necessarily in X) such that for all $n \in \mathbb{N}$, scalars $(a_i)_{i=1}^n$ with $|a_i| \leq 1$ and $n \leq k_1 < \dots < k_n$,

$$\left| \left\| \sum_{i=1}^n a_{k_i} x_{k_i} \right\| - \left\| \sum_{i=1}^n a_i \tilde{x}_i \right\| \right| < \varepsilon_n.$$

The sequence (\tilde{x}_n) is called a **spreading model** of (x_n) . If (x_n) is a seminormalized weakly null basic sequence then (\tilde{x}_n) is an unconditional basic sequence. Thus if X is an HI Banach space and (x_n) is a seminormalized basic sequence in X with spreading model (\tilde{x}_n) , then it may be easier to study (\tilde{x}_n) than to study (x_n) itself.

A similar notion to the next one was introduced in [21].

Definition 3.1. *Let (x_n) and (z_n) be two seminormalized basic sequences (not necessarily in the same Banach space), such that (z_n) is not equivalent to the unit vector basis of c_0 . We*

say that (z_n) dominates (x_n) on small coefficients, (denoted by $(x_n) \ll (z_n)$ and abbreviated as “ (z_n) s.c. dominates (x_n) ”), if

$$(10) \quad \liminf_{\varepsilon \searrow 0} \{ \|\sum a_i z_i\| : |a_i| \leq \varepsilon \text{ and } \|\sum a_i x_i\| = 1 \} = \infty,$$

where $\inf \emptyset = \infty$.

Theorem 3.2. *Let X be a Banach space containing seminormalized basic sequences $(x_i)_i$ and $(x_i^n)_i$ for all $n \in \mathbb{N}$, such that $0 < \inf_{n,i} \|x_i^n\| \leq \sup_{n,i} \|x_i^n\| < \infty$. Let $(z_i)_i$ be a basic sequence not necessarily in X . Assume that (x_i) satisfies:*

$$(11) \quad \text{The sequence } (x_i)_i \text{ has a spreading model } (\tilde{x}_i)_{i=1}^\infty \text{ such that } (\tilde{x}_i)_{i \in \mathbb{N}} \ll (z_i)_{i \in \mathbb{N}}.$$

Assume that for all $n \in \mathbb{N}$ the sequence $(x_i^n)_i$ satisfies:

$$(12) \quad \begin{aligned} &\text{The sequence } (x_i^n)_i \text{ has a spreading model } (\tilde{x}_i^n)_i \text{ such that} \\ &(z_i)_{i=1}^n \text{ is } C\text{-dominated by } (\tilde{x}_i^n)_{i=1}^n \text{ for some } C \text{ independent of } n. \end{aligned}$$

Then there exists a subspace Y of X which has a basis and an operator $T \in \mathcal{L}(Y, X)$ which is not a compact perturbation of a multiple of the inclusion map.

For the proof of Theorem 3.2 we need the following three lemmas whose proofs are postponed. For their formulation we need to introduce the following

Definition 3.3. *A seminormalized basic sequence (x_n) has Property P if for all $\rho > 0$ there exists an $M = M(\rho) \in \mathbb{N}$, such that if $\|\sum a_i z_i\| = 1$ then $|\{i : |a_i| \geq \rho\}| \leq M$.*

Obviously Property P can be equivalently stated with the equality “ $\|\sum a_i z_i\| = 1$ ” replaced by “ $\|\sum a_i z_i\| \leq 1$ ” (and everything else staying unchanged).

Lemma 3.4. *Let X be a Banach space which does not contain an isomorphic copy of ℓ_1 and let (z_i) be a basic sequence not necessarily in X . Assume that X contains basic sequences (x_i) and (x_i^n) for all $n \in \mathbb{N}$ such that $0 < \inf_{n,i} \|x_i^n\| \leq \sup_{n,i} \|x_i^n\| < \infty$ and conditions (11) and (12) of Theorem 3.2 are satisfied. Then there exist seminormalized basic sequences $(X_i)_i$ and (Z_i) not necessarily in X and for each $n \in \mathbb{N}$ there exists a basic sequence $(X_i^n)_i$ in X such that $0 < \inf_{n,i} \|X_i^n\| \leq \sup_{n,i} \|X_i^n\| < \infty$ and the following conditions are satisfied:*

- (a) *The sequence $(X_i)_i$ dominates $(\underline{x}_i)_i$, where $(\underline{x}_i)_i = (x_{m_{2i}} - x_{m_{2i-1}})$ for some increasing subsequence (m_i) of positive integers, and (\underline{x}_i) is weakly null.*
- (b) *(X_i) and (Z_i) satisfy (11) of Theorem 3.2.*
- (c) *(X_i) has Property P.*
- (d) *$(X_i^n)_i$ and (Z_i) satisfy (12) of Theorem 3.2.*
- (e) *The sequence $(X_i^n)_i$ is weakly null for all $n \in \mathbb{N}$.*

Lemma 3.5. *Let X be a Banach space, (X_i) be a seminormalized basic sequence in X having Property P and (Z_i) be a seminormalized basic sequence not necessarily in X . Assume that the sequence (X_i) and (Z_i) satisfy condition (11) of Theorem 3.2. Then for all $(\delta_n)_{n=2}^\infty \subseteq (0, \infty)$ there exists an increasing sequence $M_1 < M_2 < \dots$ of positive integers and a subsequence (X_{n_i}) of (X_i) such that for all $(a_i) \in c_{00}$,*

$$(13) \quad \|\sum a_i X_{n_i}\| \leq \sup_{n \in \mathbb{N}} \sup_{n \leq F \subseteq \mathbb{N}; |F| \leq M_n} \delta_n \|\sum_{i \in F} a_i Z_i\|,$$

for some δ_1 (where “ $n \leq F$ ” means $n \leq \min(F)$).

Lemma 3.6. *Let X be a Banach space, (δ_n) be a summable sequence of positive numbers, $(M_n)_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence of positive integers, (Z_n) be a seminormalized basic sequence (not necessarily in X) and for every $n \in \mathbb{N}$ let $(X_j^n)_{j=1}^\infty$ be a weakly null basic sequence in X having spreading model $(\tilde{X}_j^n)_{j=1}^\infty$ such that $0 < \inf_{n,j} \|X_j^n\| \leq \sup_{n,j} \|X_j^n\| < \infty$ and condition (12) of Theorem 3.2 is satisfied. Then there exists a seminormalized weakly null basic sequence (y_i) in X such that*

$$(14) \quad \frac{1}{6C} \sup_n \sup_{n \leq F \subseteq \mathbb{N}; |F| \leq M_n} \delta_n \left\| \sum_{i \in F} a_i Z_i \right\| \leq \left\| \sum a_i y_i \right\|.$$

Moreover, $\|y_j\| \geq \frac{\delta_1}{2} \inf_{n,m} \|X_m^n\|$. Furthermore, if (x_i^*) is any given sequence of functionals in X^* and $\varepsilon > 0$ we can choose (y_i) to satisfy $|x_i^* y_i| < \varepsilon$.

