POSITIVITY RESULTS FOR THE YANG-MILLS-HIGGS HESSIAN

GEORGE ANDROULAKIS AND STAMATIS DOSTOGLOU

Abstract: We study the second derivative Q_{c_λ} of the Yang-Mills-Higgs functional with structure group $SU(2)$ at a spherically symmetric critical point c_λ when the self-interaction parameter λ is not close to 0. We find that if $Q_{c_{\lambda_0}}$ is non-negative and with kernel consisting entirely of translation modes then positivity persists in a neighborhood of λ_0. In particular, we show that if translation modes always account for the whole kernel then Q_{c_λ} is always non-negative. This extends our previous results for λ in an neighborhood of 0.

1. Introduction

The Yang-Mills-Higgs functional E_λ on $\mathbb{R}^3$, with structure group $SU(2)$, is the classical static version of the functional introduced by P. Higgs in [H]. The critical points of E_λ correspond to magnetic monopoles.

It has been known since the late 70s that spherically symmetric critical points c_λ of E_λ exist for all values of the positive parameter λ. The authors of this paper have recently shown that for positive λ in a neighborhood of 0 these critical points have non-negative Hessian Q_{c_λ}, i.e. they are (weakly) stable, see [AD]. The aim of this article is to investigate the stability of c_λ for (the more relevant for physics) large values of λ.

For $\lambda = 0$ the spherically symmetric solution c_0 satisfies the first order (Bogomol'nyi) equation for global minima, [JT]. This equation is unique to the $\lambda = 0$ case and has no analogue for $\lambda \neq 0$. One of the crucial observations in [AD] for extending the positivity from $\lambda = 0$ to $\lambda \neq 0$ is that the kernel of the Hessian at 0 consists entirely of translation modes. Here we show that this is not special to $\lambda = 0$: for any positive λ_0 where $Q_{c_{\lambda_0}}$ is non-negative,

1991 *Mathematics Subject Classification.* 58E15 (53C07 81T13).

if the kernel of $Q_{c_{\lambda_0}}$ is the span of the translation modes then Q_{c_λ} is non-negative for λ in a neighborhood of λ_0. In the same neighborhood the dimension of the kernel may not increase. The first main result then is:

Theorem *The set of $\lambda \geq 0$ for which Q_{c_λ} is non-negative and has 3-dimensional kernel is open.*

In addition, since for λ close to λ_0 it is shown that Q_{c_λ} is close to $Q_{c_{\lambda_0}}$, the set of λ's for which $Q_{c_\lambda} \geq 0$ is closed, yielding

Corollary *If $\ker Q_{c_\lambda}$ is always 3-dimensional then Q_{c_λ} is non-negative for all λ.*

The proof of the main results consists of three parts. First it is shown that if $Q_{c_{\lambda_0}}$ is non-negative then, away from its kernel, it is bounded below by a strictly positive constant. Then c_λ is shown to converge to c_{λ_0} in the configuration space as $\lambda \to \lambda_0$. This implies that the Hessians Q_{c_λ} are all defined on the same Hilbert space and that they differ by a small amount. It is then shown that the subspaces N_λ spanned by the translation modes of c_λ contain the kernel of $Q_{c_{\lambda_0}}$ at the limit. This implies that directions orthogonal to N_λ are almost orthogonal to, and definitely not in, the kernel of $Q_{c_{\lambda_0}}$. For the last two steps in the proof the estimates of [AD] need to be extended from uniform estimates on some neighborhood of 0 to uniform estimates on any compact λ-interval.

In section 2 we review the basics of the theory and state the main results. The proofs are contained in section 3, where we focus on aspects that are genuinely different from the $\lambda_0 = 0$ case. It is in section 4 that we show how to improve the estimates in [AD] so that they hold on any bounded λ-interval.

2. The functional E_λ and the symmetric solutions c_λ

The Yang-Mills-Higgs functional E_λ with self-interaction parameter $\lambda \geq 0$ is defined by

$$E_\lambda(A, \Phi) = \frac{1}{2} \int_{\mathbf{R}^3} \{|F_A|^2 + |d_A \Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2\} d^3x, \tag{1}$$

on pairs $c = (A, \Phi)$. Here A is a connection on the $SU(2)$ bundle $SU(2) \times \mathbf{R}^3$ over $\mathbf{R}^3$ and Φ is a section of the associated bundle E with fiber the Lie Algebra $\mathfrak{su}(2)$, $E = \mathfrak{su}(2) \times \mathbf{R}^3$. F_A is the curvature of the connection A and $d_A\Phi$ the covariant derivative of Φ with respect to the connection A:

$$F_A = dA + \frac{1}{2}[A, A], \quad d_A\Phi = d\Phi + [A, \Phi].$$

All norms use the Killing inner product on $\mathfrak{su}(2)$ and the standard metric on $\mathbf{R}^3$.

E_λ is defined on the configuration space

$$\hat{\mathcal{C}} = \{(A, \Phi) : A \in L^2_{1,loc}, \Phi \in L^2_{1,loc}, E_\lambda(A, \Phi) < \infty.\}$$

Note here that $\hat{\mathcal{C}}$ stays the same for all $\lambda > 0$. For the special case $\lambda = 0$ see [AD]. $\hat{\mathcal{C}}$ is equipped with the $L^2_{1,loc}$ topology intersected with the topology that makes $\|d_A\Phi\|_2$ and $\|F_A\|_2$ continuous. E_λ, $\lambda \geq 0$ is invariant under the action of the gauge group

$$\mathcal{G} = \{g : \mathbf{R}^3 \to \mathrm{SU}(2), \quad \mathrm{g} \in \mathrm{L}^2_{2,\mathrm{loc}}\},$$

where $g \cdot A = gAg^{-1} + gdg^{-1}$, and $g \cdot \Phi = g\Phi g^{-1}$. Then E_λ descends to the quotient $\mathcal{C} = \hat{\mathcal{C}}/\mathcal{G}$.

