Math 122

Ann Clifton [1]

[1] University of South Carolina, Columbia, SC USA

Calculus for Business Administration and Social Sciences

Outline

1. 5.2: The Definite Integral

Outline

The Definite Integral

- In the last section, we saw that for a continuous function f on an interval $[a, b]$, the error for Left-Hand Sums and Right-Hand Sums goes to zero as the number of points in the partition becomes large.

The Definite Integral

- In the last section, we saw that for a continuous function f on an interval $[a, b]$, the error for Left-Hand Sums and Right-Hand Sums goes to zero as the number of points in the partition becomes large.
- As the error term goes to zero, the Left-Hand Sum increases towards a fixed value and the Right-Hand Sum decreases towards that same value.

THE DEFINITE INTEGRAL

- In the last section, we saw that for a continuous function f on an interval $[a, b]$, the error for Left-Hand Sums and Right-Hand Sums goes to zero as the number of points in the partition becomes large.
- As the error term goes to zero, the Left-Hand Sum increases towards a fixed value and the Right-Hand Sum decreases towards that same value.
- The common value that these sums approach is called a *limit*, and we call this particular limit the Definite Integral.

Definition 1

Assume that f is continuous on the interval $[a, b]$. The *definite integral of f from a to b* is

$$\int_a^b f(t)\,\mathrm{d}\,t = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i)\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t,$$

where the set of t-values

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

is a partition of $[a, b]$ into n intervals of length

$$\Delta t = \frac{b-a}{n}.$$

GEOMETRIC EXAMPLES

Compute $\int_{x_0}^{x_1} b \, \mathrm{d}\, x$ for $0 < b$.

Geometric Examples

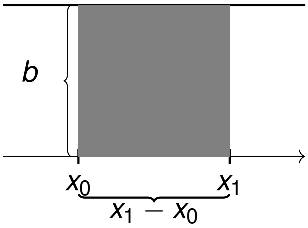
Compute $\int_{x_0}^{x_1} b \, \mathrm{d}\, x$ for $0 < b$.

The integral is just the area of the shaded region below:

Geometric Examples

Compute $\int_{x_0}^{x_1} b \, \mathrm{d}x$ for $0 < b$.

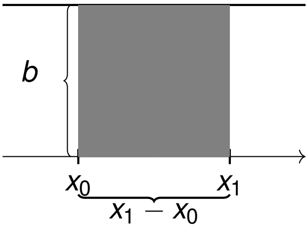
The integral is just the area of the shaded region below:



GEOMETRIC EXAMPLES

Compute $\int_{x_0}^{x_1} b\,\mathrm{d}\,x$ for $0 < b$.

The integral is just the area of the shaded region below:



$$\int_{x_0}^{x_1} b\,\mathrm{d}\,x = b(x_1 - x_0).$$

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

Geometric Examples

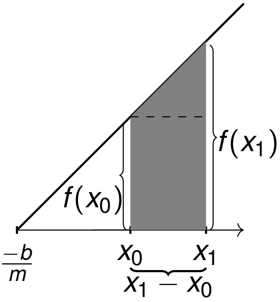
Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

The integral is just the area of the shaded region below:

GEOMETRIC EXAMPLES

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

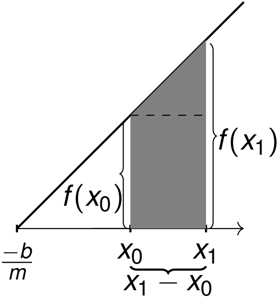
The integral is just the area of the shaded region below:



Geometric Examples

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

The integral is just the area of the shaded region below:



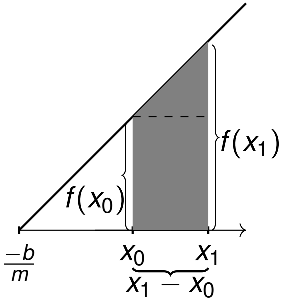
$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x =$$

MATH 122
CLIFTON
5.2: THE DEFINITE INTEGRAL
5.5: THE FUNDAMENTAL THEOREM OF CALCULUS

Geometric Examples

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

The integral is just the area of the shaded region below:

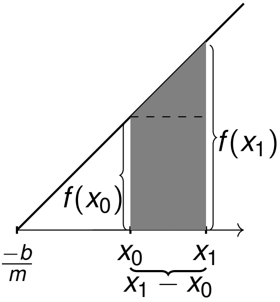


$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x = f(x_0)(x_1 - x_0) +$$

Geometric Examples

Let $f(x) = mx + b$ for $0 < b$, $0 < m$. Compute $\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x$ for $\frac{-b}{m} < x_0$.

The integral is just the area of the shaded region below:



$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}\,x = f(x_0)(x_1 - x_0) + \frac{1}{2}\left[f(x_1) - f(x_0)\right](x_1 - x_0).$$

Geometric Examples

Compute $\int_{-1}^{1} \sqrt{1 - x^2} \, d\, x$.