We now present the

Proof of Theorem 3.2. Obviously, if ℓ_1 embeds in X , (or if, more generally, X contains an unconditional basic sequence), then the conclusion of Theorem 3.2 is satisfied. Thus we can assume that X does not contain an isomorphic copy of ℓ_1 and consider the sequences (X_i) , (Z_i) , (X_i^n) and (\underline{x}_i) which satisfy the conclusion of Lemma 3.4. Let $(\delta_n)_{n=2}^\infty$ be a summable sequence of positive numbers. Apply Lemma 3.5 to obtain a subsequence (X_{n_i}) , $\delta_1 > 0$ and an increasing sequence $(M_n)_{n \in \mathbb{N}}$ of positive integers which satisfies (13). For every $i \in \mathbb{N}$ let a norm 1 functional x_i^* satisfying $x_i^* \underline{x}_{n_i} = \|\underline{x}_{n_i}\|$. Then apply Lemma 3.6 for $(\delta_n)_{n \in \mathbb{N}}$ and $(M_n)_{n \in \mathbb{N}}$ to obtain a basic sequence (y_i) which satisfies (14).

Assume also that (y_i) satisfies the ‘‘furthermore’’ part of the statement of Lemma 3.6 for the sequence (x_i^*) and $\varepsilon = \delta_1 \inf_{n,i} \|X_i^n\| \inf_i \|\underline{x}_i\| / (8 \sup_i \|\underline{x}_i\|)$. Note that if $|\lambda| \geq \frac{4 \sup_i \|\underline{x}_i\|}{\delta_1 \inf_{n,i} \|X_i^n\|}$ then

$$\|\underline{x}_{n_j} + \lambda y_j\| \geq |\lambda| \|y_j\| - \|\underline{x}_{n_j}\| \geq \frac{4 \sup_i \|\underline{x}_i\|}{\delta_1 \inf_{n,i} \|X_i^n\|} \frac{\delta_1}{2} \inf_{n,i} \|X_i^n\| - \|\underline{x}_{n_j}\| \geq \|\underline{x}_{n_j}\| \geq \inf_i \|\underline{x}_i\|$$

(by the ‘‘moreover’’ part of the statement of Lemma 3.6). Also if $|\lambda| < \frac{4 \sup_i \|\underline{x}_i\|}{\delta_1 \inf_{n,i} \|X_i^n\|}$ then

$$\|\underline{x}_{n_j} + \lambda y_j\| \geq |x_j^*(\underline{x}_{n_j} + \lambda y_j)| \geq \|\underline{x}_{n_j}\| - \frac{4 \sup_i \|\underline{x}_i\|}{\delta_1 \inf_{n,i} \|X_i^n\|} \varepsilon \geq \frac{1}{2} \inf_i \|\underline{x}_i\|.$$

Thus for all scalars λ we have

$$\|\underline{x}_{n_j} + \lambda y_j\| \geq \frac{1}{2} \inf_i \|\underline{x}_i\|.$$

Thus if we define $T : [(y_i)] \rightarrow X$ by

$$T\left(\sum a_i y_i\right) = \sum a_i \underline{x}_{n_i}$$

we have that this operator is bounded by (13), (14) and our assumption that $(\underline{x}_i)_i$ is dominated by $(X_i)_i$. We also have that for any scalar λ , $(T - \lambda i_{[(y_i)] \rightarrow X})(y_k) = \underline{x}_{n_k} - \lambda y_k$. But since (y_k) and (\underline{x}_k) are weakly null and $(x_{n_k} - \lambda y_k)$ is not norm null, $T - \lambda i_{[(y_i)] \rightarrow X}$ is not compact. In other words T is not a compact perturbation of a scalar multiple of the inclusion. \square

Proof of Lemma 3.4. Since ℓ_1 does not embed in X , by Rosenthal's ℓ_1 Theorem [18] and a diagonal argument, by passing to subsequences of (x_i) , (x_i^n) and (z_i) and relabeling we can assume that (x_i) and $(x_i^n)_i$ are weakly Cauchy for all $n \in \mathbb{N}$. Set $(\underline{x}_i)_i = (x_{2i} - x_{2i-1})_i$, $(\underline{x}_i^n)_i = (x_{2i}^n - x_{2i-1}^n)_i$ and $(\underline{z}_i)_i = (z_{2i} - z_{2i-1})_i$. For every $n \in \mathbb{N}$ the sequences (\underline{x}_i) and $(\underline{x}_i^n)_i$ are weakly null. Moreover the sequences $(\underline{x}_i)_i$ and $(\underline{z}_i)_i$ satisfy (11) and the sequences $(\underline{x}_i^n)_i$ and $(\underline{z}_i)_i$ satisfy (12) of Theorem 3.2.

Since (\underline{x}_i) and $(\underline{x}_i^n)_i$ are weakly null for all $n \in \mathbb{N}$, and satisfy conditions (11) and (12) of Theorem 3.2, we have that the spreading models of normalized weakly null sequences in X are not uniformly equivalent. Thus, (since every normalized suppression 1-unconditional basic sequence 2-dominates the unit vector basis of c_0), for every $n \in \mathbb{N}$, there exists a normalized weakly null sequence $(u_i^n)_i$ in X which has spreading model $(\tilde{u}_i^n)_i$ and there exist finitely supported scalars $(a_i^{(n)})_i$ such that $\|\sum_i a_i^{(n)} \tilde{u}_i^n\| \geq 2^{2^n}$ yet $\max_i |a_i^{(n)}| = 1$. By [2, Proposition 3.2] there exists a seminormalized weakly null sequence (u_i) in X having spreading model (\tilde{u}_i) such that (\tilde{u}_i) 2^n -dominates $(\tilde{u}_i^n)_i$. Thus $\|\sum_i a_i^{(n)} \tilde{u}_i\| \geq \frac{1}{2^n} \|\sum_i a_i^{(n)} \tilde{u}_i^n\| \geq 2^n$. Hence (\tilde{u}_i) is not equivalent to the unit vector basis of c_0 . By passing to a subsequence of (u_i) and relabeling we can assume by Theorem 1.1 that (u_i) is Schreier unconditional. Moreover, by passing to a subsequence of (u_i) and relabeling we can assume that for any Schreier set $F \subseteq \mathbb{N}$ and scalars $(a_i)_{i \in F}$,

$$(15) \quad \frac{1}{2} \left\| \sum_{i \in F} a_i \tilde{u}_i \right\| \leq \left\| \sum_{i \in F} a_i u_i \right\| \leq 2 \left\| \sum_{i \in F} a_i \tilde{u}_i \right\|.$$

Define the basic sequence (Z_i) by

$$\left\| \sum a_i Z_i \right\| := \max\left(\left\| \sum a_i \underline{z}_i \right\|, \left\| \sum a_i u_i \right\|\right),$$

(first for finitely supported sequences (a_i) and then take its completion). Obviously (Z_i) has a spreading model (\tilde{Z}_i) which satisfies

$$\left\| \sum a_i \tilde{Z}_i \right\| = \max\left(\left\| \sum a_i \tilde{\underline{z}}_i \right\|, \left\| \sum a_i \tilde{u}_i \right\|\right),$$

for finitely supported sequences of scalars (a_i) .