To define the space $T_c\mathcal{C}$ of admissible infinitesimal perturbations at $c = (A, \Phi)$ in $\mathcal{C}$, consider first the completion H_c of C_0^∞ sections on $\mathbb{R}^3$ with respect to the inner product norm

$$||(a, \phi)||_c^2 = \|\nabla_A a\|_2^2 + \|\nabla_A \phi\|_2^2 + \|[\Phi, a]\|_2^2 + \|\phi\|_2^2. \tag{2}$$

H_c contains only directions that keep E_λ finite, c.f. [T1], but it still contains deformations along the orbit of $\mathcal{G}$. This is remedied here by excluding the elements of the kernel of the (formal) adjoint of the linearization of the action

$$\partial_c(a, \phi) = -d_A^* a + [\Phi, \phi]. \tag{3}$$

We define therefore

$$T_c\mathcal{C} = \{(a, \phi) \in H_c : \partial_c(a, \phi) = 0\}.$$

The variational equations for $\lambda \geq 0$ are the Yang-Mills-Higgs equations

$$d_A^* F_A = [d_A\Phi, \Phi], \qquad d_A^* d_A\Phi = -\frac{\lambda}{2}\Phi(|\Phi|^2 - 1). \tag{4}$$

For any $c = (A, \Phi)$ in $\mathcal{C}$, the second derivative of the energy E_λ defines a bilinear form on $T_c\mathcal{C}$

$$\hat{Q}_{\lambda,c}((a_1,\phi_1),(a_2,\phi_2)) = \frac{d^2}{dsdt}\Big|_{(0,0)} E_\lambda(A + sa_1 + ta_2, \Phi + s\phi_1 + t\phi_2), \quad \lambda \geq 0, \tag{5}$$

the Hessian of the Yang-Mills-Higgs functional. Then

$$\begin{aligned}
\hat{Q}_{\lambda,c}((a_1,\phi_1),(a_2,\phi_2)) &= < F_A, [a_1,a_2] > + < d_A\Phi, [a_1,\phi_2] + [a_2,\phi_1] > + < d_Aa_1, d_Aa_2 > \\
&+ < d_A\phi_1, d_A\phi_2 > + < [a_1,\Phi], [a_2,\Phi] > + < d_A\phi_1, [a_2,\Phi] > \\
&+ < d_A\phi_2, [a_1,\Phi] > + \frac{\lambda}{2}\int_{\mathbf{R}^3} (|\Phi|^2 - 1) < \phi_1, \phi_2 > d^3x \\
&+ \lambda \int_{\mathbf{R}^3} < \Phi, \phi_1 >< \Phi, \phi_2 > d^3x.
\end{aligned}$$

and the corresponding quadratic form is

$$\begin{aligned}
Q_{\lambda,c}(a,\phi) &= \|d_Aa\|_2^2 + \|d_A\phi\|_2^2 + \|[a,\Phi]\|_2^2 \\
&+ < F_A, [a,a] > +2 < d_A\Phi, [a,\phi] > +2 < d_A\phi, [a,\Phi] > \\
&+ \frac{\lambda}{2}\int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda\int_{\mathbf{R}^3} < \Phi, \phi >^2 d^3x.
\end{aligned}$$

Now recall the following standard

Definition 2.1. *Let $\hat{Q}$ be a bilinear form a Hilbert space H. Then v_0 in H is in* $\ker\hat{Q}$ *if and only if*

$$\hat{Q}(v_0, v) = 0$$

for all v in H.

Throughout this paper whenever Q is the quadratic form associated to a bilinear form $\hat{Q}$ we use the phrase "v_0 is in $\ker Q$" to mean "v_0 is in $\ker\hat{Q}$".

The following is rewriting $Q_{\lambda,c}$ on $T_c\mathcal{C}$ as in [T2], page 246.

Lemma 2.2. *For (a, ϕ) in $T_c\mathcal{C}$,*

$$\begin{aligned} Q_{\lambda,c}(a,\phi) &= \|\nabla_A a\|_2^2 + \|\nabla_A \phi\|_2^2 + \|[a,\Phi]\|_2^2 + \|[\Phi,\phi]\|_2^2 \\ &+ 2 < F_A, [a,a] > +4 < d_A\Phi, [a,\phi] > \\ &+ \frac{\lambda}{2}\int_{\mathbf{R}^3}(|\Phi|^2-1)|\phi|^2 d^3x + \lambda\int_{\mathbf{R}^3} < \Phi,\phi >^2 d^3x. \end{aligned}$$

Proof. First seperate in $Q_{\lambda,c}$ the terms that contain λ from those that do not, using the obvious notation:

$$Q_{\lambda,c}(a,\phi) = Q_{0,c}(a,\phi) + \frac{\lambda}{2}\int_{\mathbf{R}^3}(|\Phi|^2-1)|\phi|^2 d^3x + \lambda\int_{\mathbf{R}^3} < \Phi,\phi >^2 d^3x.$$

Next use the Bochner-Weitzenböck formula for 1-forms on flat spaces, see page 95 of [L], and integrate by parts to get

$$\|d_A a\|_2^2 + \|d_A^* a\|_2^2 = \|\nabla_A a\|_2^2 + < F_A, [a,a] > .$$

Then

$$\begin{aligned} Q_{0,c}(a,\phi) &= \|\nabla_A a\|_2^2 - \|d_A^* a\|_2^2 + \|d_A\phi\|_2^2 + \|[a,\Phi]\|_2^2 \\ &+ 2 < F_A, [a,a] > +2 < d_A\Phi, [a,\phi] > +2 < d_A\phi, [a,\Phi] > . \end{aligned}$$

Therefore for (a, ϕ) in $T_c\mathcal{C}$, where $d_A^*a - [\Phi, \phi] = 0$ holds,

$$\begin{aligned} Q_{0,c}(a,\phi) &= \|\nabla_A a\|_2^2 + \|d_A\phi\|_2^2 + \|[a,\Phi]\|_2^2 + \|[\Phi,\phi]\|_2^2 \\ &- 2\|[\Phi,\phi]\|_2^2 + 2 < F_A, [a,a] > +2 < d_A\Phi, [a,\phi] > +2 < d_A\phi, [a,\Phi] > . \end{aligned}$$

