# INTEGRATION VIA SUBSTITUTION

SO FAR, WE REALLY ONLY HAVE THE POWER RULE FOR INTEGRATION (& SOME SPECIAL CASES, $e^x$ & $1/x$)

**OBJECTIVES:**

- DEFINE & PRACTICE INTEGRATION BY u-SUBSTITUTION
- RELATE THE CHAIN RULE FOR DERIVATIVES TO u-SUBSTITUTION

**Ex.** RECALL THE CHAIN RULE...

$$\frac{d}{dx}\sqrt[3]{x^2+x+7}$$

$f(x) = \text{OUT} = \sqrt[3]{x}$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$

$g(x) = \text{IN} = x^2+x+7$ $\quad g'(x) = 2x+1$

$$\frac{d}{dx}\sqrt[3]{x^2+x+7} = \frac{1}{3}(x^2+x+7)^{-2/3}\cdot(2x+1)$$

(by the chain rule!)

NOW... WE KNOW...

$$\underbrace{\int \frac{1}{3}(x^2+x+7)^{-2/3}\cdot(2x+1)\,dx}_{} = \sqrt[3]{x^2+x+7} + C$$

BUT IF I STARTED WITH THIS, HOW COULD I INTEGRATE?

Notice the following:

$\frac{d}{dx}(x^2+x+7) = 2x+1$.. so if we let $u = x^2+x+7$ then $du = 2x+1\,dx$

so we can rewrite the integral:

$$\int \frac{1}{3}u^{-2/3}\,du \leftarrow \int \frac{1}{3}(x^2+x+7)^{-2/3}(2x+1)dx$$

NOW WE CAN INTEGRATE THIS USING THE POWER RULE!

**THE SUBSTITUTION RULE** (u-SUBSTITUTION)

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du$$

WHERE $u = g(x)$

$du = g'(x)dx$

PLEASE PRACTICE THIS!

**Ex.** $\int 18x^2\sqrt[4]{6x^3+5}\,dx$

$u = 6x^3+5$

$du = 18x^2\,dx$

REWRITE INTEGRAL $\rightarrow$ $\int \sqrt[4]{u}\,du$

WHEN YOU LET $u = 6x^3+5$..

$$\int 18x^2\cdot\sqrt[4]{u}\,dx$$

THEN WHEN YOU WRITE $du = 18x^2dx$...

$$\int \sqrt[4]{u}\,du$$

$n = 1/4$ so $n+1 = 5/4$

$$\int \sqrt[4]{u}\,du = \int u^{1/4}du = \frac{4u^{5/4}}{5} + C$$

but we know what u is!

$$= \boxed{\frac{4(6x^3+5)^{5/4}}{5} + C}$$

**Ex.** $\int 2xe^{x^2+1}\,dx$

$u = x^2+1 \rightarrow \int 2xe^u dx$

$du = 2x\,dx \rightarrow \int e^u du$

then $\int e^u du = e^u + C = \boxed{e^{x^2+1} + C}$

Ex. $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ ----→ Why might we use u-sub here?

$u = 1-4x^2$

$du = \underbrace{-8x\,dx}$

IN THE INTEGRAL ABOVE, I HAVE $x\,dx$... NOT $-8x\,dx$. HOW TO FIX THIS?

$n = -\frac{1}{2}$ so $n+1 = \frac{1}{2}$

$\frac{-1}{8}\,du = x\,dx$ SO, WE REWRITE AS: $\int \frac{1}{\sqrt{u}} \cdot \left(\frac{-1}{8}\,du\right) = \frac{-1}{8}\int u^{-1/2}\,du$

$= \frac{-1}{8}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{-1}{4}u^{1/2} + C$

BUT WE KNOW WHAT $u$ IS!

SO

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \boxed{\frac{-1}{4}(1-4x^2)^{1/2} + C}$$

Ex $\int \frac{1}{x \cdot \ln(x)}\,dx$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\ln(x)| + C}$

Ex $\int \frac{3}{5x+4}\,dx$

$u = 5x+4$

$du = 5\,dx$ ⟿ BUT IN THE INTEGRAL ABOVE, I HAVE $3\,dx$.. NOT $5\,dx$.

$\Rightarrow \frac{3}{5}\,du = \frac{3}{5}\cdot 5\,dx = 3\,dx$

SO WE CAN REWRITE AS:

$$\int \frac{1}{u}\cdot\frac{3}{5}\,du = \frac{3}{5}\int\frac{1}{u}\,du = \frac{3}{5}\ln|u| + C$$

$$= \boxed{\frac{3}{5}\ln|5x+4| + C}$$