Since (u_i) and $(\underline{x}_i^n)_i$ are weakly null seminormalized sequences in X for all $n \in \mathbb{N}$, by [2, Proposition 3.2] there exist weakly null basic sequences $(X_i^n)_i$ in X such that $0 < \inf_{n,i} \|X_i^n\| \leq \sup_{n,i} \|X_i^n\| < \infty$ and the spreading model $(\tilde{X}_i^n)_i$ of $(X_i^n)_i$ 1-dominates the spreading models of both sequences $(\underline{x}_i^n)_i$ and (u_i) . By the definitions of $(X_i^n)_i$ and (Z_i) it is obvious that $(X_i^n)_i$ and (Z_i) satisfy condition (12) of Theorem 3.2. Thus we have satisfied conditions (d) and (e) of Lemma 3.4. It remains to define the basic sequence (X_i) in order to satisfy (a), (b) and (c).

Before defining (X_i) we define an auxiliary basic sequence (u_i^-) by

$$\left\| \sum_i a_i u_i^- \right\| := \sup_{1 \leq n < m} \frac{1}{\sqrt{L_{m-n}}} \left\| \sum_{i=n}^{m-1} a_i u_i \right\|,$$

where

$$L_k := \left\| \sum_{i=1}^k u_i \right\|,$$

(first for finitely supported sequences (a_i) of scalars and then take the completion). Obviously (u_i^-) is seminormalized since

$$\frac{\inf_j \|u_j\|}{\sqrt{L_1}} \leq \|u_i^-\| \leq \frac{\sup_j \|u_j\|}{\inf_j \sqrt{L_j}}, \text{ for all } i.$$

Notice that

$$\begin{aligned} \left\| \sum_{i=1}^n u_i^- \right\| &\geq \frac{\left\| \sum_{i=1}^n u_i \right\|}{\sqrt{L_n}} = \frac{L_n}{\sqrt{L_n}} = \sqrt{L_n} \\ &\geq \sqrt{\frac{1}{1+C} \left\| \sum_{i=\lceil n/2 \rceil}^n u_i \right\|^2} \text{ (where } C \text{ is the basis constant of } (u_i)) \\ &\geq \frac{1}{2} \sqrt{\frac{1}{1+C} \left\| \sum_{i=\lceil n/2 \rceil}^n \tilde{u}_i \right\|^2} \text{ (by (15)),} \end{aligned}$$

where the last quantity tends to infinity as $n \rightarrow \infty$ since the basic sequence (\tilde{u}_i) is spreading, unconditional and not equivalent to the unit vector basis of c_0 . Thus

$$(16) \quad \left\| \sum_{i=1}^n u_i^- \right\| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Claim 1: $(u_i^-) \ll (u_i)$.

In order to prove Claim 1, let $0 < \varepsilon < 1$ and choose $N \in \mathbb{N}$ such that $\sqrt{L_k} > 2C/\varepsilon$ for all $k > N$ (by (16)) where C is the basis constant of (u_i) . Let

$$\delta := \min_{1 \leq n \leq N} \frac{\varepsilon \sqrt{L_n}}{\sup_i \|u_i\| n}.$$

Let (a_i) be a finitely supported sequence of scalars such that $\left\| \sum a_i u_i^- \right\| = 1$ and $|a_i| \leq \delta$. Let $1 \leq n_0 < m_0$ such that

$$(17) \quad 1 = \left\| \sum a_i u_i^- \right\| = \frac{1}{\sqrt{L_{m_0-n_0}}} \left\| \sum_{i=n_0}^{m_0-1} a_i u_i \right\|.$$

Notice that if $m_0 - n_0 \leq N$ then

$$\begin{aligned} \frac{1}{\sqrt{L_{m_0-n_0}}} \left\| \sum_{i=n_0}^{m_0-1} a_i u_i \right\| &\leq \frac{1}{\sqrt{L_{m_0-n_0}}} (m_0 - n_0) \max_{n_0 \leq i \leq m_0-1} |a_i| \sup_i \|u_i\| \\ &\leq \frac{1}{\sqrt{L_{m_0-n_0}}} (m_0 - n_0) \delta \sup_i \|u_i\| \\ &\leq \frac{1}{\sqrt{L_{m_0-n_0}}} (m_0 - n_0) \frac{\varepsilon \sqrt{L_{m_0-n_0}}}{\sup_i \|u_i\| (m_0 - n_0)} \sup_i \|u_i\| \\ &= \varepsilon < 1, \end{aligned}$$

which contradicts (17). Thus $m_0 - n_0 > N$. Therefore, if C is the basis constant of (u_i) , equation (17) implies that

$$\left\| \sum a_i u_i \right\| \geq \frac{1}{2C} \left\| \sum_{i=n_0}^{m_0-1} a_i u_i \right\| \geq \frac{1}{2C} \sqrt{L_{m_0-n_0}} > \frac{1}{2C} \frac{2C}{\varepsilon} = \frac{1}{\varepsilon},$$

(by the choice of N and the fact that $m_0 - n_0 > N$). This finishes the proof of Claim 1.

Since (u_i) is Schreier unconditional, we easily obtain that (u_i^-) is Schreier unconditional. Also it is easy to verify that (u_i^-) has a spreading model (\tilde{u}_i^-) which satisfies

$$\left\| \sum a_i \tilde{u}_i^- \right\| = \sup_{1 \leq n < m} \frac{1}{\sqrt{L_{m-n}}} \left\| \sum_{i=n}^{m-1} a_i \tilde{u}_i \right\|,$$

for all finitely supported sequences (a_i) of scalars. Thus by (15) we obtain that for any Schreier set $F \subseteq \mathbb{N}$ and scalars $(a_i)_{i \in F}$,

$$(18) \quad \frac{1}{2} \left\| \sum_{i \in F} a_i \tilde{u}_i^- \right\| \leq \left\| \sum_{i \in F} a_i u_i^- \right\| \leq 2 \left\| \sum_{i \in F} a_i \tilde{u}_i^- \right\|.$$

Also, since (\tilde{u}_i) is unconditional we obtain that (\tilde{u}_i^-) is unconditional. Since (\tilde{u}_i^-) is spreading, unconditional and it is not equivalent to the unit vector basis of c_0 (by (16)), it is easy to see that (\tilde{u}_i^-) has Property P.

Claim 2: (u_i^-) has Property P.

This will follow from the fact that (\tilde{u}_i^-) has Property P, equation (18) and that (u_i^-) is Schreier unconditional. Indeed, let $\rho > 0$. Since (\tilde{u}_i^-) has Property P, there exists $M \in \mathbb{N}$ such that if $\left\| \sum a_i \tilde{u}_i \right\| \leq 1$ then $|\{i : |a_i| > \frac{\rho}{2C_S}\}| \leq M$ (where C_S is the Schreier unconditionality constant of (u_i^-)). Assume that $\left\| \sum b_i u_i^- \right\| \leq 1$ and let $A = \{i : |b_i| > \rho\}$. We claim that $|A| \leq 3M$ which finishes the proof of Claim 2. Indeed, if $|A| > 3M$ then let $A_1, A_2 \subseteq A$ with $A_1 \cup A_2 = A$, $a_1 < a_2$ for all $a_1 \in A_1$, $a_2 \in A_2$ and $|A_2| = \lfloor |A|/2 \rfloor$. Since $|A| > 3M$ we have that

$$(19) \quad |A_2| > M.$$

Since A_2 is a Schreier set we obtain by (18),

$$\left\| \sum_{i \in A_2} b_i \tilde{u}_i^- \right\| \leq 2 \left\| \sum_{i \in A_2} b_i u_i^- \right\| \leq 2C_S \left\| \sum_{i \in A_2} b_i u_i^- \right\| \leq 2C_S.$$

Therefore

$$\left\| \sum_{i \in A_2} \frac{b_i}{2C_S} \tilde{u}_i^- \right\| \leq 1.$$

Hence $|\{i \in A_2 : |\frac{b_i}{2C_S}| > \frac{\rho}{2C_S}\}| \leq M$ i.e. $|A_2| \leq M$ which contradicts (19). This finishes the proof of Claim 2.