Now observe that on $T_c\mathcal{C}$ we also have

$$\begin{aligned}
< d_A\Phi, [a,\phi] > &= \sum_1^3 \int_{\mathbb{R}^3} < (d_A)_i\Phi, [a_i,\phi] > \\
&= -\sum_1^3 \int_{\mathbb{R}^3} < \Phi, (d_A)_i[a_i,\phi] > \\
&= \int_{\mathbb{R}^3} < \Phi, [-\sum_1^3 (d_A)_i a_i, \phi] > - \sum_1^3 \int_{\mathbb{R}^3} < \Phi, [a_i, (d_A)_i\phi] > \\
&= \int_{\mathbb{R}^3} < \Phi, [d_A^* a, \phi] > - \int_{\mathbb{R}^3} < \Phi, [a, d_A\phi] > \\
&= \int_{\mathbb{R}^3} < \Phi, [[\Phi,\phi],\phi] > + \int_{\mathbb{R}^3} < [a,\Phi], d_A\phi > \\
&= -\|[\Phi,\phi]\|_2^2 + < d_A\phi, [a,\Phi] >,
\end{aligned}$$

to finally get

$$Q_{0,c}(a,\phi) = \|\nabla_A a\|_2^2 + \|d_A\phi\|_2^2 + \|[a,\Phi]\|_2^2 + \|[\Phi,\phi]\|_2^2 + 2 < F_A, [a,a] > +4 < d_A\Phi, [a,\phi] > .$$

□

In [tH] and [P] 't Hooft and Polyakov suggest spherically symmetric solutions for the three-dimensional Yang-Mills-Higgs equations. With respect to the standard basis e_a, $a = 1,2,3$ of $\mathfrak{su}(2)$ their Ansatz is

$$A = \varepsilon_{ija}\frac{x_j}{r^2}(1-K(r))e_a dx_i, \quad \Phi = \frac{x_\alpha}{r}\frac{H(r)}{r}e_a, \tag{6}$$

with boundary conditions $K(r) \to 0$ and $H/r \to 1$, as $r \to \infty$.

On configurations of this form, E_λ is

$$\begin{aligned}
E_\lambda(H,K) = 4\pi \int_0^\infty \{(K')^2 &+ \frac{1}{2}(H' - \frac{H}{r})^2 + \frac{K^2H^2}{r^2} + \frac{1}{2}\frac{(K^2-1)^2}{r^2} \\
&+ \frac{\lambda}{4}(\frac{H^2}{r} - r)^2\} dr.
\end{aligned}$$

A critical point (K_λ, H_λ) of the 1-dimensional integral satisfies the variational equation

$$\frac{d}{dt}\bigg|_{t=0} E_\lambda(H_\lambda + th, K_\lambda + tk) = 0$$

for all h, k with compact support on $[0,\infty)$. This yields the system of non-linear, second order, ordinary differential equations

$$K''_\lambda = \frac{H_\lambda^2 - 1 + K_\lambda^2}{r^2} K_\lambda \qquad \text{(YMH 1)}$$

$$H''_\lambda = \frac{2K_\lambda^2}{r^2} H_\lambda - 4\lambda H_\lambda (1 - \frac{H_\lambda^2}{r^2}). \qquad \text{(YMH 2)}$$

It is relatively easy to produce critical points of the 1-dimensional integral by direct minimization, see [D] for example. On the other hand it is a standard fact, referred to as "the principle of symmetric criticality" in [Pa], that due to symmetry a critical point of the 1-dimensional integral is also a critical point of E_λ overall.

Therefore for each λ there is a spherically symmetric monopole solution. Throughout this paper

$$c_\lambda = (K_\lambda, H_\lambda)$$

will denote this solution.

Remark on notation: The notation for the Hessian Q_{λ,c_λ} of E_λ over all directions in $T_{c_\lambda}\mathcal{C}$ will be shortened to Q_{c_λ}. Otherwise, $Q_{\lambda,c}$ will denote the Hessian at an arbitrary configuration c in $\mathcal{C}$.

There is no *a priori* reason why c_λ should be an overall minimum. For example, the spherically symmetric minimizer of the Skyrmion functional has (non-spherically symmetric) unstable directions, see [WB].

For $\lambda = 0$ the the point c_0 is a global minimum in the connected component of all configurations with finite E_0 energy, [M].

For $\lambda \neq 0$ and small, the main result in [AD] is

Theorem 2.3. *There is $\lambda_0 > 0$ such that $Q_{c_\lambda}(v) \geq 0$ for all $\lambda \leq \lambda_0$ and for all v in $T_{c_\lambda}\mathcal{C}$. Furthermore,*

$$\ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1,2,3 \right\rangle .$$

The behavior of Q_{c_λ} for λ away from $\lambda = 0$ is investigated here. For this, define

$$\Lambda = \{\lambda > 0 : Q_{c_\lambda} \geq 0\},$$

$$\Lambda' = \{\lambda \geq 0 : \ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, \right\rangle_{i=1,2,3} \}$$

Then Theorem 2.3 states that $\Lambda \cap \Lambda'$ is not empty and contains an interval of the form $(0, \lambda_0)$.

The main result here is:

Theorem 2.4. 1. $\Lambda \cap \Lambda'$ *is an open subset of* $(0, \infty)$.

2. Λ *is closed.*
3. Λ *contains the first connected component of* Λ'.

With this, to prove that Q_{c_λ} is non-negative for all λ reduces to the following

Conjecture 2.5. *For all positive* λ, $\ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1, 2, 3 \right\rangle$.

3. Convergence in the configuration space and Hessians

3.1. **General Observations.** The proof of the Theorem 2.4 relies on some general observations about quadratic forms from [AD]. First, in order to describe the fact that the kernel of the Hessians changes as the solutions c_λ move in the configuration space $\mathcal{C}$, adopt the following

Definition Let $(H, <, >)$ be a Hilbert space. Let V_{λ_0} be a closed subspace of H and V_λ be a one-parameter family of closed subspaces of H. V_λ *contains* V_{λ_0} *at the limit as* $\lambda \to \lambda_0$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that for any u in $V_\lambda^\perp$ of norm 1 there exists v in $V_{\lambda_0}^\perp$ of norm 1 such that $\|v - u\| < \varepsilon$.