Geometric Examples

Compute $\int_{-1}^{1} \sqrt{1 - x^2}\, \mathrm{d}\, x$.

- Observe $x^2 + y^2 = 1$ is a circle of radius 1 centered at $(0, 0)$ and the area of a cirlce of radius r is $\pi \cdot r^2$.

Geometric Examples

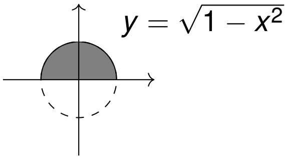
Compute $\int_{-1}^{1} \sqrt{1-x^2}\, \mathrm{d}\, x$.

- Observe $x^2 + y^2 = 1$ is a circle of radius 1 centered at $(0, 0)$ and the area of a cirlce of radius r is $\pi \cdot r^2$.
- The curve $y = \sqrt{1-x^2}$ is the top half of this circle, and the integral is the area bounded by this semicircle:

Geometric Examples

Compute $\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}\,x$.

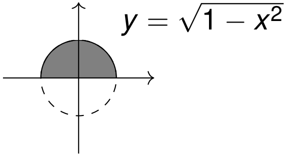
- Observe $x^2 + y^2 = 1$ is a circle of radius 1 centered at $(0,0)$ and the area of a cirlce of radius r is $\pi \cdot r^2$.
- The curve $y = \sqrt{1-x^2}$ is the top half of this circle, and the integral is the area bounded by this semicircle:



GEOMETRIC EXAMPLES

Compute $\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}\,x$.

- Observe $x^2+y^2=1$ is a circle of radius 1 centered at $(0,0)$ and the area of a cirlce of radius r is $\pi \cdot r^2$.
- The curve $y=\sqrt{1-x^2}$ is the top half of this circle, and the integral is the area bounded by this semicircle:

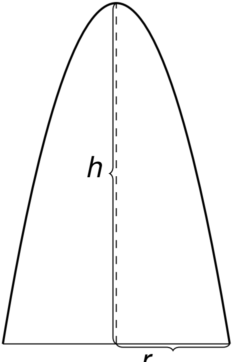


- Therefore

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}\,x = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

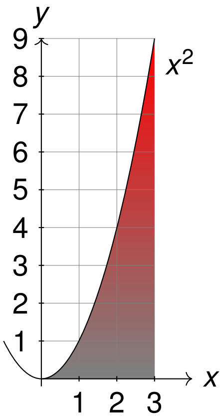
Geometric Examples

The area under the parabola



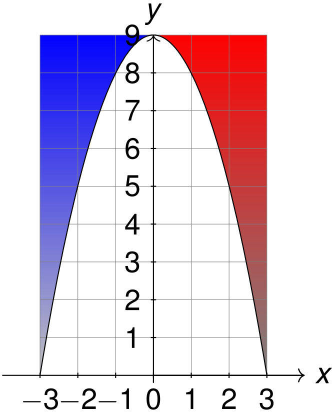
is $\frac{4}{3}rh$. Use this to compute $\int_0^3 x^2 \, dx$.

The integral is just the area under the parabola:



Geometric Examples

If we flip the picture upside down, we have the picture



And we note that the red and blue areas are, by symmetry, the same.

Geometric Examples

Hence we can compute the area using

Geometric Examples

Hence we can compute the area using

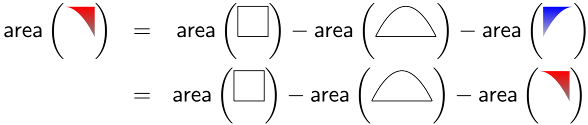
area () =

Hence we can compute the area using

area () = area () − area () − area ()