Now define the basic sequence (X_i) by

$$\left\| \sum a_i X_i \right\| := \max\left(\left\| \sum a_i \underline{x}_i \right\|, \left\| \sum a_i u_i^- \right\|\right),$$

(first for finitely supported sequences (a_i) of scalars and then take the completion). Obviously (X_i) dominates (\underline{x}_i) thus (a) is satisfied. Since (u_i^-) has Property P we obtain that (X_i) has

Property P, hence (c) is satisfied. Also, since $(\underline{x}_i) \ll (\underline{z}_i)$ and $(u_i^-) \ll (u_i)$ we have that $(X_i) \ll (Z_i)$, therefore (b) is satisfied. \square

Now we present the proof of Lemma 3.5. Our arguments resemble the ones found in [2].

Proof of Lemma 3.5. Since (X_n) has Property P, for each $\rho > 0$ we can define $M = M(\rho)$ such that if $\|\sum a_i X_i\| = 1$ then $|\{i : |a_i| > \rho\}| \leq M$.

Let $(\varepsilon_j)_{j=1}^\infty$ be such that

$$\sum_{j=2}^{\infty} \frac{\varepsilon_{j-1}}{\delta_j} \leq \frac{1}{2}.$$

Since $(Z_n) \gg (\tilde{X}_n)$ by (10) we may choose a decreasing sequence $(\rho_j)_{j=1}^\infty \subseteq (0, 1]$ such that $\sum_j \sqrt{\rho_j}(j+1) \leq 1/4$ and satisfying the following: for all $(a_i) \in c_{00}$ with $|a_i| \in [0, \sqrt{\rho_j}]$ for each i and $\|\sum a_i \tilde{x}_i\| = 1$ we have

$$(20) \quad \left\| \sum a_i \tilde{X}_i \right\| \leq \varepsilon_j \left\| \sum a_i Z_i \right\|.$$

Finally let $M_j = M(\rho_j)$ as above.

By the definition of spreading models, by passing to a subsequence of (X_i) and relabeling, we can assume that if $j \leq F$ and $|F| \leq M_j$ then for all $(a_i) \in c_{00}$,

$$(21) \quad \frac{1}{2} \left\| \sum_{i \in F} a_i X_i \right\| \leq \left\| \sum_{i \in F} a_i \tilde{X}_i \right\| \leq 2 \left\| \sum_{i \in F} a_i X_i \right\|.$$

Now fix $(a_i) \in c_{00}$ such that $\|\sum a_i X_i\| = 1$. For $j \in \mathbb{N}$ consider the vector $\tilde{y} = \sum_{i>j; \rho_j < |a_i| \leq \rho_{j-1}} a_i \tilde{X}_i$. If $\|\tilde{y}\| \geq \sqrt{\rho_{j-1}}$ then

$$\begin{aligned} \|\tilde{y}\| &= \|\tilde{y}\| \left\| \frac{\tilde{y}}{\|\tilde{y}\|} \right\| = \|\tilde{y}\| \left\| \sum_{\substack{i>j \\ \rho_j < |a_i| \leq \rho_{j-1}}} \frac{a_i}{\|\tilde{y}\|} \tilde{X}_i \right\| \\ &\leq \|\tilde{y}\| \varepsilon_{j-1} \left\| \sum_{\substack{i>j \\ \rho_j < |a_i| \leq \rho_{j-1}}} \frac{a_i}{\|\tilde{y}\|} Z_i \right\| \quad (\text{by (20) since } \left\| \sum_{\substack{i>j \\ \rho_j < |a_i| \leq \rho_{j-1}}} \frac{a_i}{\|\tilde{y}\|} \tilde{X}_i \right\| = 1) \\ &= \varepsilon_{j-1} \left\| \sum_{\substack{i>j \\ \rho_j < |a_i| \leq \rho_{j-1}}} a_i Z_i \right\|. \end{aligned}$$

Thus in general, (without assuming that $\|\tilde{y}\| \geq \sqrt{\rho_{j-1}}$), we get

$$(22) \quad \|\tilde{y}\| \leq \sqrt{\rho_{j-1}} + \varepsilon_{j-1} \left\| \sum_{i>j; \rho_j < |a_i| \leq \rho_{j-1}} a_i Z_i \right\|.$$

Let ρ_0 be twice the basis constant of (X_i) divided by the $\inf_i \|X_i\|$. Since $\|\sum a_i X_i\| = 1$, we have that $|a_i| \leq \rho_0$.

$$\begin{aligned}
1 &= \left\| \sum a_i X_i \right\| \leq \sum_{j=1}^{\infty} \left\| \sum_{\rho_j < |a_i| \leq \rho_{j-1}} a_i X_i \right\| \\
&\leq \left\| \sum_{\rho_1 < |a_i| \leq \rho_0} a_i X_i \right\| + \sum_{j=2}^{\infty} \left\| \sum_{i \leq j; \rho_j < |a_i| \leq \rho_{j-1}} a_i X_i \right\| + \sum_{j=2}^{\infty} \left\| \sum_{i > j; \rho_j < |a_i| \leq \rho_{j-1}} a_i X_i \right\| \\
&\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \left\| \sum_{i \in F} a_i Z_i \right\| + \sum_{j=2}^{\infty} j \rho_{j-1} + 2 \sum_{j=2}^{\infty} \left\| \sum_{i > j; \rho_j < |a_i| \leq \rho_{j-1}} a_i \tilde{X}_i \right\|
\end{aligned}$$

where

$$\delta_1 = \sup \left\{ \frac{\left\| \sum_{i \in F} a_i X_i \right\|}{\left\| \sum_{i \in F} a_i Z_i \right\|} : (a_i)_{i \in F} \subseteq \mathbb{C} \text{ with } |F| \leq M_1 \text{ and } (a_i)_{i \in F} \neq 0^F \right\}$$

which is clearly finite by using an ℓ_1 estimate for the numerator and an ℓ_∞ estimate for the denominator. Note the third piece of the last inequality is true by (21) since the cardinality of $\{i > j : \rho_j < |a_i| \leq \rho_{j-1}\}$ is at most M_j . Continuing the calculations from above, we get

$$\begin{aligned}
1 &\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \left\| \sum_{i \in F} a_i Z_i \right\| + \frac{1}{4} + 2 \sum_{j=2}^{\infty} \sqrt{\rho_{j-1}} + 2 \sum_{j=2}^{\infty} \varepsilon_{j-1} \left\| \sum_{i > j; \rho_j < |a_i| \leq \rho_{j-1}} a_i Z_i \right\| \text{ by (22)} \\
&\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \left\| \sum_{i \in F} a_i Z_i \right\| + \frac{1}{2} + 2 \sum_{j=2}^{\infty} \frac{\varepsilon_{j-1}}{\delta_j} \delta_j \left\| \sum_{i > j; \rho_j < |a_i| \leq \rho_{j-1}} a_i Z_i \right\| \\
&\leq \sup_{F \subseteq \mathbb{N}, |F| \leq M_1} \delta_1 \left\| \sum_{i \in F} a_i Z_i \right\| + \frac{1}{2} + \sup_{n \geq 2} \sup_{n \leq F \subseteq \mathbb{N}; |F| \leq M_n} \delta_n \left\| \sum_{i \in F} a_i Z_i \right\|.
\end{aligned}$$

Thus

$$1 \leq 2 \sup_{n \in \mathbb{N}} \sup_{n \leq F \subseteq \mathbb{N}; |F| \leq M_n} \delta_n \left\| \sum_{i \in F} a_i Z_i \right\|$$

proving the lemma. \square

Now we present the proof of Lemma 3.6.