Now, slightly abusing notation, let Q_{λ_0} be a quadratic form and Q_λ be a one-parameter family of quadratic forms on H. The following describes the steps for the proof of Theorem 2.4 in this general setting:

Proposition 3.1. *Let* Q_{λ_0} *be a quadratic form and* Q_λ *be a one-parameter family of quadratic forms defined on a Hilbert space* H *and assume that*

1. *Q_{λ_0} is uniformly continuous on the unit sphere of H.*
2. $\alpha := \inf\{Q_{\lambda_0}(v) : v \perp \ker Q_{\lambda_0}, \|v\| = 1\} \gneqq 0$
3. $\sup_{\|v\|=1} |Q_{\lambda_0}(v) - Q_\lambda(v)| \to 0$ *as* $\lambda \to \lambda_0$
4. *there are subspaces N_λ of $\ker Q_\lambda$ such that N_λ contains $\ker Q_{\lambda_0}$ at the limit as $\lambda \to \lambda_0$.*

Then there exists $\varepsilon > 0$ such that whenever $|\lambda - \lambda_0| < \varepsilon$ then

1. $\inf\{Q_\lambda(v) : v \perp N_\lambda, \|v\| = 1\} > \dfrac{\alpha}{3} \gneqq 0$
2. $N_\lambda = \ker Q_\lambda$.

Proof. If v is in $N_\lambda^\perp$ and of norm 1 then, for λ sufficiently close to λ_0, there is v' of norm 1 in $\ker Q_{c_{\lambda_0}}^\perp$ close to v. Therefore $Q_{c_\lambda}(v) \approx Q_{c_{\lambda_0}}(v) \approx Q_{c_{\lambda_0}}(v') > \alpha/3$. □

3.2. **Reduction to a single Hilbert space.** Before Proposition 3.1 can take over, one has to establish that as $\lambda \to \lambda_0$ the spaces $T_{c_\lambda}\mathcal{C}$ are isomorphic and therefore all Hessians Q_{c_λ} are defined on the same space.

Lemma 3.2. *For any c and c' in $\mathcal{C}$ with $c - c'$ in $T_c\mathcal{C}$ the following holds:*

$$|\|v\|_{c'} - \|v\|_c| \leq M_c \|v\|_c \|c - c'\|_c.$$

In particular, for $\|c - c'\|_c$ sufficiently small the identity on C_0^∞ induces an isomorphism between $T_c\mathcal{C}$ and $T_{c'}\mathcal{C}$

Proof. The proof for the norm $\|.\|_c$ as defined here, is the same as the proof of Proposition B6.2 of [T1]. □

That the conditions of this Lemma are satisfied for spherically symmetric solutions is proved in the following:

Proposition 3.3. $\|c_\lambda - c_{\lambda_0}\|_{c_{\lambda_0}} \to 0$, *as* $\lambda \to \lambda_0 > 0$.

Proof. It is a matter of straightforward calculation to show that this follows by the fact that the following norms over $[0, \infty)$ go to 0 as $\lambda \to \lambda_0$:

$$\left\| \frac{1}{r}(H_\lambda - H_{\lambda_0}) \right\|_2, \quad \|H'_\lambda - H'_{\lambda_0}\|_2, \quad \|(H_\lambda - H_{\lambda_0})\|_2. \tag{7}$$

$$\|K_\lambda - K_{\lambda_0}\|_2, \quad \left\|\frac{1}{r}(K_\lambda - K_{\lambda_0})\right\|_2, \quad \|K'_\lambda - K'_{\lambda_0}\|_2. \tag{8}$$

For these estimates work as follows:

Step 1. Estimates for H_λ and K_λ: first obtain uniform in λ pointwise bounds on the fields on $[0, r_0)$, for r_0 sufficiently small, see (23) and (24) in section 4.

Then obtain uniform in λ pointwise exponential decay estimates on the fields on $[r_1, \infty)$ for r_1 sufficiently large, see Proposition 4.2 in section 4.

For the intervals of the form $[r_0, r_1]$, Proposition 7.1 of [AD] shows that $\|F_{A_\lambda}\|_2 + \|d_{A_\lambda \Phi_\lambda}\|_2$ is an non-decreasing function of λ, therefore bounded on bounded intervals. As an elementary case of Uhlenbeck's compactness, this suffices for uniform convergence of the fields on bounded domains, see Proposition 7.4 of [AD].

Step 2. Estimates for H''_λ and K''_λ: These follow from the estimates on H_λ and K_λ after using equations (YMH1) and (YMH2) that do not involve first derivatives, see Proposition 4.1 below.

Step 3. Estimates for H'_λ and K'_λ: For these, obtain uniform in λ pointwise decay estimates on the first derivatives on $[r_1, \infty)$, c.f. Propositions 8.2, 9.4 and 9.5 of [AD]. On $[0, r_1)$ use step 2, the Poincaré inequality and the fact that $H'_\lambda(0) = K'_\lambda(0) = 0$.

The details follow from the arguments in [AD], after observing that the pointwise estimates there on the fields H_λ, K_λ and their first derivatives are valid for λ in any specified bounded interval; see section 4 below. □

Remark: $\|H_\lambda - H_{\lambda_0}\|_2 \to 0$ does not hold for $\lambda_0 = 0$ over $[0, \infty)$ but only over compact intervals. This reflects the fact that H_0 decays in power law whereas H_λ decays exponentially for all $\lambda > 0$, see Proposition 4.2.

The remaining subsections of this section show that the conditions of Proposition 3.1 hold for the Hessians Q_{c_λ}.