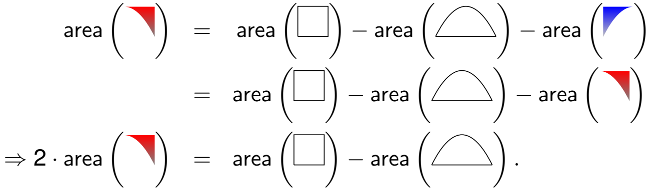
GEOMETRIC EXAMPLES

Hence we can compute the area using



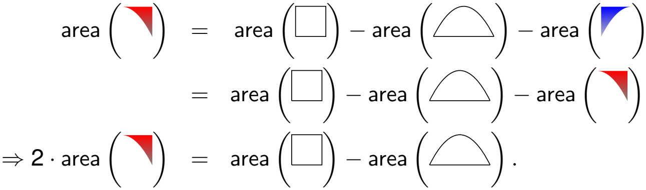
GEOMETRIC EXAMPLES

Hence we can compute the area using



GEOMETRIC EXAMPLES

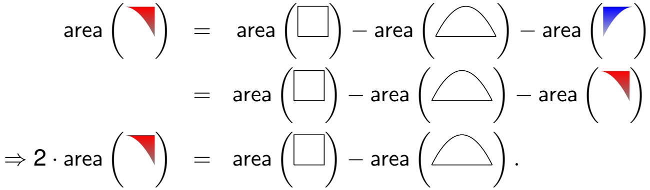
Hence we can compute the area using



Therefore

Geometric Examples

Hence we can compute the area using

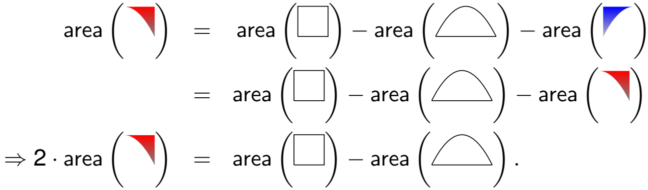


Therefore

$$\int_0^3 x^2 \, \mathrm{d}\, x \quad =$$

GEOMETRIC EXAMPLES

Hence we can compute the area using

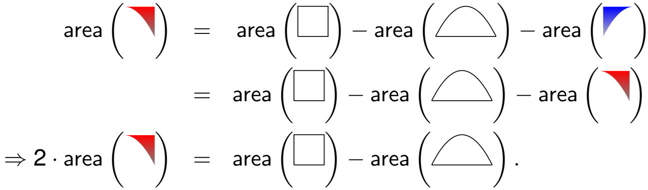


Therefore

$$\int_0^3 x^2 \, \mathrm{d}\, x = \frac{1}{2}\left[6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3\right]$$

Geometric Examples

Hence we can compute the area using

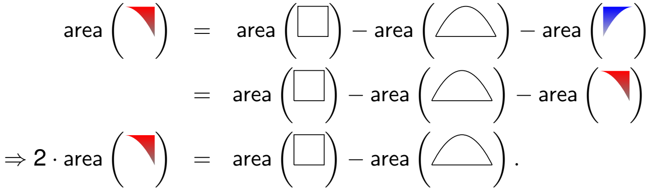


Therefore

$$\int_0^3 x^2 \, d\,x = \frac{1}{2}\left[6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3\right]$$
$$= \frac{1}{2}\left[6 \cdot 9\left(1 - \frac{2}{3}\right)\right]$$

Geometric Examples

Hence we can compute the area using



Therefore

$$\int_0^3 x^2 \, \mathrm{d}\, x = \frac{1}{2}\left[6 \cdot 9 - \frac{4}{3} \cdot 9 \cdot 3\right] = \frac{1}{2}\left[6 \cdot 9\left(1 - \frac{2}{3}\right)\right] = 9.$$

Having to estimate definite integrals is incredibly unsatisfying.

Having to estimate definite integrals is incredibly unsatisfying. Fortunately, we have the following:

Having to estimate definite integrals is incredibly unsatisfying. Fortunately, we have the following:

THEOREM 1 (FUNDAMENTAL THEOREM OF CALCULUS)

If $F'(t)$ is a continuous function on $[a, b]$, then

$$\int_a^b F'(t)\,\mathrm{d}t = F(b) - F(a).$$

Example

Let $F(t) = \frac{1}{3}x^3$.

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) =$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right)$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2\,\mathrm{d}x =$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 \, \mathrm{d}x = \int_0^3 F'(x) \, \mathrm{d}x$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 \, \mathrm{d}x = \int_0^3 F'(x) \, \mathrm{d}x = F(3) - F(0)$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 \, \mathrm{d}x = \int_0^3 F'(x) \, \mathrm{d}x = F(3) - F(0) = 3^2 - 0^2$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 \, dx = \int_0^3 F'(x) \, dx = F(3) - F(0) = 3^2 - 0^2 = 9.$$

Example

Let $F(t) = \frac{1}{3}x^3$. Differentiating F gives

$$F'(t) = \frac{1}{3}\left(3x^2\right) = x^2$$

and hence by the Fundamental Theorem of Calculus

$$\int_0^3 x^2 \, dx = \int_0^3 F'(x) \, dx = F(3) - F(0) = 3^2 - 0^2 = 9.$$

Remark 1

Essentially, this says that the area between the derivative of F and the x-axis from a and b is just the total change in F on the interval $[a, b]$.

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

EXAMPLE

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

EXAMPLE

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) =$$

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) = \int_0^b C'(q)\,\mathrm{d}q.$$

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) = \int_0^b C'(q)\,\mathrm{d}q.$$

- Total cost to produce b units:

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) = \int_0^b C'(q)\,\mathrm{d}q.$$

- Total cost to produce b units:

$$C(b) = \qquad .$$

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) = \int_0^b C'(q)\,\mathrm{d}q.$$

- Total cost to produce b units:

$$C(b) = C(b) - C(0) + C(0) \qquad .$$

Example

For a cost function, $C(q)$, the total change in the cost on $[a, b]$ is given by

$$\int_a^b C'(q)\,\mathrm{d}q.$$

Given the marginal cost $C'(q)$ and fixed costs:

- Total variable cost to produce b units:

$$C(b) - C(0) = \int_0^b C'(q)\,\mathrm{d}q.$$

- Total cost to produce b units:

$$C(b) = C(b) - C(0) + C(0) = \int_0^b C'(q)\,\mathrm{d}q + C(0).$$