Proof of Lemma 3.6. Assume that for each $n \in \mathbb{N}$ $(X_j^{M_n})_j$ has spreading model $(\tilde{X}_j^{M_n})_{j=1}^{M_n}$, $(\tilde{X}_j^{M_n})_j$ C -dominates $(Z_i)_{i=1}^{M_n}$ and moreover if $|F| \leq M_n$ and $(a_j)_{j \in F}$ are scalars then

$$(23) \quad \frac{1}{2} \left\| \sum_{j \in F} a_j \tilde{X}_j^{M_n} \right\| \leq \left\| \sum_{j \in F} a_j X_j^{M_n} \right\| \leq 2 \left\| \sum_{j \in F} a_j \tilde{X}_j^{M_n} \right\|.$$

By Remark 2.3 by passing to subsequences and relabeling, assume that $(X_j^{M_n})_{(n,j) \in I}$ forms a regular array with basis constant at most equal to 2. Apply Theorem 1.2 to $(X_j^{M_n})_{(n,j) \in I}$ to get a subarray which satisfies the conclusion of Theorem 1.2. By relabeling call $(X_j^{M_n})_{(n,j) \in I}$ the resulting subarray. Define

$$y_j = \sum_{n=1}^j \delta_n X_{\ell_j}^{M_n}$$

where (ℓ_j) is an increasing sequence of positive integers which guarantees that $|x_i^* y_i| < \varepsilon$. Note that (y_j) is weakly null since $(X_j^{M_n})_j$ is weakly null for all n and δ_n is summable. Since $(X_j^{M_n})_{(n,j) \in I}$ is regular, (y_j) is a basic sequence with

$$(24) \quad \|y_j\| \geq \frac{\delta_1}{2} \inf_{n,m} \|X_m^n\|$$

(since the basis constant of $(X_j^{M_n})_{(n,j) \in I}$ is at most equal to 2 by Remark 2.3). Since (δ_n) is summable, (y_j) is also bounded. Fix $n \in \mathbb{N}$ and let $n \leq F \subseteq \mathbb{N}$, with $|F| \leq M_n$. Then

$$\begin{aligned} \left\| \sum a_j y_j \right\| &\geq \frac{1}{3} \left\| \sum_{j \in F; j \geq n} \delta_n a_j X_{\ell_j}^{M_n} \right\| \text{ (by Theorem 1.2)} \\ &\geq \frac{1}{6} \left\| \sum_{j \in F; j \geq n} \delta_n a_j \tilde{X}_j^{M_n} \right\| \text{ (by (23))} \\ &= \frac{1}{6} \left\| \sum_{j \in F; j \geq n} \delta_n a_j \tilde{X}_{k_j}^{M_n} \right\| \\ &\quad \text{where the map } F \ni j \mapsto k_j \in \{1, 2, \dots, |F|\} \text{ is a 1-1 increasing function} \\ &\geq \frac{1}{6C} \delta_n \left\| \sum_{j \in F; j \geq n} a_j Z_j \right\| \text{ (by (12))} \end{aligned}$$

Thus

$$\left\| \sum a_j y_j \right\| \geq \sup_n \frac{1}{6C} \sup_{n \leq F \subseteq \mathbb{N}; |F| \leq M_n} \delta_n \left\| \sum_{i \in F} a_i Z_i \right\|.$$

□

An easy corollary of Theorem 3.2 is Theorem 3.8 which is obtained if we set (z_i) to be the unit vector basis of ℓ_p for some fixed $p \in [1, \infty)$. In order to formulate Theorem 3.8 we need to recall the notion of the **Krivine set** of a 1-subsymmetric (i.e. 1-equivalent to all of its subsequences and unconditional) basic sequence.

Definition 3.7. *Let (x_n) be a 1-subsymmetric basic sequence in some Banach space. The Krivine set of (x_n) is defined to be the set of all p 's in $[1, \infty]$ with the following property. For all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and scalars $(\lambda_k)_{k=1}^m$ such that for all scalars $(a_n)_{n=1}^N$,*

$$\frac{1}{1 + \varepsilon} \|(a_n)_{n=1}^N\|_p \leq \left\| \sum_{n=1}^N a_n y_n \right\| \leq (1 + \varepsilon) \|(a_n)_{n=1}^N\|_p$$

where

$$y_n = \sum_{k=1}^m \lambda_k x_{(n-1)m+k} \text{ for } n = 1, \dots, N$$

and $\|\cdot\|_p$ denotes the norm of the space ℓ_p .

The Krivine's theorem as it was proved by H. Rosenthal [17] and H. Lemberg [13] states that if (x_n) is a 1-subsymmetric basic sequence then the Krivine set of (x_n) is non-empty. In particular, if (x_n) is a seminormalized weakly null basic sequence in a Banach space having spreading model (\tilde{x}_n) then (\tilde{x}_n) is 1-subsymmetric, hence the Krivine set of (\tilde{x}_n) is non-empty.

Theorem 3.8. *Let X be a Banach space. Assume that there exist seminormalized basic sequences $(x_i), (y_i)$ in X such that (x_i) has spreading model (\tilde{x}_i) which is s.c. dominated by the unit vector basis of ℓ_p , for some $p \in [1, \infty)$ and (y_i) has spreading model (\tilde{y}_i) which is unconditional and p belongs to the Krivine set of (\tilde{y}_i) . Then there exists a subspace Y of X with a basis and an operator $T \in \mathcal{L}(Y, X)$ which is not a compact perturbation of a multiple of the inclusion map.*

For applying this result for $p = 1$, recall that by [2, Proposition 2.1], a seminormalized subsymmetric basic sequence is not equivalent to the unit vector basis of ℓ_1 if and only if it is s.c. dominated by the unit vector basis of ℓ_1 . Thus Theorem 3.8 for $p = 1$ implies [2, Theorem 6.4].