3.3. Uniform continuity of $Q_{\lambda,c}$ on the unit sphere.

Lemma 3.4. *For any λ, and for any $c = (A, \Phi)$ in $\mathcal{C}$ with Φ bounded the Hessian $Q_{\lambda,c}$: $(T_c\mathcal{C}, \|.\|_c) \to \mathbf{R}$ is uniformly continuous on the unit sphere.*

Proof. It suffices to show that $\hat{Q}_{\lambda,c}(v,w) \leq K\|v\|_c \cdot \|w\|_c$ for some constant K and all v and w. To see that this holds separately for each term of $\hat{Q}_{\lambda,c}$, use the fact that Φ is bounded and the inequalities

$$| < F_A, [a,a] > | \leq \|F_A\|_2 \|[a,a]\|_2 \leq C\|F_A\|_2 \|(a,\phi)\|_c^2, \tag{9}$$

$$| < d_A\Phi, [a,\phi] > | \leq \|d_A\Phi\|_2 \|[a,\phi]\|_2 \leq C\|d_A\Phi\|_2 \|(a,\phi)\|_c^2, \tag{10}$$

c.f. [T1]. □

Remark: It is standard to show using maximum principle that for any critical point $c = (A,\Phi)$ of E_λ, and in particular for the spherically symmetric solutions c_λ, the Higgs field Φ satisfies $|\Phi|(x) < 1$.

3.4. When $Q_{\lambda,c}$ is non-negative.

Theorem 3.5. *For any c in $\mathcal{C}$ the following fold:*

1. *If Φ is bounded then $Q_{\lambda,c} : (T_c\mathcal{C}, \|.\|_c) \to \mathbf{R}$ is continuously differentiable.*
2. *There is $\varepsilon > 0$ such that $Q_{\lambda,c} - \varepsilon\|.\|_c^2$ is weakly lower semi-continuous in $(T_c\mathcal{C}, \|.\|_c)$.*

Proof. 1. Since $\hat{Q}_{\lambda,c}$ is continuous, this follows from the fact that $2\hat{Q}_{\lambda,c}(v,.)$ is the differential of $Q_{\lambda,c}$ at v.

2. First use Lemma 2.2 to rewrite the Hessian as

$$\begin{aligned} Q_{\lambda,c}(a,\phi) &= \|\nabla_A a\|_2^2 + \|d_A\phi\|_2^2 + \|[a,\Phi]\|_2^2 + \|[\Phi,\phi]\|_2^2 \\ &+ 2 < F_A, [a,a] > +4 < d_A\Phi, [a,\phi] > \\ &+ \frac{\lambda}{2}\int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda \int_{\mathbf{R}^3} < \Phi, \phi >^2 d^3x. \end{aligned}$$

Then for $\varepsilon \leq \min(\lambda, 1)$ we have:

$$\begin{aligned} Q_{\lambda,c}(a,\phi) - \varepsilon\|(a,\phi)\|_c^2 &= (1-\varepsilon)\|\nabla_A a\|_2^2 + (1-\varepsilon)\|d_A\phi\|_2^2 \\ &+ (1-\varepsilon)\|[a,\Phi]\|_2^2 + (1-\varepsilon)\|[\phi,\Phi]\|_2^2 + (\lambda-\varepsilon)\int_{\mathbf{R}^3} <\Phi,\phi>^2 d^3x \\ &+ 2 < F_A, [a,a] > +4 < d_A\Phi, [a,\phi] > + \frac{\lambda}{2}\int_{\mathbf{R}^3}(|\Phi|^2-1)|\phi|^2 d^3x \\ &+ \varepsilon\|[\phi,\Phi]\|_2^2 + \varepsilon\int_{\mathbf{R}^3} <\Phi,\phi>^2 d^3x - \varepsilon\|\phi\|_2^2. \end{aligned}$$

Each term of the first two lines is weakly lower semi-continuous with respect to the $\|.\|_c$-norm: for the first term, for example, note that if (a_n, ϕ_n) converges weakly to (a, ϕ) in $T_c\mathcal{C}$ then $\|\nabla_A a\|_2 \leq \|(a,\phi)\|_c \leq \liminf_{n\to\infty} \|(a_n,\phi_n)\|_c$ since $\|\cdot\|_c$ is a norm and hence weakly lower semicontinuous.

The weak continuity of the terms in the third line follows as in section VI of [T2] from the fact that F_A, $d_A\Phi$ and $||\Phi|^2 - 1|$ belong to L^2 (and therefore Property (*) of [T2] is satisfied).

The terms in the fourth line are also weakly continuous since they regroup as

$$\varepsilon \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x. \tag{11}$$

□

The following is the main application of the weak lower semi-continuity offered by Theorem 3.5.

Proposition 3.6. *For any λ, if $Q_{\lambda,c}(v) \geq 0$ for all v in $T_c\mathcal{C}$ then*

$$\inf\{Q_{\lambda,c}(v) : v \perp_c \ker Q_{\lambda,c}, \|v\|_c = 1\} \gneqq 0. \tag{12}$$

Proof. Let v_n be a sequence in $\ker Q_{\lambda,c}^{\perp}$ with $\|v_n\|_c = 1$ and $Q_{\lambda,c}(v_n) \to 0$. By considering a subsequence, assume that v_n is weakly convergent and let $v \in \ker Q_{\lambda,c}^{\perp}$ be its weak limit. Since $Q_{\lambda,c}$ is weakly lower semicontinuous (by Theorem 3.5) and non-negative, obtain that $Q_{\lambda,c}(v) = 0$. Moreover, since there is an $\varepsilon > 0$ such that $Q_{\lambda,c}(\cdot) - \varepsilon\|\cdot\|_c^2$ is weakly lower semicontinuous, it follows that

$$-\varepsilon\|v\|_c^2 = Q_{\lambda,c}(v) - \varepsilon\|v\|_c^2 \leq \liminf_n (Q_{\lambda,c}(v_n) - \varepsilon\|v_n\|_c^2) = -\varepsilon$$

which implies that $\|v\|_c \geq 1$ and thus $\|v\|_c = 1$ (therefore (v_n) converges to v in the Hilbert space norm $\|\cdot\|_c$). Hence

$$Q_{\lambda,c}(v) = \min\{Q_{\lambda,c}(u) : u \perp_c \ker Q_{\lambda,c}, \|u\|_c = 1\} \tag{13}$$

Then by the differentiability and the non-negativity of $Q_{\lambda,c}$ it is easy to see using a Lagrange multiplier that for each $w \in \ker Q_{\lambda,c}^{\perp}$

$$\hat{Q}_{\lambda,c}(v,w) = Q_{\lambda,c}(v) < v, w >_c$$

which gives that v is in $\ker Q_{\lambda,c}$, a contradiction to the fact that $v \in \ker Q_{\lambda,c}^{\perp}$ of norm 1. □

