Math 141: Section 4.5 Indeterminate Forms and L'Hôpital's Rule - Notes

Indeterminate Form 0/0

Suppose we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined). Then we can examine the limit of $F(x)$ as $x \to 0$.

L'Hôpital's Rule: Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 1 The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule.

(a)

$$\lim_{x \to 0} \frac{3x - \sin x}{x}$$

(b)

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$$

(c)

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$$

(d)

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we STOP differentiating. L'Hôpital's Rule does NOT apply when either the numerator or denominator has a finite nonzero limit.

Example 2 L'Hôpital's Rule applies to one-sided limits as well.

(a)

$$\lim_{x \to 0^+} \frac{\sin x}{x^2}$$

(b)

$$\lim_{x \to 0^-} \frac{\sin x}{x^2}$$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Example 3

$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$$

Example 4

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Indeterminate Powers Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function, then using L'Hôpital's Rule, and then exponentiating the result.

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be finite or infinite.

Example 5 Apply l'Hôpital's Rule to show that

$$\lim_{x\to 0^+} (1+x)^{1/x} = e.$$