Proof of Theorem 3.8. Since p belongs to the Krivine set of (\tilde{y}_i) then for all $n \in \mathbb{N}$ there exists $(x_i^n)_{i \in \mathbb{N}}$ a block sequence of (y_i) of identically distributed blocks such that any n terms of $(x_i^n)_{i \in \mathbb{N}}$ are 2-equivalent to the unit vector basis of ℓ_p^n . Then apply Theorem 3.2 for (z_i) being the unit vector basis of ℓ_p . \square

4. AN APPLICATION OF THEOREM 3.8

Next we give an application of Theorem 3.8. As mentioned before the “multiple of the inclusion plus compact” problem is non trivial only in HI-saturated Banach spaces. The HI space to which Theorem 3.8 will be applied was constructed by N. Dew [7], and here will be denoted by D . The construction of the space D is based on the 2-convexification of the Schlumprecht space S [20] in a similar manner that the space of T.W. Gowers and B. Maurey [11] is based on S . We recall the necessary definitions.

Let X be a Banach space with a basis (e_i) . For any interval E in \mathbb{N} and a vector $x = \sum x_j e_j \in X$ define $Ex = \sum_{j \in E} x_j e_j \in X$. There is a unique norm $\|\cdot\|_S$ on c_{00} which satisfies:

$$\|x\|_S = \sup \left\{ \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i x\|_S : E_1 < \dots < E_\ell \right\} \vee \|x\|_{\ell_\infty}$$

where $f(\ell) = \log_2(\ell + 1)$. The completion of c_{00} under this norm is the Banach space S . Let S_2 be its 2-convexification. Recall if X is a Banach space having an unconditional basis (e_n) , then we can define the 2-convexification X_2 of X by the norm

$$\left\| \sum a_n \sqrt{e_n} \right\|_{X_2} := \left(\left\| \sum a_n^2 e_n \right\|_X \right)^{1/2}$$

where $(\sqrt{e_n})$ denotes the basis of X_2 , (we will talk more later about the “square root” map from X to X_2). We will show that the spreading model of the unit vector basis of D is the unit vector basis of S_2 . Before we do this we must see the definition of D .

In order to define the Banach space D , a lacunary set $J \subseteq \mathbb{N}$ is used which has the property that if $n, m \in J$ and $n < m$ then $8n^4 \leq \log \log \log m$, and $f(\min J) \geq 45^4$. Write J in increasing order as $\{j_1, j_2, \dots\}$. Now let $K \subset J$ be the set $\{j_1, j_3, j_5, \dots\}$ and $L \subset J$

be the set $\{j_2, j_4, j_6, \dots\}$. Let \mathbf{Q} be the set of scalar sequences with finite support and rational coordinates whose absolute value is at most one. Let σ be an injective function from \mathbf{Q} to L such that if z_1, \dots, z_i is such a sequence, then $(1/400)f(\sigma(z_1, \dots, z_i)^{1/40})^{\frac{1}{2}} \geq \#\text{supp}(\sum_{j=1}^i z_j)$. Then, recursively, we define a set of functionals of the unit ball of the dual space D^* as follows: Let

$$D_0^* = \{\lambda e_n^* : n \in \mathbb{N}, |\lambda| \leq 1\}.$$

Assume that D_k^* has been defined. Define the norm $\|\cdot\|_k$ on c_{00} by

$$(25) \quad \|x\|_k = \sup\{|x^*(x)| : x^* \in D_k^*\}$$

and let $\|\cdot\|_k^*$ denote its dual norm. Then D_{k+1}^* is the set of all functionals of the form $F z^*$ where $F \subseteq \mathbb{N}$ is an interval and z^* has one of the following three forms:

$$(26) \quad z^* = \sum_{i=1}^{\ell} \alpha_i z_i^*$$

where $\sum_{i=1}^{\ell} |\alpha_i| \leq 1$ and $z_i^* \in D_k^*$ for $i = 1, \dots, \ell$.

$$(27) \quad z^* = \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} \alpha_i z_i^*$$

where $\sum_{i=1}^{\ell} \alpha_i^2 \leq 1$, $z_i^* \in D_k^*$ for $i = 1, \dots, \ell$, and $z_1^* < \dots < z_{\ell}^*$.

$$(28) \quad z^* = \frac{1}{\sqrt[4]{f(\ell)}} \sum_{i=1}^{\ell} z_i^* \text{ where } z_i^* = \frac{1}{\sqrt{f(m_i)}} \sum_{j=1}^{m_i} \alpha_{i,j} \frac{E z_{i,j}^*}{\|E z_{i,j}^*\|_k^*}$$

for certain $(\alpha_{i,j})$ where $\sum_{i,j} \alpha_{i,j}^2 \leq 1$ ($\alpha_{i,j}$'s are explicitly chosen in [7], but the exact values are not needed for our purposes) where $z_{i,j}^* \in D_k^*$ for $1 \leq i \leq \ell$ and $1 \leq j \leq m_i$, $z_{1,1}^* < \dots < z_{1,m_1}^* < z_{2,1}^* < \dots < z_{\ell,m_{\ell}}^*$, $m_1 = j_{2\ell}$, βz_i^* has rational coordinates for some $\beta > 0$ (whose exact value is not needed for our purpose), $m_{i+1} = \sigma(\beta z_1^*, \dots, \beta z_i^*)$, for $i = 1, \dots, \ell-1$ and E is an interval.

Finally, the norm of D is defined by

$$\|x\|_D = \sup\{z^*(x) : z^* \in \cup_{k=0}^{\infty} D_k^*\}.$$

Proposition 4.1. *The spreading model of the unit vector basis of D is the unit vector basis of S_2 .*

Proof. From the definitions of the two spaces it is easy to see that for $(a_i) \in c_{00}$, $\|\sum a_i e_i\|_{S_2} \leq \|\sum a_i e_i\|_D$ ((e_n) will denote the bases of both spaces S_2 and D but there will be no confusion about which space we consider at each moment). Thus to show Proposition 4.1 we need only show for any given $\varepsilon > 0$, and finitely many scalars $(a_i)_{i=1}^N$ there exists n_0 such that for any $n_1, n_2, \dots, n_N \in \mathbb{N}$ with $n_0 < n_1 < n_2 < \dots < n_N$, we have

$$\|x\|_D \leq \|y\|_{S_2} + \varepsilon.$$

where $x = \sum_{i=1}^N a_i e_{n_i} \in D$ and $y = \sum_{i=1}^N a_i e_i \in S_2$. This will follow immediately once we show by induction on n that for any $n \in \mathbb{N}$, $\varepsilon > 0$ and scalars $(a_i)_{i=1}^N$ we have

$$(29) \quad \|x\|_n \leq \|y\|_{S_2} + \varepsilon$$

where $\|\cdot\|_n$ is defined in (25). For “ $n = 0$ ” we have $\|x\|_0 = \max_{1 \leq i \leq N} |a_i| \leq \|y\|_{S_2}$. Now for the inductive step assume that (29) is valid for n . Let $\varepsilon > 0$ and scalars $(a_i)_{i=1}^N$.