3.5. **Uniform convergence on the sphere.**

Lemma 3.7. *For $c_i = (A_i, \Phi_i)$ and $c = (A, \Phi)$ in $\mathcal{C}$ assume that $\|c_i - c\|_c \to 0$. If λ_i converge to λ, $|\Phi_i - \Phi|$ tends to zero uniformly on $\mathbf{R}^3$ and $|\Phi|$ is bounded on $\mathbf{R}^3$, then*

$$\sup_{\|v\|_c=1} |Q_{\lambda_i,c_i}(v) - Q_{\lambda,c}(v)| \to 0. \tag{14}$$

Proof. For the part of the Hessian with no λ coefficient, see statement (5) of Proposition A.4.3. of [T3]. For the remaining terms note that

$$\begin{aligned}
&|\lambda_i \left(\frac{1}{2}\int(|\Phi_i|^2-1)|\phi|^2 d^3x + \int <\Phi_i,\phi>^2\, d^3x\right)\\
&-\lambda\left(\frac{1}{2}\int(|\Phi|^2-1)|\phi|^2 d^3x + \int <\Phi,\phi>^2\, d^3x\right)|\\
&\leq |\lambda_i-\lambda|\frac{1}{2}\int(|\Phi|^2-1)|\phi|^2 d^3x + |\lambda_i|\frac{1}{2}\int \left||\Phi_i|^2-|\Phi|^2\right||\phi|^2 d^3x\\
&+|\lambda_i-\lambda|\int <\Phi,\phi>^2\, d^3x + |\lambda_i|\int | <\Phi_i,\phi>^2 - <\Phi,\phi>^2 | d^3x.
\end{aligned}$$

The first and third term in the last expression obviously tend to zero. The second term tends to zero since Φ_i tends to Φ uniformly on $\mathbf{R}^3$ and $\|\phi\|_2 \leq 1$. The fourth term tends to zero since

$$\int | <\Phi_i,\phi>^2 - <\Phi,\phi>^2 | d^3x \leq \int |\Phi_i - \Phi|\, |\phi|^2\, |\Phi_i + \Phi|\, d^3x \tag{15}$$

and $|\Phi_i + \Phi|$ is bounded on $\mathbf{R}^3$. □

Lemma 3.8. *For $\lambda_0 \geq 0$, Φ_λ tends to Φ_{λ_0} uniformly on $\mathbf{R}^3$ as $\lambda \to \lambda_0$.*

Proof. First note that in terms of H_λ it is enough to show that as $\lambda \to \lambda_0$

$$\left\| \frac{H_\lambda}{r} - \frac{H_{\lambda_0}}{r} \right\|_\infty \to 0. \tag{16}$$

For this use Corollary 4.3 for large r, estimate (24) as it appears in the proof of Proposition 4.1 for small r, and the uniform convergence on compact intervals (as in Step 1, Proposition 3.3) in between. □

3.6. **λ-subspaces containing the kernel of $Q_{c_{\lambda_0}}$ at the limit.** For $\lambda \geq 0$ consider the subspace of $T_{c_\lambda}\mathcal{C}$ spanned by the translation modes, i.e. the partial derivatives of c_λ:

$$N_\lambda = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1, 2, 3 \right\rangle. \tag{17}$$

As a result of a straight-forward calculation

$$\left.\frac{d^2}{dsdt}\right|_{t=0,s=0} E_\lambda(c(x+te_i) + sv(x+te_i)) = \hat{Q}_{\lambda,c}\left(\frac{\partial c}{\partial x_i}, v\right) + \nabla(E_\lambda)_c\left(\frac{\partial v}{\partial x_i}\right) \tag{18}$$

for any v in $T_c\mathcal{C}$. Since E_λ is translation invariant

$$\frac{d}{dt}E_\lambda(c(x+te_i) + sv(x+te_i)) = 0. \tag{19}$$

In addition the first variation of E_λ vanishes at c_λ, therefore

$$Q_{c_\lambda}\left(\frac{\partial c_\lambda}{\partial x_i}, v\right) = 0, \tag{20}$$

which shows that N_λ is a subspace $\ker Q_{c_\lambda}$.

Proposition 3.9. *N_λ contain S_{λ_0} at the limit in $(T_{c_{\lambda_0}}\mathcal{C}, \|.\|_{c_{\lambda_0}})$ as $\lambda \to \lambda_0 \neq 0$.*

Proof. As a matter of a straightforward calculation using that $|\Phi_{\lambda_0}|(x) < 1$, and that $|A_{\lambda_0}|$ is uniformly bounded by [AD],

$$\left\| \frac{\partial c_\lambda}{\partial x_a} - \frac{\partial c_{\lambda_0}}{\partial x_a} \right\|^2_{c_{\lambda_0}} \to 0 \tag{21}$$

if, in addition to the estimates in the proof of 3.3, the following norms over $[0,\infty)$ tend to 0 as $\lambda \to \lambda_0$:

$$\left\| \frac{1}{r^2}(H_\lambda - H_{\lambda_0}) \right\|_2, \|H''_\lambda - H''_{\lambda_0}\|_2, \left\| \frac{1}{r}(H'_\lambda - H'_{\lambda_0}) \right\|_2, \|K''_\lambda - K''_{\lambda_0}\|_2. \tag{22}$$

Observe that this involves estimates on the fields, their first and their second derivatives. These follow again as in [AD] using the fact that the pointwise estimates on the fields and their derivatives hold for λ on any specified bounded interval, see section 4. ☐

Proof or Theorem 2.4 Fix λ_0 in $\Lambda \cap \Lambda_0$. Then subsections 3.2 to 3.6 show that $Q_{c_{\lambda_0}}$ satisfies the conditions 1. to 4. respectively of Proposition 3.1, and that for λ close to λ_0 all the Hessians Q_{c_λ} are defined on the same space as $Q_{c_{\lambda_0}}$. Then Proposition 3.1 applies to give part 1. of Theorem 2.4.

