Math 141: Section 4.5 Indeterminate Forms and L'Hôpital's Rule - Notes

Indeterminate Form 0/0

Suppose we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined). Then we can examine the limit of $F(x)$ as $x \to 0$.

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \to \frac{0}{0}$$

L'Hôpital's Rule: Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 1 The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule.

(a)

$$\lim_{x \to 0} \frac{3x - \sin x}{x} \to \frac{0}{0}$$

$$\underline{\underline{L'H}} \quad \lim_{x \to 0} \frac{3 - \cos x}{1}$$

$$= 3 - 1 = 2$$

(b)

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \rightarrow \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-1/2}}{1}$$

$$= \frac{1}{2}(1+0)^{-1/2}$$

$$= 1/2$$

(c)

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2} \rightarrow \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - 1/2}{2x} \rightarrow \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2}$$

$$= \frac{-1/4}{2/1} = -1/8$$

(d)

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{1-\cos x}{3x^2} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sin x}{6x} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

Using L'Hôpital's Rule

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we STOP differentiating. L'Hôpital's Rule does NOT apply when either the numerator or denominator has a finite nonzero limit.

$$\lim_{x\to 0} \frac{x^{20}}{x^{30}} = \lim_{x\to 0} \frac{1}{x^{10}}$$

Applying L'Hopital is valid, but not the best choice. Always try to simplify first.

Example 2 L'Hôpital's Rule applies to one-sided limits as well.

(a)

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} \rightarrow \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\cos x}{2x} \rightarrow \frac{1}{\text{small, positive}}$$

$$= \infty$$

(b)

$$\lim_{x\to 0^-} \frac{\sin x}{x^2}$$

$$\overset{L'H}{=} \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Example 3

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \rightarrow \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^{1/2}}} = \lim_{x\to\infty} \frac{x^{1/2}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{x^{1/2}} = 0$$

Example 4

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \to \infty - \infty$$

Before applying L'H, we must have an indeterminate form of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} \to \frac{0}{0}$$

$$\stackrel{L'H}{=} \lim_{x\to 0} \frac{1-\cos x}{\sin x + \underline{x\cos x}} \to \frac{0}{0}$$

$$\stackrel{L'H}{=} \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x + -x\sin x} = \frac{0}{2} = 0$$

Indeterminate Powers Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function, then using L'Hôpital's Rule, and then exponentiating the result.

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be finite or infinite.

Example 5 Apply l'Hôpital's Rule to show that

$$\lim_{x\to 0^+} (1+x)^{1/x} = e.$$

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} \to 1^\infty$$

$$\ln\left((1+x)^{1/x}\right) = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x} \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \lim_{x\to 0^+} \frac{1}{1+x} = 1$$

So, $\lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right) = 1$

which implies

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} e^{\ln((1+x)^{1/x})} = e^1 = e$$

Ex 6: Find $\lim_{x \to \infty} x^{1/x} \to \infty^0$

$$\ln(x^{1/x}) = \frac{1}{x}\ln(x) = \frac{\ln(x)}{x}$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x} \to \frac{\infty}{\infty}$$

$$\stackrel{L'H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

So, $\lim_{x \to \infty} x^{1/x} = e^0 = 1$

$$\left(\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{1/n} = 1\right)$$