First note that by the induction hypothesis there exists $n_0^1 \in \mathbb{N}$ such that for all $n_0^1 < n_1 < n_2 < \dots < n_N$ we have

$$(30) \quad \|x\|_n \leq \|y\|_{S_2} + \varepsilon.$$

Secondly, by the inductive hypothesis there exists $n_0^2 \in \mathbb{N}$ such that for all $1 \leq i_0 \leq j_0 \leq N$ and $n_0^2 < n_1 < n_2 < \dots < n_N$ we have

$$(31) \quad \left\| \sum_{i=i_0}^{j_0} a_i e_{n_i} \right\|_n \leq \left\| \sum_{i=i_0}^{j_0} a_i e_i \right\|_{S_2} + \frac{\varepsilon}{\sqrt{N}}.$$

And finally there exists $j_0 \in L$ such that $\frac{1}{\sqrt{f(j_0)}} N \|y\|_{S_2} \leq \varepsilon$. Let $G = \sigma^{-1}(\{1, 2, \dots, j_0 - 1\})$, So G is a finite subset of finite sequences of vectors with rational coefficients. Let n_0^3 be the maximum of the support of any vector in any sequence in G . And then set $n_0 = \max\{n_0^1, n_0^2, n_0^3\}$. Recall the norming vectors $z^* \in D_{n+1}^*$ can be one of three different types. Each type of functional will present us with a different case.

CASE 1: Let z^* be given by (26). Then

$$\begin{aligned} |z^*(x)| &\leq \sum_{i=1}^{\ell} |\alpha_i| |z_i^*(x)| \\ &\leq \sum_{i=1}^{\ell} |\alpha_i| (\|x\|_{S_2} + \varepsilon) \quad (\text{by the (30)}) \\ &= \|x\|_{S_2} + \varepsilon. \end{aligned}$$

CASE 2: Let z^* be given by (27). Thus for $n_0 < n_1 < n_2 < \dots < n_N$ we have

$$\begin{aligned} |z^*(x)| &\leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| |z_j^*(x)| \\ &\leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| |z_j^*(E_j x)| \end{aligned}$$

where E_j is the smallest interval containing the support of z_j^* intersected with the support of x . Continuing the above calculation we have

$$\begin{aligned}
|z^*(x)| &\leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| \|E_j x\|_n \\
&\leq \frac{1}{\sqrt{f(\ell)}} \sum_{j=1}^{\ell} |\alpha_j| (\|E_j x\|_{s_2} + \frac{\varepsilon}{\sqrt{N}}) \text{ (by (31))} \\
&\leq \frac{1}{\sqrt{f(\ell)}} (\sum_{j=1}^{\ell} |\alpha_j|^2)^{1/2} (\sum_{j=1}^{\ell} \|E_j x\|_{s_2}^2)^{1/2} + \varepsilon \\
&\text{(by Cauchy-Schwarz' inequality since there are at most } N \text{ many nonempty } E_j \text{'s)} \\
&\leq \|x\|_{s_2} + \varepsilon.
\end{aligned}$$

CASE 3: Let z^* be given by (28). We can of course assume $z^*(x) \neq 0$ thus let i_0 be the smallest natural number such that

$$\bigcup_{i=1}^{i_0} \text{supp}(z_i^*) \cap \{n_0 + 1, n_0 + 2, \dots\} \neq \emptyset.$$

Then by the definition of n_0^3 we have that $\sigma(\beta z_1^*, \beta z_2^*, \dots, \beta z_{i_0}^*) > j_0$ for $i_0 + 1 \leq i \leq \ell$. Thus

$$\begin{aligned}
(32) \quad |z^*(x)| &\leq \frac{1}{\sqrt[4]{f(\ell)}} \left| \frac{1}{\sqrt{f(m_{i_0})}} \sum_{j=1}^{m_{i_0}} \alpha_{i_0,j} z_{i_0,j}^*(x) \right| + \frac{1}{\sqrt[4]{f(\ell)}} \frac{1}{\sqrt{f(m_{i_0+1})}} \sum_{j=1}^{m_{i_0+1}} |\alpha_{i_0+1,j}| |z_{i_0+1,j}^*(x)| + \dots \\
&\leq \frac{1}{\sqrt[4]{f(\ell)}} \|y\|_{s_2} + \varepsilon + \frac{1}{\sqrt[4]{f(\ell)}} \frac{1}{\sqrt{f(m_{i_0+1})}} \sum_{\{1 \leq j \leq m_{i_0+1} : z_{i_0+1,j}^* x \neq 0\}} |\alpha_{i_0+1,j}| |z_{i_0+1,j}^*(x)| + \dots \text{ (by CASE 2)} \\
&\leq \frac{1}{\sqrt[4]{f(\ell)}} \|y\|_{s_2} + \varepsilon + \frac{1}{\sqrt[4]{f(\ell)}} \frac{1}{\sqrt{f(j_0)}} N (\|y\|_{s_2} + \varepsilon),
\end{aligned}$$

where the last inequality is valid since $|\alpha_{i,j}| \leq 1$, $|z_{i,j}^*(x)| \leq \|z_{i,j}^*\|_n^* \|x\|_n \leq \|x\|_n \leq \|y\|_{s_2} + \varepsilon$, $m_i \geq j_0$ for all $i_0 + 1 \leq i \leq \ell$, (by the choice of n_0), and there are at most N many indexes (i, j) for which $z_{i,j}^*(x) \neq 0$. The last expression in equation (32) is at most equal to $\|y\|_{s_2} + 3\varepsilon$ which finishes the proof. \square

For the following remark we will need a bit more notation. If X is a Banach space having an unconditional basis (e_n) and X_2 is the 2-convexification of X then for $x = \sum a_n e_n \in X$ then there is a canonical image of x in X_2 which we define as $\sqrt{x} = \sum \text{sign}(a_n) \sqrt{|a_n|} \sqrt{e_n} \in X_2$ (where $(\sqrt{e_n})$ denotes the basis of X_2).

Remark 4.2. Let $1 \leq p < \infty$. Let X be a Banach space with an unconditional basis $(e_i)_i$ and let $(\sqrt{e_i})_i$ be the basis of X_2 . Then p is in the Krivine set of (e_i) if and only if $2p$ is in the Krivine set of $(\sqrt{e_i})$.

Proof. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Since p is in the Krivine set of X , there exists a block sequence $(v_i)_{i=1}^n$ of (e_i) such that for any scalars $(a_i)_{i=1}^n$ we have that

$$\frac{1}{1+\varepsilon} \left(\sum_{i=1}^n a_i^p \right)^{1/p} \leq \left\| \sum_{i=1}^n a_i v_i \right\| \leq (1+\varepsilon) \left(\sum_{i=1}^n a_i^p \right)^{1/p}$$

Let $w_i = \sqrt{v_i} \in X_2$, and scalars $(a_i)_{i=1}^n$ then

$$\left\| \sum_{i=1}^n a_i w_i \right\|_{X_2} = \left\| \sum_{i=1}^n a_i^2 v_i \right\|_X^{\frac{1}{2}} \leq (1+\varepsilon) \left(\sum_{i=1}^n a_i^{2p} \right)^{\frac{1}{2p}}$$

and

$$\left\| \sum_{i=1}^n a_i w_i \right\|_{X_2} = \left\| \sum_{i=1}^n a_i^2 v_i \right\|_X^{\frac{1}{2}} \geq \frac{1}{1+\varepsilon} \left(\sum_{i=1}^n a_i^{2p} \right)^{\frac{1}{2p}}.$$

The proof of the converse is similar. \square

It is known [20] that the Krivine set of the unit vector basis of S consists of the singleton $\{1\}$. Thus by Remark 4.2 we have:

Remark 4.3. *The Krivine set of the unit vector basis of S_2 consists of the singleton $\{2\}$.*