The second part of Theorem 2.4 follows immediately from Lemma 3.7, which shows that the complement of Λ is open.

Now for the third part of Theorem 2.4 argue as follows: by Thorem 2.3, both the first connected component Λ_0 of Λ and the first connected component Λ'_0 of Λ' are intervals starting from 0. If Λ'_0 is not contained in Λ then Λ_0 is a proper subset of Λ'_0. Let λ_0 be the supremum of Λ_0, which by the second part of the theorem belongs to Λ_0, i.e. λ_0 is in the intersection of Λ and Λ'. This contradicts the first part of the theorem. The proof of theorem 2.4 is now complete.

4. Estimates

This section will substantiate the claim that the estimates of [AD] for λ in a neighborhood of 0 can be extended to estimates on any bounded λ-interval. Proposition 4.1 is typical of the L^2-norm estimates required. Proposition 4.2 is typical of the uniform in λ and pointwise in r estimates required.

Proposition 4.1. $\|H''_\lambda - H''_{\lambda_0}\|_{L^2[0,\infty)} \to 0,$ *as* $\lambda \to \lambda_0 \neq 0.$

Proof. For this, first note that it follows as in section 10 of [AD] that there is a constant C such that for all $\lambda < \lambda_0 + 1$

$$|K_\lambda(r)| \leq C \tag{23}$$

for all r and

$$\left|\frac{H_\lambda(r)}{r^2}\right| \leq C \tag{24}$$

for r in $[0,1]$. Then

$$\begin{aligned}
\|H''_\lambda - H''_{\lambda_0}\|_{L^2[0,\infty)} &= \left\|\frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2} - 4\lambda H_\lambda(1-\frac{H_\lambda^2}{r^2}) - 4\lambda_0 H_{\lambda_0}(1-\frac{H_{\lambda_0}^2}{r^2})\right\|_{L^2[0,\infty)} \\
&\leq \left\|\frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2}\right\|_{L^2[0,\infty)} \\
&\quad + \left\|4\lambda H_\lambda(1-\frac{H_\lambda^2}{r^2}) - 4\lambda_0 H_{\lambda_0}(1-\frac{H_{\lambda_0}^2}{r^2})\right\|_{L^2[0,\infty)}.
\end{aligned}$$

For the first term in this sum find appropriately small r_0 and large r_1 such that:

$$\begin{aligned}
\left\|\frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2}\right\|_{L^2[0,\infty)} &\leq \left\|\frac{K_\lambda^2 H_\lambda}{r^2}\right\|_{L^2[0,r_0]} + \left\|\frac{K_{\lambda_0}^2 H_{\lambda_0}}{r^2}\right\|_{L^2[0,r_0]} \\
&\quad + \left\|\frac{K_\lambda^2 (H_\lambda - H_{\lambda_0})}{r^2}\right\|_{L^2[r_0,r_1]} + \left\|\frac{(K_\lambda^2 - K_{\lambda_0}^2) H_{\lambda_0}}{r^2}\right\|_{L^2[r_0,r_1]} \\
&\quad + 2\left\|\alpha\frac{e^{-r/2}}{r}\right\|_{L_2[r_1,\infty)} \\
&\leq C^3 M\varepsilon + C^3\varepsilon \\
&\quad + \frac{C^2}{r_0^2}\|H_\lambda - H_{\lambda_0}\|_{L^2[r_0,r_1]} + \frac{r_1}{r_0^2}\|K_\lambda^2 - K_{\lambda_0}^2\|_{L^2[r_0,r_1]} \\
&\quad + 2\varepsilon.
\end{aligned}$$

As already remarked in step 1. of Proposition 3.3 above, E_0 is an increasing function of λ, and therefore bounded on a bounded interval, and that this is enough to give uniform convergence of H_λ to H_0 and of K_λ to K_0 on bounded domains. Therefore the last expression becomes smaller than 5ε if $|\lambda - \lambda_0|$ is small enough, by the uniform convergence of K_λ to K_{λ_0} and H_{λ_0} to H_0 on the interval $[r_0, r_1]$.

For the second term and for a choice of $0 < r_0 < r_1$,

$$
\begin{aligned}
\left\|4\lambda H_\lambda(1-\frac{H_\lambda^2}{r^2}) \right. &- \left. 4\lambda_0 H_{\lambda_0}(1-\frac{H_{\lambda_0}^2}{r^2})\right\|_{L^2[0,\infty)} \\
&\leq \left\|4\lambda H_\lambda(1-\frac{H_\lambda^2}{r^2})\right\|_{L^2[0,r_0]} + \left\|4\lambda_0 H_{\lambda_0}(1-\frac{H_{\lambda_0}^2}{r^2})\right\|_{L^2[0,r_0]} \\
&\quad +4\left\|(\lambda-\lambda_0)H_\lambda(1-\frac{H_\lambda^2}{r^2})\right\|_{L^2[r_0,r_1]} \\
&\quad +4\left\|\lambda_0(H_\lambda-H_{\lambda_0})(1-\frac{H_\lambda^2}{r^2})\right\|_{L^2[r_0,r_1]} \\
&\quad +4\left\|\lambda_0\frac{H_{\lambda_0}}{r^2}(H_{\lambda_0}^2-H_\lambda^2)\right\|_{L^2[r_0,r_1]} \\
&\quad +4\left\|\lambda_0(H_\lambda-H_{\lambda_0})(1-\frac{H_\lambda^2}{r^2})\right\|_{L^2[r_1,\infty)} \\
&\quad +4\left\|\lambda_0\frac{H_{\lambda_0}}{r}\left((1-\frac{H_\lambda}{r})(1-\frac{H_{\lambda_0}}{r})\right)(H_{\lambda_0}+H_\lambda)\right\|_{L^2[r_1,\infty)}
\end{aligned}
$$

Obviously, r_0 can be chosen small enough to make the first two terms arbitrarily small. Then r_1 can be chosen large enough to make the last two terms arbitrarily small (for the last term use triangle inequality of the norm and Proposition 4.2; for the anti-penultimate term use Corollary 4.3 and Proposition 4.2). The rest of the terms tend to zero as $\lambda \to \lambda_0$ by uniform convergence on compact intervals. □