Proposition 4.4. *Let X, Y be Banach spaces with unconditional bases, (x_n) be a basic sequence in X , (y_n) be a basic sequence in Y such that $(x_n) \gg (y_n)$. Then $(\sqrt{x_n}) \gg (\sqrt{y_n})$ where $(\sqrt{x_n})$ and $(\sqrt{y_n})$ are the canonical images of (x_n) and (y_n) in X_2 and Y_2 respectively.*

Proof. Note that

$$(33) \quad \liminf_{n \rightarrow \infty} \inf_{A \subseteq \mathbb{N}; |A|=n} \left\| \sum_{i \in A} \sqrt{x_i} \right\|_{X_2} = \liminf_{n \rightarrow \infty} \inf_{A \subseteq \mathbb{N}; |A|=n} \left\| \sum_{i \in A} x_i \right\|_X^{\frac{1}{2}} = \infty.$$

Also

$$(34) \quad \begin{aligned} \Delta_{(\sqrt{x_n}), (\sqrt{y_n})}(\varepsilon) &= \sup \left\{ \left\| \sum a_i \sqrt{y_i} \right\|_{Y_2} : |a_i| \leq \varepsilon \text{ and } \left\| \sum a_i \sqrt{x_i} \right\|_{X_2} = 1 \right\} \\ &= \left(\sup \left\{ \left\| \sum a_i^2 y_i \right\|_Y : |a_i|^2 \leq \varepsilon^2 \text{ and } \left\| \sum a_i^2 x_i \right\|_X = 1 \right\} \right)^{\frac{1}{2}} \\ &\leq \left(\Delta_{(x_n), (y_n)}(\varepsilon^2) \right)^{\frac{1}{2}}. \end{aligned}$$

The result follows immediately from (33) and (34). \square

It has been shown in [2, Proposition 2.1] that if (e_n) is the unit vector basis of ℓ_1 and (f_n) is a normalized subsymmetric basic sequence which is not equivalent to (e_n) then $(e_n) \gg (f_n)$. Thus since the unit vector basis of S is normalized and subsymmetric we have that the unit vector basis of ℓ_1 s.c. dominates the unit vector basis of S . Thus Proposition 4.4 gives:

Remark 4.5. *The unit vector basis of ℓ_2 s.c. dominates the unit vector basis of S_2 .*

Theorem 4.6. *There exists an infinite dimensional subspace Y of D having a basis and $T \in \mathcal{L}(Y, D)$ such that $T \notin \mathbb{C}i_{Y \rightarrow D} + \mathcal{K}(Y, D)$.*

Proof. We will refer to (s_i) as the unit vector basis of the Schlumprecht space S and $(\sqrt{s_i})$ as the unit vector basis of S_2 . Then apply Theorem 3.8 for $p = 2$ and $(x_i)_i = (y_i)_i =$ the unit vector basis of D . By Proposition 4.1 we have that $(\sqrt{s_i})$ is the spreading model of the unit vector basis of D . By Remark 4.3 we have that 2 is in the Krivine set of $(\sqrt{s_i})$. By Remark 4.5 we have that the unit vector basis of ℓ_2 s.c. dominates $(\sqrt{s_i})$. Thus

by Theorem 3.8 we have that there exists an infinite dimensional subspace Y of D and $T \in \mathcal{L}(Y, D)$ such that $T \notin Ci_{Y \rightarrow D} + \mathcal{K}(Y, D)$. \square

REFERENCES

- [1] Albiac, F; Kalton, N. *Topics in Banach space theory*. Graduate Texts in Mathematics, **233**. Springer, New York, 2006.
- [2] Androulakis, G., Odell, E., Schlumprecht, Th., Tomczak-Jaegermann, N. *On the structure of the spreading models of a Banach space*. *Canad. J. Math.* **57** (2005), no. 4, 673–707.
- [3] Argyros, S.A, Mercourakis, S., Tsarpalias, A. *Convex unconditionality and summability of weakly null sequences*, *Israel J. Math.* **107**, (1998) 157–193.
- [4] Beauzamy, B., Lapreste, J.-T. *Modeles etaes des espaces de Banach*. Travaux en Cours. Hermann, Paris, 1984.
- [5] Brunel, A., Sucheston, L. *On J -convexity and some ergodic super-properties of Banach spaces*. *Trans. Amer. Math. Soc.* **204** (1975), 79–90.
- [6] Brunel, A., Sucheston, L. *On B -convex Banach spaces*. *Math. Systems Theory* **7** (1974), no. 4, 294–299.
- [7] Dew, N. *Asymptotic Structure of Banach Spaces*, Ph.D dissertation, University of Oxford, St. Johns College (2002) (available at <http://eprints.maths.ox.ac.uk/49/01/dew.pdf>).
- [8] Dilworth, S.J., Odell, E., Schlumprecht, Th., A. Zsak. *Partial Unconditionality*, preprint.
- [9] Gowers, W. T. *An infinite Ramsey theorem and some Banach-space dichotomies*. *Ann. of Math.* (2) **156** (2002), no. 3, 797–833.
- [10] Gowers, W. T. *A remark about the scalar-plus-compact problem*. *Convex geometric analysis* (Berkeley, CA, 1996), 111–115, *Math. Sci. Res. Inst. Publ.*, **34**, Cambridge Univ. Press, Cambridge, 1999.
- [11] Gowers, W. T.; Maurey, B. *The unconditional basic sequence problem*. *J. Amer. Math. Soc.* **6** (1993), no. 4, 851–874.
- [12] Halbeisen, L.; Odell, E. *On asymptotic models in Banach spaces*. *Israel J. Math.* **139** (2004), 253–291.
- [13] Lemberg, H. *Nouvelle demonstration d’un théorème de J.-L. Krivine sur la finie representation de ℓ_p dans un espace de Banach*. *Israel J. Math.* **39** (1981), no. 4, 341–348.
- [14] Lindenstrauss, J.; Tzafriri, L. *Classical Banach spaces. I. Sequence spaces*. (1977) *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Vol. 92.
- [15] Maurey, B.; Rosenthal, H. P. *Normalized weakly null sequence with no unconditional subsequence*, *Studia Math.* **61** (1977), no. 1, 77–98.
- [16] Odell, E. *On Schreier Unconditional Sequences*. *Contemp. Math.* **144** (1993), 197–201.
- [17] Rosenthal, H. P. *On a theorem of J. L. Krivine concerning block finite representability of ℓ_p in general Banach spaces*. *J. Funct. Anal.* **28** (1978), no. 2, 197–225.
- [18] Rosenthal, H. P. *A characterization of Banach spaces containing ℓ_1* , *Proc. Nat. Acad. Sci. USA* **71** (1974), 2411–2413.
- [19] Rosenthal, H.P. *Some remarks concerning unconditional basic sequences*, Longhorn Notes, The University of Texas, Texas Functional Analysis seminar, (1982-1983), 15–48.
- [20] Schlumprecht, Th. *An arbitrarily distortable Banach space*. *Israel J. Math.* **76** (1991), no. 1-2, 81–95.
- [21] Schlumprecht, Th. *How many operators exist on a Banach space?* Trends in Banach spaces and operator theory (Memphis, TN, 2001), 295–333, *Contemp. Math.*, **321**, Amer. Math. Soc., Providence, RI, 2003.

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