Proposition 4.2. *For all $\lambda_0 \geq 0$ there exist $\alpha > 0$, $r_0 > 0$ such that for all $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$ and for all $r \geq r_0$*

$$\left|1 - \frac{H_\lambda(r)}{r}\right| \leq \alpha e^{-\min(2\sqrt{\lambda},1)r}. \tag{25}$$

Proof. Let $u_\lambda(r) = 1 - \dfrac{H_\lambda(r)}{r}$. Differentiate twice to obtain

$$u_\lambda'' + \frac{2}{r}u_\lambda' = -\frac{H_\lambda''}{r}. \tag{26}$$

Replacing H_λ'' from (YMH-2) yields

$$u_\lambda'' + \frac{2}{r}u_\lambda' - \frac{4\lambda H_\lambda}{r}\left(1 + \frac{H_\lambda}{r}\right)u_\lambda = -\frac{2K_\lambda^2 H_\lambda}{r^3}. \tag{27}$$

Now let

$$s(r) = \alpha e^{-\min(2\sqrt{\lambda},1)r}$$

to be the test function. The aim is to show that there exist $\alpha \geq 1$ and $r_0 > 0$ such that

$$|u_\lambda(r)| \leq s(r),$$

for all $r \geq r_0$ and for all λ in an appropriate range. Since

$$\left|1 - \frac{H_\lambda(r)}{r}\right| \leq \sqrt{\frac{C}{r}}$$

and

$$|K_\lambda(r)| \leq \alpha e^{-r/2},$$

(see [AD]), for $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$

$$\begin{aligned}
& (s \pm u_\lambda)'' + \frac{2}{r}(s \pm u)' - \frac{4\lambda H_\lambda}{r}(1 + \frac{H_\lambda}{r})(s \pm u_\lambda) \\
= \; & \mp \frac{2K_\lambda^2 H_\lambda}{r^3} + \alpha \min(4\lambda, 1) e^{-\min(2\sqrt{\lambda},1)r} - \frac{2}{r}\alpha \min(2\sqrt{\lambda}, 1) e^{-\min(2\sqrt{\lambda},1)r} \\
& - \frac{4\lambda H_\lambda}{r}(1 + \frac{H_\lambda}{r})\alpha e^{-\min(2\sqrt{\lambda},1)r} \\
\leq \; & \frac{2e^{-r}}{r^2} + \alpha e^{-\min(2\sqrt{\lambda},1)r} - 4\lambda \left(1 - \sqrt{\frac{C}{r}}\right)\left(1 + 1 - \sqrt{\frac{C}{r}}\right)\alpha e^{-\min(2\sqrt{\lambda},1)r} \\
\leq \; & \frac{2e^{-r}}{r^2} + \alpha e^{-\min(2\sqrt{\lambda},1)r} - 8\lambda \alpha e^{-\min(2\sqrt{\lambda},1)r} + 12(\lambda_0 + 1)\sqrt{\frac{C}{r}}\alpha e^{-\min(2\sqrt{\lambda},1)r}
\end{aligned}$$

Note that the term $-8\lambda\alpha e^{-\min(2\sqrt{\lambda},1)r}$ is negative and it eventually makes the above expression negative as well since $|\lambda - \lambda_0| \leq 1$, i.e. there exists $r_0 > 0$ such that the last expression is negative for all $r \geq r_0$ and for all $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$. Since

$$u_\lambda(r_0) \leq 1 + \sqrt{\frac{C}{r_0}}$$

for all λ in the range $[\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$, choose $\alpha \geq 1$ such that

$$s_\lambda(r_0) \pm u_\lambda(r_0) > 0$$

for all such λ's. □

Corollary 4.3. *For every* $\lambda_0 > 0$ *there exists* $M > 0$ *such that for* λ *in* $[0, \lambda_0]$ *the following holds for* $r \geq 0$

$$|H_\lambda(r) - H_{\lambda_0}(r)| \leq M. \tag{28}$$

References

[AD] Androulakis G.; Dostoglou S., *On the Stability of Monopole Solutions* Nonlinearity **11**, no 3 (1998) 377–408

[D] Dostoglou S. *Remarks on the Existence of Solutions of the Yang-Mills-Higgs Equations with Arbitrary Higgs Potential*, Preprint available at http://www.math.missouri.edu/people/sdostoglou.html

[H] Higgs P. *Spontaneous Symmetry Breakdown without Massless Boson* Phys. Rev. **145** (1966), 1156-1163.

[JT] Jaffe, A.; Taubes C.H., *Vortices and Monopoles*, Birkhauser, 1980.

[L] Lawson, H. B. *The theory of gauge fields in four dimensions*, CBMS Regional Conference Series in Mathematics, **58**. American Mathematical Society, Providence, R.I., 1985.

[M] Maison D. *Uniqueness of the Prasad-Sommerfield Monopole Solution*, Nucl. Phys.B **182**, 144–150.

[tH] 't Hooft G. *Magnetic monopoles in unified gauge theories* Nucl. Phys. B **79**, 1976, 276

[P] Polyakov A. M.: JETP Letters **20**, 1974, 194.

[T1] Taubes, C.H. *Monopoles and Maps from* S^2 *to* S^2*; the Topology of the Configuration Space*, Commun. Math. Phys. **95** (1984), 345-391.

[T2] Taubes, C.H. *Stability in Yang-Mills Theories*, Commun. Math. Phys. **91** (1983), 235 -263.

[T3] Taubes C. H. *Existence of a Non-Minimal Solutions to the SU(2) Yang-Mills-Higgs Equations, Part I* Comm. Math. Phys. **86**, 1982, 257–298.

[Pa] Palais, R. S. The principle of symmetric criticality. Comm. Math. Phys. **69** (1979), no. 1, 19–30.

[WB] Wirzba, A.; Bang H. *The mode spectrum and stability analysis of skyrmions on the* 3*-sphere* Nuclear Phys. A **515** (1990), no 4, 571- 598.

Mathematical Sciences Building, University of Missouri-Columbia, Columbia MO 65211

E-mail address: giorgis@math.missouri.edu, stamatis@euler.math.missouri.edu